Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 1 (2017), pp. 11-19

DOI: 10.4208/ata.2017.v33.n1.2

On the Relation of Shadowing and Expansivity in Nonautonomous Discrete Systems

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Received 2 December 2014; Accepted (in revised version) 23 September 2016

Abstract. In this paper we study shadowing property for sequences of mappings on compact metric spaces, i.e., nonautonomous discrete dynamical systems. We investigate the relations of various expansivity properties with shadowing and h-shadowing property.

Key Words: Shadowing, h-shadowing, locally expanding, uniformly weak expanding, locally weak expanding.

AMS Subject Classifications: 54H20, 37B55, 39A05

1 Introduction

Let (X,d) be a compact metric space, and f be a continuous map on X. We consider the associated autonomous difference equation of the following form:

$$x_{i+1}=f(x_i). \tag{1.1}$$

A finite or infinite sequence $\{x_0,x_1,\cdots\}$ of points in X is called a δ-pseudo-orbit ($\delta>0$) of (1.1) if $d(f(x_{i-1}),x_i)<\delta$ for all $i\geq 1$. We say that Eq. (1.1), (or f) has usual shadowing property if for every $\varepsilon>0$, there exists $\delta>0$ such that for every δ-pseudo-orbit $\{x_0,x_1,\cdots\}$, there exists $y\in X$ with $d(f^i(y),x_i)<\varepsilon$ for all $i\geq 0$. The notion of pseudo-orbits appeared in several branches of dynamical systems theory, and various types of the shadowing property were presented and investigated extensively, see [5, 6, 11, 12].

In this paper we study shadowing property of nonautonomous discrete systems. We consider the compact metric space X and a sequence $f_{1,\infty}=\{f_i\}_{i=1}^{\infty}$ in which each f_i :

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$X \to X$ is continuous. We call the pair $(X, f_{1,\infty})$ a nonautonomous discrete system (on X). For further simplicity we use only $f_{1,\infty}$ in the sequel. The associated nonautonomous difference equation has the following form:

$$x_{i+1} = f_i(x_i). \tag{1.2}$$

For every $n \geq i \geq 1$, we write $f_i^n = f_n \circ f_{n-1} \circ \cdots \circ f_i$.

Orbit of a nonautonomous system $f_{1,\infty}$ in a point x is the following sequence:

$$O(x) = \{x, f_1(x), f_2 \circ f_1(x), \cdots, f_n \circ \cdots \circ f_1(x), \cdots\}.$$

On the other hand a pseudo-orbit of the system is as follows:

Definition 1.1. A finite or infinite sequence $\{x_0, x_1, \cdots\}$ of points in X is called a δ-pseudo-orbit ($\delta > 0$) of (1.2), if $d(f_i(x_{i-1}), x_i) < \delta$ for all $i \geq 1$.

In the nonautonomous case the standard definition of shadowing has the following form, see [12]:

Definition 1.2. We say that $f_{1,\infty}$ has shadowing property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, \cdots\}$, there exists $y \in X$ with $d(y, x_0) < \varepsilon$ and $d(f_1^i(y), x_i) < \varepsilon$, for all $i \geq 1$.

In this paper we investigate the relation of various expansivity such as positively expansive, locally expanding, weakly locally expanding, $\cdots$, with shadowing and h-shadowing property.

2 Shadowing and expansivity

First we prove the following simple lemma.

Lemma 2.1. *The sequence $f_{1,\infty}$ has shadowing property if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite δ-pseudo-orbit is ε-shadowed.*

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be such that every finite δ-pseudo-orbit, $\frac{\varepsilon}{2}$-shadowed. Let $\{x_i\}_{i=1}^{\infty}$ be a δ-pseudo-orbit. For every $n \geq 1$, $\{x_0, x_1, \cdots, x_n\}$, $\frac{\varepsilon}{2}$-shadowed by $y_n \epsilon X$ and there is a subsequence $\{y_{n_k}\}_{k \geq 0}$ and a point $y \epsilon X$ such that $y_{n_k} \to y$ as $k \to \infty$. Now for each $i \geq 1$, there is a $n_k > i$ such that $d(f_1^i(y_{n_k}), f_1^i(y)) < \frac{\varepsilon}{2}$. Therefore

$$d(f_1^i(y), x_i) \leq d(f_1^i(y), f_1^i(y_{n_k})) + d(f_1^i(y_{n_k}), x_i) < \varepsilon$$

and hence $f_{1,\infty}$ has the shadowing property. □

There are several variants of shadowing property, we define a stronger form which is called h-shadowing, see [2, 9].