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Simultaneous Optimal Controls for Non-Stationary Stokes Systems

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Abstract. This paper deal with optimal control problems for a non-stationary Stokes system. We study a simultaneous distributed-boundary optimal control problem with distributed observation. We prove the existence and uniqueness of a simultaneous optimal control and we give the first order optimality condition for this problem. We also consider a distributed optimal control problem and a boundary optimal control problem and we obtain estimations between the simultaneous optimal control and the optimal controls of these last ones. Finally, some regularity results are presented.

Key Words: Simultaneous optimal controls, unsteady Stokes system, optimality condition.

AMS Subject Classifications: 49J20, 76D07, 65K10

1 Introduction

Let Ω be a bounded domain (i.e., connected and open set) of $\mathbb{R}^3$ with $\partial\Omega$ of class $\mathcal{C}^2$. We consider the following unsteady Stokes system

$$\begin{cases} \dfrac{\partial \boldsymbol{y}}{\partial t} - \operatorname{div}\sigma(\boldsymbol{y},p) = \boldsymbol{u} & \text{in } \ \Omega\times(0,T), \\ \operatorname{div}\boldsymbol{y} = 0 & \text{in } \ \Omega\times(0,T), \\ \boldsymbol{y} = \boldsymbol{g} & \text{on } \ \partial\Omega\times(0,T), \\ \boldsymbol{y}(0) = \boldsymbol{a} & \text{in } \ \Omega. \end{cases} \tag{1.1}$$

In this paper, we will use the notation in bold for vector functions. Here, $(\boldsymbol{y},p) = (y_1,y_2,y_3,p)$ are the velocity and the pressure of the fluid and $\sigma(\boldsymbol{y},p)$ denotes the Cauchy

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stress tensor, which is defined by Stokes law $\boldsymbol{\sigma}(\boldsymbol{y},p)=-p\ \boldsymbol{Id}+2\nu\boldsymbol{D}(\boldsymbol{y})$, where $\boldsymbol{Id}$ is the identity matrix of order 3, ν is the kinematic viscosity of the fluid and $\boldsymbol{D}(\boldsymbol{y})$ is the strain tensor defined by

$$[\boldsymbol{D}(\boldsymbol{y})]_{kl}=\frac{1}{2}\left(\frac{\partial y_k}{\partial x_l}+\frac{\partial y_l}{\partial x_k}\right).$$

Since $\operatorname{div}\boldsymbol{y}=0$, we have $-\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{y},p)=-\nu\Delta\boldsymbol{y}+\nabla p$ in Ω. The System (1.1) admits a unique (up to a constant for p) solution $(\boldsymbol{y},p)$ with

$$(\boldsymbol{y},p)\in L^2(0,T;H^1(\Omega))\cap C^0(0,T;L^2(\Omega))\times L^2(0,T;L^2(\Omega))/\mathbb{R} \tag{1.2}$$

(see below for the notation of these spaces), provided that $\boldsymbol{u}\in L^2(0,T;L^2(\Omega))$ and $\boldsymbol{g}\in L^2(0,T;H^{1/2}(\partial\Omega))\cap C^0(0,T;H^{-1/2}(\partial\Omega))$ satisfies the compatibility conditions

$$\int_{\partial\Omega}\boldsymbol{g}\cdot\boldsymbol{n}\,d\gamma=0 \quad\text{and}\quad \boldsymbol{g}(0)=\boldsymbol{a} \quad\text{on }\partial\Omega$$

with $\boldsymbol{a}\in H(\operatorname{div};\Omega)$, where

$$H(\operatorname{div};\Omega)=\{\boldsymbol{a}\in L^2(\Omega)\ \text{ such that }\ \operatorname{div}\boldsymbol{a}=0\},$$

properties of this space can be found in [7]. Moreover, there exists a constant K depending of Ω and ν such that

$$\|\boldsymbol{y}\|_{L^2(H^1(\Omega))}+\|p\|_{L^2(L^2(\Omega))}\le K\Big(\|\boldsymbol{u}\|_{L^2(L^2(\Omega))}+\|\boldsymbol{g}\|_{L^2(H^{1/2}(\partial\Omega))}+\|\boldsymbol{a}\|_{L^2(\Omega)}\Big) \tag{1.3}$$

for results on the existence, uniqueness and regularity of solutions for Stokes equations with non homogeneous data, we refer to [4,5,8,13,15].

Let X be a Banach space, we will denote by $L^p(0,T;X)$ the space of the all measurable functions $\boldsymbol{y}$ such that $\boldsymbol{y}:[0,T]\to X$ defined by $\boldsymbol{y}(t)(x)=\boldsymbol{y}(t,x)$ satisfy

$$\|\boldsymbol{y}\|_{L^p(0,T;X)}=\left(\int_0^T\|\boldsymbol{y}(t)\|_X^p\,dt\right)^{1/p}<+\infty, \qquad \text{if } p\in[1,+\infty),$$
$$\|\boldsymbol{y}\|_{L^\infty(0,T;X)}=ess\sup_{0\le t\le T}\|\boldsymbol{y}(t)\|_X<+\infty, \qquad \text{if } p=+\infty.$$

For the sake of simplicity, we will often use $L^p(X)$ instead of $L^p(0,T,X)$. In what follows, we will denote $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\partial\Omega}$ the usual scalar products in $L^2(L^2(\Omega))$ and $L^2(L^2(\partial\Omega))$ respectively; and we also write X^* the dual vectorial space of X and $\langle\cdot,\cdot\rangle$ the duality pairing.

In this work we will consider $\boldsymbol{u}$ and $\boldsymbol{g}$ as control variables and we fix the initial condition $\boldsymbol{a}\in H(\operatorname{div};\Omega)$.

Now, we formulate the optimal control problems with distributed observation that we will study in this paper.