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On M–Term Approximations of the Nikol'skii–Besov Class in the Lorentz Spaces

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Abstract. In this paper, we consider a Lorentz space with a mixed norm of periodic functions of many variables. We obtain the exact estimation of the best M-term approximations of Nikol'skii's and Besov's classes in the Lorentz space with the mixed norm.

Key Words: Lorentz space, Besov's class, approximation.

AMS Subject Classifications: 41A10, 41A25

1 Introduction

Let $\overline{x}=(x_1,\cdots,x_m)\in\mathbb{T}^m=[0,2\pi]^m$ and $p_j\in(1,+\infty)$, $\theta_j\in[1,+\infty)$, $j=1,\cdots,m$. Let $L_{\bar{p},\bar{\theta}}(\mathbb{T}^m)$ denotes the space of Lebesgue measureable functions $f(\bar{x})$ defined on $\mathbb{R}^m$, which have 2π–period with respect to each variable such that

$$\|f\|_{\overline{p},\overline{\theta}}=\|\cdots\|f\|_{p_1,\theta_1}\cdots\|_{p_m,\theta_m}<+\infty,$$

where

$$\|g\|_{p,\theta}=\left\{\int_0^{2\pi}(g^*(t))^\theta t^{\frac{\theta}{p}-1}dt\right\}^{\frac{1}{\theta}},$$

where g^* a non-increasing rearrangement of the function $|g|$ (see [1]).

It is known that if $\theta_j=p_j$, $j=1,\cdots,m$, then $L_{\bar{p},\bar{\theta}}(\mathbb{T}^m)=L_{\bar{p}}(\mathbb{T}^m)$ the Lebesgue measurable space of functions $f(\bar{x})$ defined on $\mathbb{R}^m$, which have 2π–period with respect to each variable with the norm

$$\|f\|_{\bar{p}}=\left[\int_0^{2\pi}\left[\cdots\left[\int_0^{2\pi}|f(\bar{x})|^{p_1}dx_1\right]^{\frac{p_2}{p_1}}\cdots\right]^{\frac{p_m}{p_{m-1}}}dx_m\right]^{\frac{1}{p_m}}<+\infty,$$

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where $\overline{p}=(p_1,\cdots,p_m)$, $1\le p_j<+\infty$, $j=1,\cdots,m$ (see [2]).

Any function $f\in L_1(\mathbb{T}^m)=L(\mathbb{T}^m)$ can be expanded to the Fourier series

$$\sum_{\overline{n}\in\mathbb{Z}^m} a_{\overline{n}}(f)e^{i\langle\overline{n},\overline{x}\rangle},$$

where $a_{\overline{n}}(f)$ Fourier coefficients of $f\in L_1(\mathbb{T}^m)$ with respect to multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}_{\bar{n}\in\mathbb{Z}^m}$, and $\mathbb{Z}^m$ is the space of points in $\mathbb{R}^m$ with integer coordinates.

For a function $f\in L(\mathbb{T}^m)$ and a number $s\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}$ let us introduce the notation

$$\delta_0(f,\bar{x})=a_0(f),\quad \delta_s(f,\overline{x})=\sum_{\overline{n}\in\rho(s)} a_{\overline{n}}(f)e^{i\langle\overline{n},\overline{x}\rangle},$$

where

$$\langle\bar{y},\bar{x}\rangle=\sum_{j=1}^{m} y_j x_j,\quad \rho(s)=\left\{\overline{k}=(k_1,\cdots,k_m)\in\mathbb{Z}^m:[2^{s-1}]\le\max_{j=1,\cdots,m}|k_j|<2^s\right\},$$

where $[a]$ is the integer part of the number a.

Let us consider Nikol'skii, Besov classes (see [2, 3]). Let $1<p_j<+\infty, 1<\theta_j<+\infty$, $j=1,\cdots,m$, $1\le\tau\le\infty$, and $r>0$

$$H^r_{\bar{p},\bar{\theta}}=\left\{f\in L_{\bar{p},\bar{\theta}}(\mathbb{T}^m):\sup_{s\in\mathbb{Z}_+}2^{sr}\|\delta_s(f)\|_{\bar{p},\bar{\theta}}\le 1\right\},$$

$$B^r_{\bar{p},\bar{\theta},\tau}=\left\{f\in L_{\bar{p},\bar{\theta}}(\mathbb{T}^m):\left(\sum_{s\in\mathbb{Z}_+}2^{sr\tau}\|\delta_s(f)\|^{\tau}_{\bar{p},\bar{\theta}}\right)^{\frac{1}{\tau}}\le 1\right\}.$$

It is known that for $1\le\tau\le\infty$ the following holds

$$B^r_{\bar{p},\bar{\theta},1}\subset B^r_{\bar{p},\bar{\theta},\tau}\subset B^r_{\bar{p},\bar{\theta},\infty}=H^r_{\bar{p},\bar{\theta}}.$$

Let $f\in L_{\bar{p},\bar{\theta}}(\mathbb{T}^m)$ and $\left\{\bar{k}^{(j)}\right\}_{j=1}^{M}$ be a system of vectors $\bar{k}^{(j)}=(k_1^{(j)},\cdots,k_m^{(j)})$ with integer coordinates. Consider the quantity

$$e_M(f)_{\bar{p},\bar{\theta}}=\inf_{\bar{k}^{(j)},b_j}\left\|f-\sum_{j=1}^{M}b_j e^{\langle i\bar{k}^{(j)},\bar{x}\rangle}\right\|_{\bar{p},\bar{\theta}},$$

where b_j are arbitrary numbers.

The quantity $e_M(f)_{\bar{p},\bar{\theta}}$ is called the best M–term approximation of a function $f\in L_{\bar{p},\bar{\theta}}(\mathbb{T}^m)$. For a given class $F\subset L_{\bar{p},\bar{\theta}}(\mathbb{T}^m)$ let

$$e_M(F)_{\bar{p},\bar{\theta}}=\sup_{f\in F}e_M(f)_{\bar{p},\bar{\theta}}.$$