Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 4 (2017), pp. 333-354

DOI: 10.4208/ata.2017.v33.n4.4

A Generalized Lyapunov-Sylvester Computational Method for Numerical Solutions of NLS Equation with Singular Potential

Riadh Chteoui[1,3] and Anouar Ben Mabrouk[2,3,4,*]

[1] *Departement de Mathématiques, Faculté des Sciences de Tunis, Université de Tunis ElManar, Tunisie*
[2] *Department of Mathematics, Higher Institute of Applied Mathematics and Informatics, Street of Assad Ibn Alfourat, Kairouan University, 3100 Kairouan, Tunisia*
[3] *UR11ES50–Algebra, Number Theory and Nonlinear Analysis, Department of Mathematics, Faculty of Sciences, Monastir University, 5000 Monastir, Tunisia*
[4] *Department of Mathematics, College of Sciences, University of Tabuk, Saudi Arabia Kingdom*

Received 16 February 2017; Accepted (in revised version) 5 August 2017

Abstract. In the present paper a numerical method is developed to approximate the solution of two-dimensional Nonlinear Schrödinger equation in the presence of a singular potential. The method leads to generalized Lyapunov-Sylvester algebraic operators that are shown to be invertible using original topological and differential calculus issued methods. The numerical scheme is proved to be consistent, convergent and stable using the Lyapunov criterion, lax equivalence theorem and the properties of the generalized Lyapunov-Sylvester operators.

Key Words: NLS equation, finite-difference scheme, stability analysis, Lyapunov criterion, consistency, convergence, error estimates, Lyapunov operator.

AMS Subject Classifications: 35B05, 65M06

1 Introduction

The Schrödinger equation is widely studied from both numerical and theoretical points of view. This is due to its relation to the modeling of real physical phenomena such as Newton's laws and conservation of energy in classical mechanics, behaviour of dynamical

*Corresponding author. *Email addresses:* `rchteoui@yahoo.fr` (R. Chteoui), `anouar.benmabrouk@fsm.rnu.tn` (A. Ben Mabrouk)

systems, the description of a particle in a non-relativistic setting in quantum mechanics, etc. The Schrödinger's linear equation states that

$$\Delta\psi+\frac{8\pi^2 m}{\hbar^2}(E-V(x))\psi=0, \tag{1.1}$$

where ψ is the Schrödinger wave function, m is the mass, $\hbar$ denotes Planck's constant, E is the energy and V is the potential energy. This equation is a prototypical dispersive linear partial differential equation related to Bose-Einstein condensates and nonlinear optics [10], propagation of electric fields in optical fibers [22,28], self-focusing and collapse of Langmuir waves in plasma physics [34], behaviour of rogue waves in oceans [30].

Based upon the analogy between mechanics and optics, Schrödinger established the classical derivation of his equation. By developing a perturbation method, he proved the equivalence between his wave mechanics equation and Heisenberg's matrix one and thus introduced the time dependent version stated hereafter with a cubic nonlinearity

$$i\hbar\psi_t=-\frac{\hbar^2}{2m}\Delta\psi+V(x)\psi-\gamma|\psi|^2\psi \quad \text{in } \mathbb{R}^N, \quad (N\geq 2). \tag{1.2}$$

However, in the nonlinear case such as (1.2), the structure of the nonlinear Schrödinger equation is more complicated. The nonlinear Schrödinger equation is also related to electromagnetic, ferromagnetic fields as well as magnums, high-power ultra-short laser self-channelling in matter, condensed matter theory, dissipative quantum mechanics, [2], film equations, etc (see [1,32]).

In [19] and [29] the potential V is assumed to be bounded with a non-degenerate critical point at $x=0$. More precisely, V belongs to the class V_a, for some real parameter a (see [26]). With suitable assumptions it is proved in [29] with Lyapunov-Schmidt type reduction the existence of standing wave solutions of problem (1.2), of the form

$$\psi(x,t)=e^{-iEt/\hbar}u(x). \tag{1.3}$$

The nonlinear Schrödinger equation (1.2) is thus reduced to the semilinear elliptic equation

$$-\frac{\hbar^2}{2m}\Delta u+(V(x)-E)u=|u|^2u.$$

Setting $y=\hbar^{-1}x$ and replacing y by x we get

$$-\Delta u+2m(V_\hbar(x)-E)u=|u|^2u \quad \text{in|} \ \mathbb{R}^N, \tag{1.4}$$

where $V_\hbar(x)=V(\hbar x)$.

If for some $\xi\in\mathbb{R}^N\setminus\{0\}$, $V(x+s\xi)=V(x)$ for all $s\in\mathbb{R}$, Eq. (1.2) is invariant under the Galilean transformation

$$\psi(x,t)\longmapsto\exp\left(i\xi\cdot x/\hbar-\frac{1}{2}i|\xi|^2t/\hbar\right)\psi(x-\xi t,t).$$