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On an Axiomatic about Functional Means

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Abstract. In this paper, we introduce an axiomatic approach about functional means. This includes that of operator means already introduced in the literature.

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1 Introduction

For over the last centuries, the mean-theory has been the subject of intensive research. Scalar and operator means arise in various contexts and have multiple applications in theoretical point of view as well as in practical purposes.

In recent few years, extension of operator means from the case that the variables are positive linear operators to the case that the variables are convex functionals has been investigated by many authors. Such extensions were introduced in the sense that if $m(T,S)$ is an operator mean between two positive linear operator T and S then its extension $\mathcal{F}(f,g)$ for two functional variables f and g satisfies the following connection-relationship

$$\mathcal{F}(f_T,f_S)=f_{m(T,S)}.$$

Here the notation f_T refers to the convex quadratic function generated by the positive linear operator T acting on a Hilbert space H i.e., $f_T(x)=(1/2)\langle Tx, x\rangle$ for all $x\in H$.

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In the aim to recall some standard examples of functional means, we need some notation. Let H be a complex Hilbert space and $f:H\longrightarrow\mathbb{R}\cup\{-\infty,\infty\}$ be a (convex) functional. We denote by f^* the conjugate of f defined through

$$\forall x^*\in H,\qquad f^*(x^*):=\sup_{x\in H}\Big(Re\langle x^*,\,x\rangle-f(x)\Big).\tag{1.1}$$

The arithmetic and harmonic functional means of $f,g:H\longrightarrow\mathbb{R}\cup\{\infty\}$ were introduced in [1,9,10] as follows

$$\mathcal{A}(f,g):=\frac{f+g}{2},\quad \mathcal{H}(f,g):=\Big(\frac{f^*+g^*}{2}\Big)^*=\Big(\mathcal{A}\big(f^*,g^*\big)\Big)^*.\tag{1.2}$$

The geometric functional mean of f and g, denoted here by $\mathcal{G}(f,g)$, was firstly introduced in [1] as the point-wise limit of an iterative process descending from the arithmetic and harmonic functional means. Since then some papers were written about $\mathcal{G}(f,g)$ in other points of view, see [4,8–10]. Throughout this paper, we adopt as definition of $\mathcal{G}(f,g)$ the following one, see [9] for instance:

$$\mathcal{G}(f,g)=\frac{1}{\pi}\int_0^1\frac{1}{\sqrt{t(1-t)}}\Big((1-t)f^*+tg^*\Big)^*dt,\tag{1.3}$$

which, under convenient assumptions on f and g, is equivalent to that introduced in [1].

The above three functional means satisfy the next double inequality, see [9,10]

$$\mathcal{H}(f,g)\le\mathcal{G}(f,g)\le\mathcal{A}(f,g).\tag{1.4}$$

The logarithmic functional mean of f and g was defined by the author in [10] via the following relation:

$$\mathcal{L}(f,g):=\Big(\int_0^1\Big((1-t)f+tg\Big)^*dt\Big)^*.\tag{1.5}$$

The logarithmic functional mean $\mathcal{L}(f,g)$ interpolates $\mathcal{H}(f,g)$ and $\mathcal{A}(f,g)$ in the sense that, see [10]

$$\mathcal{H}(f,g)\le\mathcal{L}(f,g)\le\mathcal{A}(f,g).\tag{1.6}$$

As far as we know, there is no inequality stated in the literature about comparison between $\mathcal{G}(f,g)$ and $\mathcal{L}(f,g)$.

For further details about the above functional means, and more other functional means, we indicate the references [4,8–10]. We also refer the reader to [6] for some interesting discussion about the geometric functional mean as well as the construction of a parameterized algorithm extending that of $\mathcal{G}(f,g)$.

The fundamental goal of the present paper is to introduce a general definition for means involving (convex) functional arguments f and g. In the case where f and g are both quadratic functions generated by positive operators, we immediately obtain in a simple way the operator version of the present functional approach.