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Commutators of Singular Integral Operators Related to Magnetic Schrödinger Operators

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Abstract. Let $A:=-(\nabla-i\vec{a})\cdot(\nabla-i\vec{a})+V$ be a magnetic Schrödinger operator on $L^2(\mathbb{R}^n)$, $n\geq 2$, where $\vec{a}:=(a_1,\cdots,a_n)\in L^2_{\rm loc}(\mathbb{R}^n,\mathbb{R}^n)$ and $0\leq V\in L^1_{\rm loc}(\mathbb{R}^n)$. In this paper, we show that for a function b in Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ with $\alpha\in(0,1)$, the commutator $[b,V^{1/2}A^{-1/2}]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $p,q\in(1,2]$ and $1/p-1/q=\alpha/n$.

Key Words: Commutator, Lipschitz space, the sharp maxical function, magnetic Schrödinger operator, Hölder inequality.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

Let b be a locally integrable function on $\mathbb{R}^n$ and T be a linear operator. For a suitable function f, the commutator is defined by $[b,T]f=bT(f)-T(bf)$. It is well known that Coifman, Rochberg and Weiss [3] proved that $[b,T]$ is a bounded operator on L^p for $1<p<\infty$ if and only if $b\in \mathrm{BMO}(\mathbb{R}^n)$, when T is a Calderón-Zygmund operator. Janson [4] proved that the commutator $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, $1<p<q<\infty$, if and only if $b\in\mathrm{Lip}_\alpha(\mathbb{R}^n)$ with $\alpha=(\frac{1}{p}-\frac{1}{q})n$, where the Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ consists of the functions f satisfying

$$\|f\|_{\mathrm{Lip}_\alpha}:=\sup_{x,y\in\mathbb{R}^n,x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty,\quad 0<\alpha<1.$$

Furthermore, Lu, Wu and Yang studied the boundedness properties of the commutator $[b,T]$ on the classical Hardy spaces when $b\in\mathrm{Lip}_\alpha(\mathbb{R}^n)$ in [12].

In recent years, more scholars pay attention to the boundedness of the commutators $[b,T]$ when T are the singular integral operators associated with the Schrödinger operator

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(cf. [1, 6–11]). When the potential V satisfies the weaker condition, the operator T may not be a Calderón-Zygmund operator. In this paper we focus on the boundedness of the commutators $[b,T]$ when T are the singular integral operators associated with the magnetic Schrödinger operator based on the research in [5] and [16].

Consider a real vector potential $\vec{a}=(a_1,\cdots,a_n)$ and an electric potential V. In this paper, we assume that

$$\begin{aligned} &a_k\in L^2_{\rm loc}(\mathbb{R}^n), \quad \forall k=1,\cdots,n,\\ &0\le V\in L^1_{\rm loc}(\mathbb{R}^n). \end{aligned}$$

Let $L_k=\partial/\partial x_k-ia_k$. We adopt the same notation as in [5] and define the *sesquilinear form* Q by

$$Q(f,g):=\sum_{k=1}^n\int_{\mathbb{R}^n}L_kf\overline{L_kg}dx+\int_{\mathbb{R}^n}Vf\overline{g}dx,$$

with domain

$$D(Q):=\{f\in L^2(\mathbb{R}^n):L_kf\in L^2(\mathbb{R}^n),\, k\in 1,\cdots,n,\sqrt{V}f\in L^2(\mathbb{R}^n)\}.$$

It is known that Q is closed and symmetric. So the magnetic Schrödinger operator A is a self-adjoint operator associated with Q.

The domain of A is given by

$$D(A)=\Big\{f\in D(Q),\,\exists g\in L^2(\mathbb{R}^n)\text{ such that }Q(f,\varphi)=\int_{\mathbb{R}^n}g\bar{\varphi}dx,\,\forall\varphi\in D(Q)\Big\},$$

and A is formally given by the following expression

$$Af=\sum_{k=1}^nL_k^*L_kf+Vf$$

or $A=-(\nabla-i\vec{a})\cdot(\nabla-i\vec{a})+V$, where L_k^* is the adjoint operator of L_k. For $k=1,\cdots,n$, the operators $L_kA^{-1/2}$ and $V^{1/2}A^{-1/2}$ are called the Riesz transforms associated with A. Moreover, it was proved in [14] that for each $k=1,\cdots,n$, the Riesz transform $L_kA^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for all $1<p\le 2$. Namely, there exists a constant $C>0$ such that

$$\|V^{1/2}A^{-1/2}f\|_{L^p(\mathbb{R}^n)}+\sum_{k=1}^n\|L_kA^{-1/2}f\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)},\quad 1<p\le 2.$$

Furthermore, in [5] Duong and Yan proved that the commutators $[b,V^{1/2}A^{-1/2}]$ and $[b,L_kA^{-1/2}]$ are bounded on L^p for $1<p\le 2$, that is, there exists a constant $C>0$ such that

$$\|[b,V^{1/2}A^{-1/2}]f\|_{L^p(\mathbb{R}^n)}+\|[b,L_kA^{-1/2}]f\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)},\quad\text{where } b\in{\rm BMO}(\mathbb{R}^n).$$