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On Quasi-Chebyshevity Subsets of Unital Banach Algebras

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Abstract. In this paper, first, we consider closed convex and bounded subsets of infinite-dimensional unital Banach algebras and show with regard to the general conditions that these sets are not quasi-Chebyshev and pseudo-Chebyshev. Examples of those algebras are given including the algebras of continuous functions on compact sets. We also see some results in C^*-algebras and Hilbert C^*-modules. Next, by considering some conditions, we study Chebyshev of subalgebras in C^*-algebras.

Key Words: Best approximation, Quasi-Chebyshev sets, Pseudo-Chebyshev, C^*-algebras, Hilbert C^*-modules.

AMS Subject Classifications: 41A50, 41A65, 41A99, 46L05

1 Introduction

The subject of approximation theory is an old branch of analysis and has attracted the attention of several mathematicians during last years. This theory which has many important applications in mathematics and some other sciences has been studied by many authors, e.g., [4, 15]. A basic problem in the theory is "Given a point x and a set W in normed space X, determine a point w_0 of W which is at a minimum distance from x" i.e. to find $w_0 \in W$ such that

$$\|x-w_0\| = d_W(x) = \inf_{w\in W}\|x-w\|. \tag{1.1}$$

The set of all best approximations to x from W is denoted by $\mathscr{P}_W(x)$. Thus

$$\mathscr{P}_W(x) := \{w\in W \mid \|x-w\| = d_W(x)\}. \tag{1.2}$$

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If each $x\in X$ has at least one best approximation in W, then W is called a proximinal set and W is said to be non-proximinal if $\mathscr{P}_W(x)=\varnothing$ for some $x\in X\backslash W$. A problem which has been intensively studied is to check whether a Banach space X does or does not contain bounded closed non-proximinal sets. The results in general Banach spaces can be found in [1,5,6]. A subset W of a Banach space X is called quasi-Chebyshev if $\mathscr{P}_W(x)$ is a non-empty and compact set in X for every $x\in X$ (see [10]). Some results on characterizations of quasi-Chebyshev subspaces in Banach spaces can be found in [9,10]. In the paper, we introduce the problem exist non-quasi-Chebyshev and non-pseudo -Chebyshev sets in unital Banach algebras. All this works done by applying the related fixed point and approximation theory results. We give characterizations of quasi-Chebyshev subalgebras in C^*-algebras in terms of substate function. The structure of this paper is as follows. In Section 2 we records some facts about Banach algebras, spectral properties of C^*-algebras $\mathbb{A}$ and Hilbert C^*-modules. In Section 3, we approach the question on the existence of non-quasi-Chebyshev sets in unital abelian Banach algebras by using the related fixed points and invariant approximation results. As a consequence, we obtain some results on the algebra of continuous functions $C(S)$, where S is a compact set. We show that every closed bounded convex set in a C^*-algebra $\mathbb{A}$ is quasi-Chebyshev if and only if $\mathbb{A}$ be finite dimensional. Similarly, we get some results for Hilbert C^*-modules. Best approximation and quasi-Chebyshev of subalgebra in C^*-algebras, is discussed and characterized in Section 4.

2 Preliminaries

Let us start with some basic definitions, which will be used later. Consider $\mathbb{A}$ as a unital algebra with the unit e. If $\mathbb{A}$ is a Banach space with respect to a norm which satisfies the multiplicative inequality

$$\|xy\|\le\|x\|\|y\|\,(x,y\in\mathbb{A}), \tag{2.1}$$

then the pair $(\mathbb{A},\|\cdot\|)$ is called a normed algebra. A complete unital normed algebra is called unital Banach algebra. $a\in\mathbb{A}$ is said to be invertible if there is an element b in $\mathbb{A}$ such that $ab=ba=e$. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbol F denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. The spectrum of an element x of a unital algebra $\mathbb{A}$ over F is the set

$$\sigma(x)=\{\lambda\in F:x-\lambda \text{ is non-invertible}\}. \tag{2.2}$$

The spectral radius of x is defined by

$$r(x)=\sup_{\lambda\in\sigma(x)}|\lambda|. \tag{2.3}$$

A nonzero homomorphism $\tau:\mathbb{A}\to F$, where $\mathbb{A}$ is a unital algebra over F, is called a character. We denote by $\Omega(\mathbb{A})$ the set of all characters on $\mathbb{A}$. If $\mathbb{A}$ is a unital abelian