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Weighted Boundedness of Commutators of Generalized Calderón-Zygmund Operators

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Abstract. $[b,T]$ denotes the commutator of generalized Calderón-Zygmund operators T with Lipschitz function b, where $b\in \mathrm{Lip}_\beta(R^n)$, $(0<\beta\le 1)$ and T is a $\theta(t)-$type Calderón-Zygmund operator. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T]f(x)=b(x)Tf(x)-T(bf)(x)=\int k(x,y)(b(x)-b(y))f(y)dy.$$

In this paper, the authors discuss the boundedness of the commutator $[b,T]$ on weighted Hardy spaces and weighted Herz type Hardy spaces and prove that $[b,T]$ is bounded from $H^p(\omega^p)$ to $L^q(\omega^q)$, and from $H\dot{K}^{\alpha,p}_{q_1}(\omega_1,\omega_2^{q_1})$ to $\dot{K}^{\alpha,p}_{q_2}(\omega_1,\omega_2^{q_2})$. The results extend and generalize the well-known ones in [7].

Key Words: Commutator, Lipschitz function, weighted hardy space, Herz space.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

The singular integral operators and their commutators have been extensively studied in recent years, and the results are plentiful and substantial. But the commutators of $\theta(t)-$type Calderón-Zygmund operators which were introduced in [1] by Yabuta and in [2] by Lizhong Peng, have not been discussed extensively. However, the introduction of this kind of operators has profound background of partial differential equation. Lin Haibo [3] obtained weighted estimates for commutators generated by the multilinear Calderón-Zygmund operators and $RMO(\mu)$ functions. In 2014, Hu Guoen [4] gave

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the weighted norm inequalities for multilinear Calderón-Zygmund operators on non-homogeneous metric measure spaces. The studies of Calderón-Zygmund operators and commutators see [5,6]. In [7], Zhao Kai etc. studied the boundedness of commutators of generalized Calderón-Zygmund operators on classical Hardy spaces and Herz type Hardy spaces.

Inspired by the results in [7] and other papers, we study the commutators generated by generalized Calderón-Zygmund operators and Lipschitz functions. By the Minkowski integral inequality, Holder inequality and Jensen inequality to control some inequalities, we obatin the boundedness of the commutator $[b,T]$ generated by $\theta(t)-$type Calderón-Zygmund operators T and Lipschitz function b on weighted Hardy spaces and weighted Herz type Hardy spaces.

First of all, let us introduce the definitions of $\theta(t)-$type Calderón-Zygmund operators.

Definition 1.1. Suppose that T is a bounded operator from Schwartz class $S(R^n)$ to its dual $S'(R^n)$, satisfying the following conditions:

(i) There exists $C>0$, such that for any $f\in C_0^\infty(R^n)$, $||Tf||_{L^2(R^n)}\le C||f||_{L^2(R^n)}$.

(ii) There exists a continuous function $k(x,y)$ defined on $\Omega=\{(x,y)\in R^n\times R^n : x\neq y\}$ and $C>0$ such that

a) $k(x,y)\le C|x-y|^{-n}$ for all $(x,y)\in\Omega$.

b) for all $x,x_0,y\in R^n$ with $2|x-x_0|<|y-x_0|$,

$$|k(x,y)-k(x_0,y)|+|k(y,x)-k(y,x_0)|\le C\theta\left(\frac{|x_0-x|}{|x_0-y|}\right)|x_0-y|^{-n},$$

where $\theta(t)$ is a nonnegative nondecreasing function on $[0,\infty)$ with $\int_0^1\frac{\theta(t)}{t}dt<\infty$ and $\theta(0)=0$, $\theta(2t)\le C\theta(t)$.

c) $Tf(x)=\int k(x,y)f(y)dy$, a.e. $x\notin suppf$.

Then T is said to be a $\theta(t)-$type Calderón-Zygmund operator.

For some properties of $\theta(t)-$type Calderón-Zygmund operator defined above, especially the boundedness on some spaces, see [1] and [2] etc for details.

Similar to the definitions of other commutators, we introduce the definition of commutator generated by $\theta(t)-$type Calderón-Zygmund operator and Lipschitz function as follows.

Definition 1.2. Let $b\in$ Lip$_\beta(R^n)$, and T be a $\theta(t)-$type Calderón-Zygmund operator. The commutator $[b,T]$ generated by b and T is defined by

$$[b,T]f(x)=b(x)Tf(x)-T(bf)(x)=\int k(x,y)(b(x)-b(y))f(y)dy.$$