Analysis in Theory and Applications
Anal. Theory Appl., Vol. **34**, No. 3 (2018), pp. 225-231
DOI: 10.4208/ata.OA-2017-0059

Some Characterizations of Bloch Functions

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Received 4 September 2017; Accepted (in revised version) 24 January 2018

Abstract. We define Bloch-type functions of $\mathcal{C}^1(\mathbb{D})$ on the unit disk of complex plane $\mathbb{C}$ and characterize them in terms of weighted Lipschitz functions. We also discuss the boundedness of a composition operator C_ϕ acting between two Bloch-type spaces. These obtained results generalize the corresponding known ones to the setting of $\mathcal{C}^1(\mathbb{D})$.

Key Words: Bloch space, majorant, composition operator.

AMS Subject Classifications: 11K70, 32A70

1 Introduction

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk of the complex plane $\mathbb{C}$, and $\mathcal{C}^1(\mathbb{D})$ be the set of all complex-valued functions having continuous partial derivatives on $\mathbb{D}$. For $\alpha>0$, a function $f\in\mathcal{C}^1(\mathbb{D})$ is called α-Bloch if

$$\|f\|_\alpha=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha(|f_z(z)|+|f_{\bar{z}}(z)|)<\infty.$$

It is readily seen that the set of all α-Bloch functions on $\mathbb{D}$ is a Banach space $\mathcal{B}^\alpha$ with the norm $\|f\|_{\mathcal{B}^\alpha}=|f(0)|+\|f\|_\alpha$.

Let $\omega:[0,+\infty)\to[0,+\infty)$ be an increasing function with $\omega(0)=0$, we say that ω is a *majorant* if $\omega(t)/t$ is non-increasing for $t>0$ (cf. [4]). Following [5], given a majorant ω and $\alpha>0$, the *ω-α-Bloch space* $\mathcal{B}^\alpha_\omega$ consists of all functions $f\in\mathcal{C}^1(\mathbb{D})$ such that

$$\|f\|_{\omega,\alpha}=\sup_{z\in\mathbb{D}}\omega\big((1-|z|^2)^\alpha\big)(|f_z(z)|+|f_{\bar{z}}(z)|)<\infty$$

and the *little ω-α-Bloch space* $\mathcal{B}^\alpha_{\omega,0}$ consists of the functions $f\in\mathcal{B}^\alpha_\omega$ such that

$$\lim_{|z|\to1^-}\omega\big((1-|z|^2)^\alpha\big)(|f_z(z)|+|f_{\bar{z}}(z)|)=0.$$

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For $0<\alpha\leq 1$, the weighted hyperbolic metric ds_α of $\mathbb{D}$, introduced in [1] is defined as

$$ds_\alpha^2=\frac{|dz|^2}{(1-|z|^2)^{2\alpha}}.$$

Suppose that $\gamma(t)$ $(0\leq t\leq 1)$ is a continuous and piecewise smooth curve in $\mathbb{D}$. Then the length of $\gamma(t)$ with respect to the weighted hyperbolic metric ds_α is equal to

$$L_{h_\alpha}(\gamma)=\int_\gamma ds_\alpha=\int_0^1\frac{|\gamma'(t)|}{[(1-|\gamma(t)|^2)]^\alpha}dt.$$

Consequently, the associated distance between z and w in $\mathbb{D}$ is

$$h_\alpha(z,w)=\inf\{L_{h_\alpha}(\gamma):\gamma(0)=z,\ \gamma(1)=w\},$$

where γ is a continuous and piecewise smooth curve in $\mathbb{D}$. Note that h_1 $(\alpha=1)$ is the hyperbolic distance on $\mathbb{D}$.

Let $s,t\geq 0$ and f be a continuous function in $\mathbb{D}$. If there exists a constant C such that

$$(1-|z|^2)^s(1-|w|^2)^t|f(z)-f(w)|\leq C|z-w| \quad (\text{resp. } \leq Ch_\alpha(z,w)),$$

for any $z,w\in\mathbb{D}$, then we say that f is a *weighted Euclidian (resp. hyperbolic) Lipschitz function* of indices (s,t). In particular, when $s=t=0$, we say that f is a *Euclidian (resp. hyperbolic) Lipschitz function* (cf. [12]).

In the theory of function spaces, the relationship between Bloch spaces and (weighted) Lipschitz functions has attracted much attention. For instance, in 1986, Holland and Walsh [7] established a classical criterion for analytic Bloch space in the unit disc $\mathbb{D}$ in terms of weighted Euclidian Lipschitz functions of indices $(\frac{1}{2},\frac{1}{2})$. Ren and Tu [13] extended the criterion to the Bloch space in the unit ball of $\mathbb{C}^n$, Li and Wulan [8], Zhao [15] characterized holomorphic α-Bloch space in terms of

$$(1-|z|^2)^\beta(1-|w|^2)^{\alpha-\beta}|f(z)-f(w)|/|z-w|.$$

In [16, 17], Zhu investigated the relationship between Bloch spaces and Bergman Lipschitz functions and proved that a holomorphic function belongs to Bloch space if and only if it is Bergman Lipschitz. For the related results of harmonic functions, we refer to [2,3,5,6,12,14] and the references therein.

Motivated by the known results mentioned above, we consider the corresponding problems in the setting of $\mathcal{C}^1(\mathbb{D})$ in this paper. In Section 2, we collect some known results that will be needed in the sequel. The main results and their proofs are presented in Sections 3 and 4.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A\asymp B$ means that there is a positive constant C such that $B/C\leq A\leq CB$.