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Domain of Euler Mean in the Space of Absolutely p-Summable Double Sequences with $0<p<1$

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Abstract. In this study, as the domain of four dimensional Euler mean $E(r,s)$ of orders r,s in the space $\mathcal{L}_p$ for $0<p<1$, we examine the double sequence space $\mathcal{E}_p^{r,s}$ and some properties of four dimensional Euler mean. We determine the α- and $\beta(bp)$-duals of the space $\mathcal{E}_p^{r,s}$, and characterize the classes $(\mathcal{E}_p^{r,s}:\mathcal{M}_u)$, $(\mathcal{E}_p^{r,s}:\mathcal{C}_{bp})$ and $(\mathcal{E}_p^{r,s}:\mathcal{L}_q)$ of four dimensional matrix transformations, where $1\leq q<\infty$. Finally, we shortly emphasize on the Euler spaces of single and double sequences, and note some further suggestions.

Key Words: Summability theory, double sequences, double series, alpha-, beta- and gamma-duals, matrix domain of 4-dimensional matrices, matrix transformations.

AMS Subject Classifications: 46A45, 40C05

1 Introduction

We denote the set of all complex valued double sequences by Ω which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as *a double sequence space*. A double sequence $x=(x_{mn})$ of complex numbers is said to be *bounded* if $\|x\|_\infty=\sup_{m,n\in\mathbb{N}}|x_{mn}|<\infty$, where $\mathbb{N}=\{0,1,2,\cdots\}$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$ which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the sequence $x=(x_{mn})\in\Omega$. If for every $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)\in\mathbb{N}$ and $l\in\mathbb{C}$ such that $|x_{mn}-l|<\varepsilon$ for all $m,n>n_0$, then we call that the double sequence x is *convergent* in the *Pringsheim's sense* to the limit l and write $p-\lim_{m,n\to\infty}x_{mn}=l$; where $\mathbb{C}$ denotes the complex field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed following Boos [7, pp. 16], if we define the sequence

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$x=(x_{mn})$ by

$$x_{mn}:=\begin{cases} n, & m=0, \quad n\in\mathbb{N}, \\ 0, & m\geq 1, \quad n\in\mathbb{N}, \end{cases}$$

then it is trivial that $x\in\mathcal{C}_p-\mathcal{M}_u$, since $p-\lim_{m,n\to\infty}x_{mn}=0$ but $\|x\|_\infty=\infty$. So, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp}=\mathcal{C}_p\cap\mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the space of all such sequences by $\mathcal{C}_r$. Also by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, we denote the spaces of all double sequences converging to 0 contained in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Móricz [12] proved that $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|\cdot\|_\infty$.

Let λ be a space of double sequences, converging with respect to some linear convergence rule $\vartheta-\lim:\lambda\to\mathbb{C}$. The sum of a double series $\sum_{i,j}x_{ij}$ with respect to this rule is defined by $\vartheta-\sum_{i,j}x_{ij}=\vartheta-\lim_{m,n\to\infty}\sum_{i,j=0}^{m,n}x_{ij}$. For short, throughout the text the summations without limits run from 0 to ∞, for example $\sum_{i,j}x_{ij}$ means that $\sum_{i,j=0}^{\infty}x_{ij}$.

The *α-dual* λ^α, *$\beta(\vartheta)$-dual* $\lambda^{\beta(\vartheta)}$ with respect to the ϑ-convergence and the *γ-dual* λ^γ of a double sequence space λ are respectively defined by

$$\begin{aligned} \lambda^\alpha &:=\Big\{(a_{kl})\in\Omega:\sum_{k,l}|a_{kl}x_{kl}|<\infty \text{ for all } (x_{kl})\in\lambda\Big\}, \\ \lambda^{\beta(\vartheta)} &:=\Big\{(a_{kl})\in\Omega:\vartheta-\sum_{k,l}a_{kl}x_{kl} \text{ exists for all } (x_{kl})\in\lambda\Big\}, \\ \lambda^\gamma &:=\Big\{(a_{kl})\in\Omega:\sup_{m,n\in\mathbb{N}}\Big|\sum_{k,l=0}^{m,n}a_{kl}x_{kl}\Big|<\infty \text{ for all } (x_{kl})\in\lambda\Big\}. \end{aligned}$$

It is easy to see for any two spaces λ, μ of double sequences that $\mu^\alpha\subset\lambda^\alpha$ whenever $\lambda\subset\mu$ and $\lambda^\alpha\subset\lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha\subset\lambda^{\beta(\vartheta)}$ holds while the inclusion $\lambda^{\beta(\vartheta)}\subset\lambda^\gamma$ does not hold, since the ϑ-convergence of a sequence of partial sums of a double series does not imply its boundedness.

Let λ and μ be two double sequence spaces, and $A=(a_{mnkl})$ be any four-dimensional complex infinite matrix. Then, we say that A defines a *matrix mapping* from λ into μ and we write $A:\lambda\to\mu$, if for every sequence $x=(x_{kl})\in\lambda$ the A-transform $Ax=\{(Ax)_{mn}\}_{m,n\in\mathbb{N}}$ of x exists and is in μ; where

$$(Ax)_{mn}=\vartheta-\sum_{k,l}a_{mnnk}x_{kl} \quad \text{for each } m,n\in\mathbb{N}. \tag{1.1}$$