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Harmonic Polynomials Via Differentiation

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Abstract. It is well-known that if p is a homogeneous polynomial of degree k in n variables, $p\in\mathcal{P}_k$, then the ordinary derivative $p(\nabla)\left(r^{2-n}\right)$ has the form $A_{n,k}Y(\mathbf{x})r^{2-n-2k}$ where $A_{n,k}$ is a constant and where Y is a harmonic homogeneous polynomial of degree k, $Y\in\mathcal{H}_k$, actually the projection of p onto $\mathcal{H}_k$. Here we study the distributional derivative $p\left(\overline{\nabla}\right)\left(r^{2-n}\right)$ and show that the ordinary part is still a multiple of Y, but that the delta part is independent of Y, that is, it depends only on $p-Y$. We also show that the exponent $2-n$ is special in the sense that the corresponding results for $p(\nabla)(r^{\alpha})$ do not hold if $\alpha\neq 2-n$.
Furthermore, we establish that harmonic polynomials appear as multiples of $r^{2-n-2k-2k'}$ when $p(\nabla)$ is applied to harmonic multipoles of the form $Y'(\mathbf{x})r^{2-n-2k'}$ for some $Y'\in\mathcal{H}_k$.

Key Words: Harmonic functions, harmonic polynomials, distributions, multipoles.

AMS Subject Classifications: 46F10, 33C55

1 Introduction

It is well known [1,7,19] that any homogeneous polynomial of degree $k, p\in\mathcal{P}_k$, can be decomposed, in a unique fashion, as

$$p=Y+r^2q, \tag{1.1}$$

where

$$Y=\pi_k(p)\in\mathcal{H}_k,\quad q=\chi_k(p)\in\mathcal{P}_{k-2}, \tag{1.2}$$

the notation $\mathcal{H}_k$ being used to denote the harmonic homogeneous polynomials of degree k.

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One can easily find the projections $\pi_k(p)$ and $\chi_k(p)$. For example, if we apply the Laplacian to (1.1) we readily obtain $\Delta p=\Delta\left(r^2q\right)=2nq+4(k-2)q=2(n+2k-4)q$, so that

$$q=\frac{\Delta p}{2(n+2k-4)},\quad Y=p-\frac{r^2\Delta p}{2(n+2k-4)}. \tag{1.3}$$

Interestingly, these projections appear in other, somewhat surprising places. Indeed, as explained in the section Spherical Harmonics via Differentiation of [1, Chapter 5], whenever a homogeneous differential operator of degree k is applied to r^{2-n} in $\mathbb{R}^n$ one obtains an expression of the form $u(\mathbf{x})r^{2-n-2k}$ where u is not just homogeneous of degree k, but actually belongs to $\mathcal{H}_k$. In fact, more is true, since $u=(2-n)(-n)\cdots(-n-2k+4)Y$, that is, if $p\in\mathcal{P}_k$ and we denote $(2-n)(-n)\cdots(-n-2k+4)$ as $A_{n,k}$ then

$$p(\nabla)\left(\frac{1}{r^{n-2}}\right)=A_{n,k}\frac{Y(\mathbf{x})}{r^{n+2k-2}}, \tag{1.4}$$

and in particular if $Y\in\mathcal{H}_k$ then

$$Y(\nabla)\left(\frac{1}{r^{n-2}}\right)=A_{n,k}\frac{Y(\mathbf{x})}{r^{n+2k-2}}. \tag{1.5}$$

Several further questions arise, however. First, since the function r^{2-n} is singular at the origin, these formulas hold in $\mathbb{R}^n\setminus\{\mathbf{0}\}$ but not in all $\mathbb{R}^n$, so what are the corresponding formulas for the distributional derivatives[†] $p\left(\overline{\nabla}\right)\left(r^{2-n}\right)$ and $Y\left(\overline{\nabla}\right)\left(r^{2-n}\right)$?, that is, the corresponding formulas in the whole space[‡]. Curiously, while in general $p\left(\overline{\nabla}\right)\left(r^{2-n}\right)$ will contain extra terms, namely a delta part, the distributional expression $Y\left(\overline{\nabla}\right)\left(r^{2-n}\right)$ remains basically equal to (1.5) since $Y\left(\overline{\nabla}\right)\left(r^{2-n}\right)$ does not have a delta part; delta parts and ordinary parts of a distribution are explained in Section 2. We give two different proofs of the formula for $Y\left(\overline{\nabla}\right)\left(r^{2-n}\right)$, one by induction in Section 3 and another in Section 5. We also consider the distributional derivative $p\left(\overline{\nabla}\right)\left(r^{2-n}\right)$ in Section 4, showing that in general the ordinary part of this derivative depends only on Y, while the delta part depends only on q.

Furthermore, we show that harmonic polynomials are also obtained when we take the derivatives of multipoles[§] of the form $Y'(\mathbf{x})/r^{2k'+n-2}$ for some harmonic polynomial $Y'\in\mathcal{H}_{k'}$. Indeed we obtain formulas for the derivatives $p\left(\overline{\nabla}\right)\left(p.v.\left(Y'(\mathbf{x})/r^{2k'+n-2}\right)\right)$ of the principal value distribution $p.v.\left(Y'(\mathbf{x})/r^{2k'+n-2}\right)$ and show that the ordinary part is a multipole of the form $Z(\mathbf{x})/r^{2k'+2k+n-2}$ for some $Z\in\mathcal{H}_{k+k'}$.

[†]Following Farassat [6] we denote distributional derivatives with an overbar, namely, $\overline{\nabla}_i$, $\overline{\Delta}$, $\overline{\partial}/\partial x_i$, and so on.

[‡]Distributional derivatives of this kind play an important role in Physics; the distributional derivatives $\overline{\nabla}_i\overline{\nabla}_j(1/r)$ were given by Frahm [8], and can be found in the textbooks [14].

[§]Such harmonic multipoles have received increasing attention in recent years [2]; see also [18]. They play a fundamental role in the ideas of the late professor Stora on convergent Feyman amplitudes [17,21].