Abstract. The purpose of this paper is to verify the Smulyan lemma for the support function, and also the Gateaux differentiability of the support function is studied on its domain. Moreover, we provide a characterization of Frechet differentiability of the support function on the extremal points.

Key Words: Frechet and Gateaux differentiability, support function, strict convexity, Smulyan lemma.

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1 Introduction

The problem of differentiability and subdifferentiability of a convex function on a Banach space $X$ are important in the theory of optimization (specially in economics) and geometry of Banach spaces. Recently, this issue has been discussed for specific convex functions known as support functions. In fact, they play a fundamental role in the development of optimization and variational analysis.

In economics, maximization of linear functionals on the subsets of Banach spaces has special importance in optimizing the price and profit. Shephard’s lemma is one of the most important results in economics. It is also associated with the differentiability of the cost function (see [10]) defined by

$$g: \mathbb{R}_+^p \to \mathbb{R}, \quad g(x) := \inf_{a \in A} x(a),$$

where $p$ is a positive integer, $\mathbb{R}_+^p$ is the $p$-dimensional Euclidean space and $A$ is a subset of $\mathbb{R}_+^p$ (the positive cone of $\mathbb{R}^p$).
Let $X$ be a Banach space and $A$ be a subset of $X$. Support function of the set $A$ is defined by

$$
\sigma_A : X^* \to \bar{\mathbb{R}}, \quad \sigma_A(x^*) := \sup_{t \in A} x^*(t).
$$

Clearly, when $\dim X = p$, the cost function $g$ is strongly related to the support function $\sigma_A$. In fact, any property of the support function $\sigma_A$ can be translated to a corresponding property of the cost function. See [11, 12] and the long list of references therein.

This article is organized as follows. In Section 2, we present some preliminaries. In Section 3, we state Smulyan lemma for the support function and we establish some results regarding Smulyan lemma on the Gateaux and Frechet differentiability of support function. In Section 4, we show that the support function $\sigma_A$ is Gateaux differentiable on the interior of its domain $\text{int}(\text{dom} \sigma_A)$, which is an extension of [11, Theorem 6] into the infinite dimensional case.

## 2 Preliminaries

Throughout this paper, $(X, \| \cdot \|)$ is a real Banach space whose dual $X^*$ is endowed with the dual norm, denoted also by $\| \cdot \|$. We consider $A \subset X$ a nonempty set. As usual, we denote the interior of $A$, the cone generated by $A$, the affine hull of $A$, the linear space parallel to $\text{aff} A$, the relative interior of $A$, the relative boundary of $A$, the closure of $A$, the convex hull of $A$ and the polar set of $A$ by $\text{int} A$, $\text{cone} A$, $\text{aff} A$, $\text{lin} 0 A$, $\text{rint} A$, $\text{rbd} A$, $\text{cl} A$, $\text{conv} A$ and $A^0$, respectively.

Let $U$ be an open subset of the Banach space $X$ and $f : U \to \mathbb{R}$ be a real valued function. We say that $f$ is Gateaux differentiable at $x \in U$, if for every $h \in X$,

$$
\lim_{t \to 0} \frac{f(x + th) - f(x)}{t}
$$

exists in $\mathbb{R}$ and $f'(x)$ is a linear continuous function at $h$ (i.e., $f'(x) \in X^*$). The functional $f'(x)$ is then called the Gateaux derivative or Gateaux differential of $f$ at $x$. If, in addition, the above limit is uniform at $h \in S_X$ (where $S_X$ denotes the unit sphere in $X$), we say that $f$ is Frechet differentiable at $x$. See [4] for more details.

We recall that the domain of a convex extended-valued function $f : X \to \bar{\mathbb{R}}$ is the set

$$
\text{dom} f := \{ x \in X : f(x) < \infty \}.
$$

A convex extended-valued function $f$ is proper if and only if $\text{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for each $x \in X$ [1]. The subdifferential of a proper function $f$ at $x \in \text{dom} f$ is

$$
\partial f(x) := \{ x^* \in X^* : x^*(y - x) \leq f(y) - f(x), \quad \forall y \in X \},
$$

and the domain of $\partial f$ is defined by

$$
\text{dom} \partial f = \{ x \in X : \partial f(x) \neq \emptyset \} \subset \text{dom} f,
$$
(see [3]). For nonempty subsets $A \subset X$ and $B \subset X^*$, we define the support function of the set $A$ by
\[ \sigma_A : X^* \to \mathbb{R}, \quad \sigma_A(x^*) = \sup_{t \in A} x^*(t), \]
and the support function of the set $B$ by
\[ h_B : X \to \mathbb{R}, \quad h_B(x) = \sup_{x^* \in B} x^*(x). \]
It is well-known that $\sigma_A = \sigma_{\text{conv} A} = \sigma_{\text{cl} A} = \sigma_{\text{cl}(\text{conv} A)}$ (the same equality holds for $h_B$). Therefore, we assume that $A$ and $B$ are nonempty closed and convex sets. Moreover, for a nonempty closed convex subset $B$ of $X^*$,
\[ \partial h_B(x) = \{ u^* \in B : u^*(x) = h_B(x) \}, \]
and when the space is reflexive, for a nonempty closed convex subset $A$ of $X$,
\[ \partial \sigma_A(x^*) = \{ u \in A : x^*(u) = \sigma_A(x^*) \}, \]
also $\sigma_A(0) = A$ and $h_B(0) = B$ (see [1, 5] for more details). It is well-known that a lower semi-continuous proper convex function $f$ on $X$ is continuous at $x \in \text{dom} f$ if and only if $x \in \text{int}(\text{dom} f)$ (see [2, Proposition 4.1.5]). Also, $\partial f(x) \neq \emptyset$ for every $x \in \text{int}(\text{dom} f)$ and $f$ is Gateaux differentiable at $x \in \text{int}(\text{dom} f)$ if and only if $\partial f(x)$ is a singleton [5, theorem 7.17]. Note that $h_B$ and $\sigma_A$ are lower semi-continuous proper convex support functions. Hence, they are continuous and subdifferentiable on the interior of their domains.

### 3 Strict convexity of a set and Smulyan lemma

Let $A$ be a nonempty closed convex subset of the Banach space $X$ with nonempty interior ($\text{int} A \neq \emptyset$), and $x \notin \text{int} A$. By the separation theorem, there exists a nonzero bounded linear functional $x^* \in X^*$ so that $\sup_{t \in A} x^*(t) \leq x^*(x)$. But the interior of $A$ is dense in $A$ (see [1, Lemma 5.28]), so $\sup_{t \in A} x^*(t) \leq x^*(x)$. That is $x^*$ supports $A$ at $x$. Letting $y^* = \frac{x^*}{x^*(x)}$, we get $\sup_{t \in A} y^*(t) = y^*(x) = 1$ and $y^* \in A^0$. Also, $h_{A^0}(y) \leq 1$ for every $y \in A$. Hence,
\[ h_{A^0}(x) = x^*(x) = 1. \]
This can be summarized as follows.

**Lemma 3.1.** Let $A$ be a closed convex subset of a real Banach space $X$ with nonempty interior ($\text{int} A \neq \emptyset$). Then $h_{A^0}(x) = 1$ for every $x \in \text{bd} A$, where $\text{bd} A$ denotes the boundary points of $A$.

**Definition 3.1.** A nonempty subset $A$ of a Banach space $X$ is said to be r-strictly convex (strictly convex) if every relative boundary point of $A$ (boundary point of $A$) is an extreme point.
Proposition 3.1. Let $A$ be a closed convex subset of the Banach space $X$. Then, the following are equivalent:

$(d_1)$ $A$ is strictly convex.

$(d_2)$ $\forall x, y \in \text{bd}A, \forall \lambda \in (0,1); \lambda x + (1-\lambda)y \in \text{bd}A \Rightarrow x = y.$

In addition, if $\text{int}A \neq \emptyset$, then $(d_1)$ and $(d_2)$ are equivalent to the following:

$(d_3)$ $\forall x, y \in \text{bd}A; h_{A^0}(x+y) = h_{A^0}(x) + h_{A^0}(y) \Rightarrow x = y,$

$(d_4)$ $\forall x, y \in \text{bd}A; h_{A^0}^2(x+y) = 2h_{A^0}^2(x) + 2h_{A^0}^2(y) \Rightarrow x = y.$

When $A$ is a bounded neighborhood of zero, the following replaces $(d_4)$.

$(d'_4)$ $\forall x, y \in X; h_{A^0}^2(x+y) = 2h_{A^0}^2(x) + 2h_{A^0}^2(y) \Rightarrow x = y.$

Proof. It is easy to check that $(d_1) \iff (d_2)$. For $(d_1) \iff (d_3)$, let $x, y \in \text{bd}A$ and

$$h_{A^0}(x+y) = h_{A^0}(x) + h_{A^0}(y),$$

where $x \neq y$. From Lemma 3.1, $h_{A^0}(x) = h_{A^0}(y) = 1$. Hence, $h_{A^0}(x+y) = 1$. Since $A$ is strictly convex ($x \neq y$ and $x, y \in \text{bd}A = \text{ext}A$, where $\text{ext}A$ denotes the extreme points of $A$), we get $x+y \not\in \text{bd}A$. Therefore, $x+y \neq \sup t \in A x^*(t)$ for all $x^* \in X^*$. So, $x^*(x+y) < \sup t \in A x^*(t) \leq 1$ for each $x^* \in A^0$. Hence, $h_{A^0}(x+y) < 1$, which is a contradiction.

On the other hand, let $x \in \text{bd}A \setminus \text{ext}A$. Then, there exist $y, z \in \text{bd}A$ so that $y+z = x$. From Theorem 3.1, $h_{A^0}(x) = h_{A^0}(y) = h_{A^0}(z) = 1$. So, $h_{A^0}(y+z) = h_{A^0}(y) + h_{A^0}(z)$ and from $(d_3)$, the contradiction $y = z$ is obtained.

By Lemma 3.1, $(d_3) \iff (d'_4)$. It remains to show that $(d_3) \Rightarrow (d'_4)$. Let $x, y \in X$ satisfy

$$h_{A^0}^2(x+y) = 2h_{A^0}^2(x) + 2h_{A^0}^2(y). \quad (3.1)$$

Also,

$$2h_{A^0}^2(x) + 2h_{A^0}^2(y) - h_{A^0}^2(y+x)$$

$$> 2h_{A^0}^2(x) + 2h_{A^0}^2(y) - (h_{A^0}(y) + h_{A^0}(x))^2$$

$$= (h_{A^0}(x) - h_{A^0}(y))^2$$

$$\geq 0.$$

Therefore, $h_{A^0}(x) = h_{A^0}(y)$. Since $A$ is a bounded neighborhood of zero with $\text{int}A \neq \emptyset$, there exist $\beta, \gamma > 0$ and $x_1, y_1 \in \text{bd}A$ so that $\beta x = x_1$ and $\gamma y = y_1$. According to Lemma 3.1, $h_{A^0}(x_1) = h_{A^0}(y_1) = 1$, which means $\beta = \gamma$. Hence, by equality (3.1):

$$h_{A^0}^2(x_1 + y_1) = 2,$$

and from $(d_3), x_1 = y_1$. The proof is complete, because $\beta = \gamma$. \qed
Let $A$ be a closed convex neighborhood of zero. By the Bipolar theorem, $A^{00} = A$. Therefore, according to [5, Theorem 7.18],

$$
P_{A^0}(x^*) = \sup_{x^* \in A^{00}} x^{**}(x^*) = \sigma_{x^* \in A}(x^*),
$$
$$
P_A(x) = \sup_{x^* \in A^0} x^{*}(x) = h_{A^0}(x).
$$

Lemma 7.19 of [5] states that when $A$ is a closed convex neighborhood of zero, for $\varepsilon \geq 0$ and $x_0 \in X$,

$$
\partial_t P_A(x_0) = \left\{ x^* \in A^0 : x^*(x_0) \geq \sup_{u^* \in A^0} u^*(x_0) - t \right\}.
$$

Thus,

$$
\partial_t \sigma_A(x_0^*) = \left\{ x \in A : x_0^*(x) \geq \sigma_A(x_0^*) - \varepsilon \right\}, \quad (3.2a)
$$
$$
\partial_t h_{A^0}(x_0) = \left\{ x^* \in A^0 : x^*(x_0) \geq h_{A^0}(x_0) - \varepsilon \right\}. \quad (3.2b)
$$

These results lead us to write the Smulyan lemma [5, Lemma 7.20] for $\sigma_A$ and $h_{A^0}$, as follows.

**Theorem 3.1.** Let $A$ be a closed convex neighborhood of zero. Then

1. $\sigma_A$ is Frechet differentiable at $x_0^* \in X^*$ if and only if $||x_n - y_n|| \rightarrow 0$ whenever $x_n, y_n \in A$ satisfy $\lim x_0^*(x_n) = \lim x_0^*(y_n) = \sigma_A(x_0^*)$ if and only if $\{x_n\} \subset A$ is convergent whenever $x_0^*(x_n) = \sigma_A(x_0^*)$.

2. $\sigma_A$ is Gateaux differentiable at $x_0^* \in X^*$ if and only if $x_n - y_n \rightarrow 0$ whenever $x_n, y_n \in A$ satisfy $\lim x_0^*(x_n) = \lim x_0^*(y_n) = \sigma_A(x_0^*)$ if and only if there exists a unique $x \in A$ which satisfies $x_n^*(x) = \sigma_A(x_0^*)$.

3. $h_{A^0}$ is Frechet differentiable at $x \in X$ if and only if $||f_n - g_n|| \rightarrow 0$ whenever $f_n, g_n \in A$ satisfy $\lim f_n(x) = \lim g_n(x) = h_{A^0}(x)$ if and only if $\{f_n\} \subset A^0$ is convergent whenever $\lim f_n(x) = h_{A^0}(x)$.

4. $h_{A^0}$ is Gateaux differentiable at $x \in X$ if and only if $f_n - g_n \rightarrow 0$ whenever $f_n, g_n \in A^0$ satisfy $\lim f_n(x) = \lim g_n(x) = h_{A^0}(x)$ if and only if there exists a unique $f \in A^0$ which satisfies $f(x) = h_{A^0}(x)$.

**Remark 3.1.** (f1) Differentiability conditions for $\sigma_A$ and $h_{A^0}$ are homogeneous, indeed $h_{A^0}$ is differentiable at $x$ if it is differentiable at $\lambda x$ for some scalar $\lambda$. Also $\sigma_A$ is differentiable at $x_0^*$ if it is differentiable at $\lambda x^*$ for some scalar $\lambda$. Consequently, it is enough to check the differentiability at points of a bounded neighborhood of zero.

(f2) Let $A$ ($B$) be a closed convex subset of $X$ ($X^*$). It is easy to check that $\partial \sigma_A(0) = A$ ($\partial \sigma_B(0) = B$). So, $\sigma_A$ ($h_B$) is Gateaux differentiable at 0 if and only if $A$ ($B$) is a singleton. In this case, $\sigma_A$ ($h_B$) is Gateaux differentiable on $X^*$ ($X$).
(f₃) It is clear that $x^* \in X^*$ is constant on $A$ if and only if $x^* \in (\text{lin} \, 0 \, A)^\perp$. It means that 
$\partial \sigma_A(x^*) = A$ for every $x^* \in (\text{lin} \, 0 \, A)^\perp$, and $\sigma_A$ is not Gateaux differentiable at any $x^* \in (\text{lin} \, 0 \, A)^\perp$ unless $A$ is a singleton. Thus, when we speak about differentiability of $\sigma_A$ on $X^*$, we mean that it is differentiable on the set $\text{dom} \sigma_A \setminus (\text{lin} \, 0 \, A)^\perp$.

(f₄) Based on the fact that $\sigma_A(x^*_0) = \sigma_A(x^*_0) - x^*_0(x)$ for every $x \in X$ and $A \subset X$, from (e₁) of Theorem 3.1, $\sigma_A$ is Frechet differentiable at $x^*_0$ if and only if $\sigma_A - x$ is Frechet differentiable at $x^*_0$. From (e₂) of Theorem 3.1, $\sigma_A$ is Gateaux differentiable at $x^*_0$ if and only if $\sigma_A - x$ is Gateaux differentiable at $x^*_0$. Also, $\text{dom}(\partial \sigma_A) = \text{dom}(\partial \sigma_A - y)$ for all $y \in X$.

**Theorem 3.2.** Let $A$ be a closed convex neighborhood of zero. If $A^0$ is strictly convex, then $h_{A^0}$ is Gateaux differentiable on $\text{int}(\text{dom} h_{A^0}) \setminus \{0\}$.

**Proof.** The support function $h_{A^0}$ is a lower semi-continuous proper convex function that is subdifferentiable on the interior of its domain. Suppose that $x_0 \in \text{int}(\text{dom} h_{A^0})$ and $f, g \in \partial h_{A^0}(x_0)$. From the equality (3.2),

$\partial h_{A^0}(x_0) = \{ f \in A^0 : f(x_0) = h_{A^0}(x_0) \}$.

Hence, $f, g \in A^0$ and $f(x_0) = g(x_0) = \frac{f + g}{2}(x_0) = h_{A^0}(x_0)$. But, from Bishop-Phelps theorem [1, Lemma 7.7], every support points of $A^0$ is a boundary point of $A^0$. Therefore, $f, g$ and $\frac{f + g}{2} \in \text{bd} A^0$ and under the assumption of strict convexity of $A^0$, we get $f = g$. Hence, the set $\partial h_{A^0}(x)$ is a singleton and by (e₂) of the Smulyan lemma, $h_{A^0}$ is Gateaux differentiable.

**Corollary 3.1.** Let $A$ be a closed strictly convex subset of the Banach space $X$ with nonempty interior ($\text{int} \, A \neq \emptyset$). Then, $\sigma_A$ is Gateaux differentiable on the $\text{int}(\text{dom} \sigma_A) \setminus \{0\}$.

**Proof.** Let $z \in \text{int} \, A$. By (f₄) of Remark 3.1, $\sigma_A$ is Gateaux differentiable at $x^*_0 \in X^*$ if and only if $\sigma_A - z$ is Gateaux differentiable at $x^*_0$. So, without loss of generality, assume that $0 \in A$. From the Bipolar theorem, we get $(A)^{00} = A$. Since $A^0$ is a neighborhood of zero, applying Theorem 3.2 for $A^0$, we conclude that $\sigma_A$ is Gateaux differentiable on the $\text{int}(\text{dom} \sigma_A) \setminus \{0\}$. □

In [7], Klee showed that every separable nonreflexive Banach space $(X, || \cdot ||)$ can be equivalently renormed so that the new norm is Gateaux differentiable but its dual norm is not strictly convex. So the inverse of Theorem 3.2 is not true in general.

**Theorem 3.3.** Let $A$ be a nonempty closed convex subset of a Banach space $X$ with nonempty interior. Then, $\sigma_A$ is Gateaux differentiable on the $\text{dom} \partial \sigma_A \setminus \{0\}$ if and only if $A$ is strictly convex.
Proof. Without loss of generality, we assume that $0 \in A$ (Remark 3.1, (f)). For the if part, let $x, y$ and $\frac{x+y}{2} \in \text{bd} A$. Applying the Separation theorem for $\{\frac{x+y}{2}\}$ and $\text{int} A$, we have a nonzero linear bounded functional $f_0 \in X^*$ so that $f_0(\frac{x+y}{2}) = \sigma_A(f_0)$. It follows that 

$$f_0(x) = f_0(y) = \sigma_A(f_0),$$

and $f_0 \in \text{dom} \partial \sigma_A$ which implies that $\sigma_A$ is Gateaux differentiable at $f_0$. So, from (e) of Smulyan lemma, we get $x = y$. Therefore, $A$ is strictly convex.

Finally, let $f \in \text{dom} \partial \sigma_A \setminus \{0\}$. Let $x, y \in \partial \sigma_A(f)$, then $f(\frac{x+y}{2}) = \sigma_A(f)$ (from the Eq. (3.2)). From the Bishop phelps theorem, every support point of $A$ is a boundary point of $A$. Hence, $\frac{x+y}{2} \in \text{bd} A$ and under the assumption of strict convexity of $A$, we have $x = y$. Therefore, $\partial \sigma_A(f)$ is a singleton and from Smulyan lemma, $\sigma_A$ is Gateaux differentiable at $f$.

Note that if a Banach space $X$ and its closed subspace $Y$ are generated by weakly compact sets, then $Y$ is complemented in $X$. In particular, reflexive Banach spaces have this property [8]. Using the latter, we have the following result.

**Theorem 3.4.** Let $A$ be a nonempty closed convex subset of a reflexive Banach space $X$ with nonempty relative interior. Then $\sigma_A$ is Gateaux differentiable on the $\text{dom} \partial \sigma_A \setminus (\text{lin} A)^\perp$ if and only if $A$ is $r$-strictly convex.

Proof. Let $\text{int} A \neq \emptyset$. Then $\text{lin} A = X$, $(\text{lin} A)^\perp = \{0\}$. Let $\text{rint} A = \text{int} A$. So, Theorem 3.3 completes the proof in this case. Now, let $\text{int} A = \emptyset$ and set $X_0 = \text{lin} A$. If $A$ is a singleton, then $X_0 = \{0\}$ and both assertions are true. Let $\dim X_0 \geq 1$. By a translation, we assume that $0 \in \text{aff} A$ and $X_0 = \text{aff} A \neq \{0\}$. From reflexivity of the space, there exists a subspace $X_1$ of $X$ so that $X = X_0 \times X_1$. So, $A := A_0 \times \{0\}$, where $A_0$ is a closed convex subset of $X_0$ with nonempty interior. Hence, $\text{bd} A = \text{bd} A \times \{0\}$ and $A$ is $r$-strictly convex if and only if $A_0$ is strictly convex. Now, we claim that $\sigma_A$ is Gateaux differentiable on the $\text{dom} \partial \sigma_A \setminus (\text{lin} A)^\perp$ if and only if $\sigma_{A_0}$ is Gateaux differentiable on $\text{dom} \partial \sigma_{A_0} \setminus \{0\}$. To prove this claim, using the same argument as in [11, Theorem 1], we get

$$\sigma_A(x^*, y^*) = \sigma_{A_0}(x^*),$$

$$\partial \sigma_A(x^*, y^*) = \partial \sigma_{A_0}(x^*) \times \{0\},$$

for each $(x^*, y^*) \in X_0^* \times X_1^*$, and

$$\text{dom} \partial \sigma_A = \text{dom} \partial \sigma_{A_0} \times X_1^*,$$

$$\text{dom} \partial \sigma_A \setminus (\text{lin} A)^\perp = \text{dom} \partial \sigma_{A_0} \setminus \{0\} \times X_1^*.$$

With applying Theorem 3.3 for $C_0$ the proof is complete. \qed

**Remark 3.2.** (g1) Let $A$ be a nonempty closed convex subset of a finite dimensional Banach space $X$. Zalinescu showed that $\sigma_A$ is differentiable on $\text{dom} \partial \sigma_A \setminus (\text{lin} A)^\perp$ if and only if $A$ is $r$-strictly convex (see [11, Theorem 2]). In fact, Theorem 3.4 is a generalization of Zalinescu’s theorem in infinite dimensional case.
In Theorem 3.4, when $A$ is compact, $\sigma_A$ is Gateaux differentiable on $X^* \setminus (\text{lin}_0 A)^\perp$ if and only if $A$ is strictly convex. Also, when $\text{int} A \neq \emptyset$, we have $\text{lin}_0 A = X$. Hence, $(\text{lin}_0 A)^\perp = \{0\}$ and $\sigma_A$ is Gateaux differentiable on $X^* \setminus \{0\}$ if and only if $A$ is r-strictly convex.

In [5], it is shown that for a bounded set $A$, a functional $x^*_0 \in X^*$ strongly exposes $A$ if and only if the support function $\sigma_A$ is Frechet differentiable at $x^*_0$. If we replace bounded sets with closed convex sets, the theorem still remains true.

**Theorem 3.5.** Let $A$ be a closed convex subset of the Banach space $X$. A point $x^*_0 \in X^*$ strongly exposes $A \subset X$ if and only if the support function $\sigma_A$ is Frechet differentiable at $x^*_0$.

**Proof.** Let $x \in X$. Based on $(e_4)$ of Remark 3.1, $\sigma_A$ is Frechet differentiable at $x^*_0 \in X^*$ if and only if $\sigma_{A-x}$ is Frechet differentiable at $x^*_0$. Also $x^*_0$ strongly exposes $A$, if and only if $x^*_0$ strongly exposes $A-x$. So, we may assume that $0 \in A$.

By the Bipolar theorem [5, Theorem 3.38], $A = A^{00}$ and:

$$\sigma_A(x^*_0) = \sigma_{A^{00}}(x^*_0) = \sup \{ F(x^*_0) : F \in A^{00} \} = P_{A^0}(x^*_0).$$

Therefore, it is enough to prove the theorem for Minkowski functional $P_{A^0}$. From [5, Corollary 7.20], $x^*_0$ strongly exposes $F$ on $A^{00}$ if and only if $\sigma_A$ is Frechet differentiable at $x^*_0$. The Bipolar theorem again, shows that $A = A^{00}$, which completes the proof. $\square$

## 4 Differentiability of $\sigma_A$ on $\text{int}(\text{dom} \sigma_A)$

Let $A$ be a nonempty closed convex subset of a Banach space $X$ such that

$$\text{int}(\text{dom} \sigma_A) \neq \emptyset.$$ 

The natural question is that if $\sigma_A$ is Gateaux differentiable on $\text{int}(\text{dom} \sigma_A) \setminus (\text{lin}_0 A)^\perp$.

**Proposition 4.1.** Let $A$ be a nonempty closed, bounded and convex subset of a reflexive Banach space $X$ and $\text{int}(\text{dom} \sigma_A) \neq \emptyset$. Then, $\sigma_A$ is Gateaux differentiable on $\text{int}(\text{dom} \sigma_A) \setminus (\text{lin}_0 A)^\perp$ if and only if $A$ is r-strictly convex.

**Proof.** Since $A$ is a closed bounded convex subset of $X$, it is $w$-compact and from the James theorem [1, Theorem 6.36], every continuous linear functional attains its supremum on $A$. Hence,

$$X^* \setminus (\text{lin}_0 A)^\perp = \text{int}(\text{dom} \sigma_A) \setminus (\text{lin}_0 A)^\perp = \text{dom}(\partial \sigma_A) \setminus (\text{lin}_0 A)^\perp.$$

Therefore, by Theorem 3.4, the proof is completed. $\square$
Let

\[ S_A := \partial \sigma_A(\text{int}(\text{dom} \sigma_A)) \]
\[ E_A := A \setminus [A + (A \setminus \{0\})]. \]

In [11], it is shown that \( S_A \subseteq E_A \subseteq \text{rbd} A \) and if \( X \) is a finite dimensional Banach space and \( A \) is an unbounded subset of \( X \) for which

\[ \text{int}(\text{dom} \sigma_A) \neq \emptyset, \]

then \( \sigma_A \) is differentiable on \( \text{int}(\text{dom} \sigma_A) \) if and only if

\[ \forall x, x' \in S_A, \ x \neq x', \ \forall \lambda \in (0,1), \ \lambda x + (1-\lambda)x' \notin S_A. \quad (4.1) \]

What follows is a generalization of the above theorem in infinite dimensional case.

**Theorem 4.1.** Let \( A \) be a nonempty subset of a reflexive Banach space \( X \). If \( \text{int}(\text{dom} \sigma_A) \neq \emptyset \), then \( \sigma_A \) is Gateaux differentiable on \( \text{int}(\text{dom} \sigma_A) \) if and only if \((4.1)\) holds.

**Proof.** Let \( x, x' \in S_A \) \( (x \neq x') \) and \( \lambda \in (0,1) \) be such that \( \lambda x + (1-\lambda)x' := x'' \in S_A \). Then, there exists \( x^* \in \text{int}(\text{dom} \sigma_A) \) such that \( x'' \in \partial \sigma_A(x^*) \). Thus, \( x^*(x), x^*(x') \leq \sigma_A(x^*) \) and \( \sigma_A(x^*) = x^*(x'') \). It follows that \( x^*(x) = x^*(x') = \sigma_A(x^*) \). Therefore, \( x, x' \in \partial \sigma_A(x^*) \). But this contradicts Gateaux differentiability of \( \sigma_A \). Now, assume that \( \sigma_A \) is not differentiable on \( x^* \in \text{int}(\text{dom} \sigma_A) \). By the assumptions, \( \text{int}(\text{dom} \sigma_A) \neq \emptyset \). Also, as we mentioned in preliminary section, \( \partial \sigma_C(x^*) \neq \emptyset \) for every \( x^* \in \text{int}(\text{dom} \sigma_C) \). Hence, there exist \( x_1, x_2 \in \partial \sigma_C(x^*) \) so that \( x_1 \neq x_2 \). Now, from the convexity of \( \partial \sigma_C(x^*) \), we get the contradiction \( \frac{x_1 + x_2}{2} \in \partial \sigma_C(x^*). \)

Let \( X \) be a finite dimensional Banach space and \( A \) be a subset of \( X \) so that \( \text{int}(\text{dom} \sigma_A) \neq \emptyset \). Then, the following two assertions are equivalent [11],

\[ \forall x, x' \in A, \ x \neq x', \ \forall \lambda \in (0,1), \ \lambda x + (1-\lambda)x' \in A + (A \setminus \{0\}), \quad (4.2) \]

and

\[ \forall x, x' \in E_A, \ x \neq x', \ \forall \lambda \in (0,1), \ \lambda x + (1-\lambda)x' \notin E_A. \quad (4.3) \]

Since this equivalence remains true in infinite dimensional reflexive Banach spaces, we obtain the following result.

**Corollary 4.1.** When the conditions \((4.2)\) or \((4.3)\) hold, then \( \sigma_A \) is Gateaux differentiable on \( \text{int}(\text{dom} \sigma_A) \).

**Proof.** Since \( E_A \subseteq S_A \), so \((4.1) \Rightarrow \!(4.2)\). \qed
References