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On Conformal Metrics with Constant Q-Curvature

Andrea Malchiodi*

Scuola Normale Superiore, Piazza dei Cavalieri 7, 50126 Pisa, Italy

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Abstract. We review some recent results in the literature concerning existence of conformal metrics with constant Q-curvature. The problem is rather similar to the classical Yamabe problem: however it is characterized by a fourth-order operator that might lack in general a maximum principle. For several years existence of geometrically admissible solutions was known only in particular cases. Recently, there has been instead progress in this direction for some general classes of conformal metrics.

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1 Introduction

A classical problem in conformal geometry is the *Yamabe problem*, consisting in deforming a background metric on a compact manifold (M,g) so that its scalar curvature becomes constant. This can be considered as an extension of the classical uniformization problem for two-dimensional surfaces and has received a lot of attention in the literature, see [34] for a general introduction to the problem.

The scalar curvature of a manifold transforms conformally according to the law

$$L_g u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}}, \quad \tilde{g} = u^{\frac{4}{n-2}} g, \tag{1.1}$$

where L_g is the *conformal laplacian*, defined by

$$L_g \phi = -\frac{4(n-1)}{(n-2)} \Delta_g \phi + R_g \phi.$$

The latter operator transforms covariantly, namely one has

$$L_g(u\phi) = u^{\frac{n+2}{n-2}} L_{\tilde{g}}(\phi). \tag{1.2}$$

*Corresponding author. *Email address:* `andrea.malchiodi@sns.it` (A. Malchiodi)

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The Yamabe problem then amounts to finding a positive solution to (1.1) with $R_{\tilde{g}}$ equal to a constant. This constant can be viewed as a Lagrange multiplier when considering the following minimization problem

$$Y(g) = \inf_{u\in W^{1,2}(M,g), u\neq 0} \frac{\int_M u L_g u d\mu_g}{\left(\int_M |u|^{\frac{2n}{n-2}} d\mu_g\right)^{\frac{n-2}{n}}}. \tag{1.3}$$

It can be proved using (1.1) and (1.2) that the latter quantity is conformally invariant.

Since the Sobolev embedding $W^{1,2}(M,g) \hookrightarrow L^{\frac{2n}{n-2}}(M,g)$ is not compact, it is a-priori not clear whether a minimizer exists. In [44] the problem was attacked by lowering the exponent by a small amount and trying to pass to the limit: however the original proof of convergence was faulty. In [41] existence of minimizers was shown provided $Y(g)$ is smaller than a given positive dimensional constant (and in particular when it is negative or zero). In [1] it was shown via a compactness argument that minimizers exist provided $Y(g) < Y(g_{S^n})$, which was verified in dimension $n\geq 6$ if (M,g) is not conformally equivalent to the standard sphere. Under this latter assumption, in [39] the same strict inequality was proved in the complementary cases, i.e., for $n=3,4,5$ or (M,g) locally conformally flat. While the argument in [1] exploited a local geometric expansion involving the Weyl tensor, the one in [39] relied on the *Positive Mass Theorem* in general relativity.

We next discuss some higher-order analogue of the above problem. In [2] T.Branson introduced the following fourth-order operator in dimension $n\geq 5$:

$$P_g u = \Delta_g^2 u - div_g\left(a_n R_g g + b_n Ric_g\right) du + \frac{n-4}{2} Q_g u,$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2}.$$

The function

$$Q_g = \frac{1}{2(n-1)}\Delta_g R_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2}|Ric_g|^2$$

is the so-called *Q-curvature* (see [14, 18, 29] for more general operators and formulas). As for L_g, the operator P_g is conformally covariant: if $\tilde{g} = u^{\frac{4}{n-4}} g$ is a conformal metric to g, then for all $\phi \in C^\infty(M)$ we have

$$P_g(u\phi) = u^{\frac{n+4}{n-4}} P_{\tilde{g}}(\phi). \tag{1.4}$$

Moreover one has the following conformal transformation law

$$P_g u = Q_{\tilde{g}} u^{\frac{n+4}{n-4}}, \quad \tilde{g} = u^{\frac{4}{n-4}} g, \tag{1.5}$$

analogous to (1.1).

Formulas (1.4) and (1.5) naturally suggest a higher order version of the Yamabe problem, i.e., given (M^n,g) does there exists a conformal metric of constant Q-curvature? Due to (1.5), this amounts to finding a positive solution of

$$P_g u = \lambda u^{\frac{n+4}{n-4}}, \tag{1.6}$$

with $\lambda \in \mathbb{R}$.

In dimension four the corresponding equation is

$$P_g w + 2Q_g = \lambda e^{4w}, \tag{1.7}$$

with conformal metric written as $\tilde{g} = e^{2w} g$. In this case the conformal factor is always positive and the progress on this problem was obtained earlier compared to the higher-dimensional case, see [3,8,10,23,33,36]. For prescribed variable curvature, we also mention the papers [7,15,21,37].

In dimensions $n \geq 5$, there were some results in particular cases. In [9] some compactness results for solutions of (1.6) were proven, assuming that that P_g with constant coefficients (as it happens for Einstein metrics), which allows to factorize the Paneitz operator as product of monomials in the Laplacian and to apply the maximum principle iteratively (see also [42]). In [13], for manifolds of dimension $n \geq 8$ and non locally-conformally flat (as a counterpart of [1]), solutions to the following equation were found

$$P_g u = \lambda |u|^{\frac{8}{n-4}} u.$$

However the authors were not able to determine the sign of solutions, so these could have been possibly non geometrically admissible.

In [38] the authors were able to find positive solutions to (1.6) on locally conformally flat manifolds of positive scalar curvature. This was done using the approach in [40] via the developing map, under the assumption that the Poincaré exponent of the manifold is less than $(n-4)/2$. In [25] the authors proved a compactness result for (1.6) on locally conformally flat manifolds with positive scalar curvature, assuming positivity of the Paneitz operator and of its Green function. They also needed a suitable version of the positive mass theorem for the Green's function of the Paneitz operator, verified later in [26]. Other results about the prescription of Q-curvature can be found in [4–6,11,12,20,24,31].

In this note we are going to describe some recent progress on (1.6), from papers where suitable maximum principles were proved for the Paneitz operator. In [19], the following pointwise conditions were imposed on the Q-curvature and the scalar curvature

$$\begin{cases} Q_g \geq 0 \text{ and } Q_g > 0 \text{ somewhere}, \\ R_g \geq 0. \end{cases} \tag{1.8}$$

Under these assumptions it was proved that the Paneitz operator enjoys a maximum principle, namely that if a smooth function u satisfies $P_g u \geq 0$, then either $u > 0$ or $u \equiv 0$ on

M^n. This result was proved by showing, naively, non-negativity of the scalar curvature and then for the conformal factor (the rigorous proof using a homotopy argument).

As an extension of a result from [22] (concerning the four-dimensional case), under the same assumptions it was also proved that the Paneitz operator is positive-definite. As a consequence of this fact one has that the Green's function with pole at p, denoted $G_P(p,\cdot)$, exist and it is positive away from p. In fact, something more precise can be deduced: if $n \leq 7$ or if (M,g) is locally conformally flat, it can be proved that G_P satisfies

$$G_P(p,\cdot)(x) = \frac{c_n}{d_{\tilde{g}}(x,p)^{n-4}} + \alpha + \mathcal{O}^{(4)}(r), \tag{1.9}$$

with $\alpha \geq 0$ and $\alpha = 0$ if and only if the manifold is conformally equivalent to the round sphere, a fourth-order version of the positive mass theorem.

Consider then the quantity

$$\mathcal{F}_{g_0}[u] = \frac{\int_M u(P_{g_0}u)\, d\mu_{g_0}}{\left(\int_M u^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}}}.$$

When M is not conformal to the round sphere and either $n=5,6,7$ or M is locally conformally flat, the above property was used to prove that there exist conformal factors for which Q is non-negative (and not identically zero), the scalar curvature is positive and such that $\mathcal{F}_{g_0}[u] < S_n$, the spherical Sobolev constant (see (4.1)). Exploiting a result in [13], the same was shown for dimension $n \geq 8$ still for non locally conformally flat manifolds. In [19] it was then introduced a parabolic flow that preserves positivity of the conformal factor and of the scalar curvature and that sequentially converges to a (geometrically admissible) solution of (1.6).

After [19] appeared, in other papers the above results were made progressively conformally invariant. In [28] a maximum principle was proved for the Paneitz operator of manifolds with positive Yamabe invariant and non-negative (and non-zero) Q-curvature. Under the same assumptions, problem (1.6) was attacked in [27] using a dual formulation of $\mathcal{F}_{g_0}$. In particular, the authors considered the following quantity

$$\Theta_4(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M f G_P f d\mu_g}{\|f\|^2_{L^{\frac{2n}{n+4}}(M,g)}}. \tag{1.10}$$

They showed that if (M,g) is not conformally equivalent to the round sphere, then $\Theta_4(g)$ is strictly larger than $\Theta_4(g_{S^n})$ and that maximizers exist and lead to solutions of (1.6) (in relation to a result in [1]). Finally, in [16] the following quantity was introduced:

$$Y_4^*(g) = \frac{n-4}{2} \inf_{\tilde{g} \in [g], R_{\tilde{g}} > 0} \frac{\int_M Q_{\tilde{g}} d\mu_{\tilde{g}}}{Vol_{\tilde{g}}(M)^{\frac{n-4}{n}}}.$$

It was shown via a continuity argument that if $n \geq 6$, $Y(g) > 0$ and $Y_4^*(g) > 0$, then it is possible to find a metric conformal to g with positive scalar curvature and positive Q-curvature. By the results in [19], this implies the existence of a conformal metric with constant and positive Q-curvature.

The restriction on dimension is believed to be just of technical nature and it is expected that it could be removed. It would also be interesting to find such kinds of uniformization results in cases when either the Yamabe quotient or the Q-curvature (or both) are negative. Another interesting aspect is to analyse the compactness of solutions of (1.6): apart from those in [25], some results in this direction are available in [43] and [35].

The plan of the paper is the following. In Section 2 we describe the maximum principle proved in [19], as well as the extension of the positive mass theorem in [26] to arbitrary manifolds not globally conformal to the round sphere. In Section 3 we describe a conformal flow introduced in [19] that preserves the positivity of the conformal factor and satisfies suitable monotonicity properties. In Section 4 we prove sequential sequence of this flow, once we choose suitable initial data with low Sobolev quotient. Finally in Section 5 we describe some conformally (or partially-conformally) invariant extensions that provide existence of metrics with positive and constant Q-curvature.

2 Maximum principle for the Paneitz operator and its Green's function

In this section we discuss a maximum principle for the Paneitz operator and study some relevant properties of its Green's function.

Before discussing the maximum principle, we state a preliminary lemma relating the Q-curvature and the scalar curvature via an iteration of the classical maximum principle.

Lemma 2.1. *Suppose (M^n,g) is a compact manifold, with $n \geq 5$. Assume that $Q_g \geq 0$ with $Q_g > 0$ somewhere and that $R_g \geq 0$. Then we have the strict inequality $R_g > 0$.*

Proof. By definition of Q_g, the Q-curvature can be written as

$$Q_g = -\frac{1}{2(n-1)}\Delta_g R_g + c_1(n) R_g^2 - c_2(n)|Ric_g|^2, \tag{2.1}$$

with $c_1(n), c_2(n) > 0$. Since Q_g is non-negative, we have that

$$\frac{1}{2(n-1)}\Delta_g R_g \leq c_1(n) R_g^2. \tag{2.2}$$

By the strong maximum principle, it follows that either $R_g > 0$ or $R_g \equiv 0$. In the latter case, (2.1) would imply

$$Q_g = -c_2(n)|Ric_g|^2 \leq 0, \tag{2.3}$$

a contradiction. □

We have next the main result of this section.

Theorem 2.1. *Suppose (M^n,g) is a compact manifold, with $n\geq 5$. Assume also that $Q_g\geq 0$ with $Q_g>0$ and that moreover $R_g\geq 0$. If $u\in C^4$ satisfies*

$$P_g u\geq 0, \tag{2.4}$$

then either $u>0$ or $u\equiv 0$ on M^n. Moreover, if $u>0$ and if $h=u^{\frac{4}{n-4}}g$, then $Q_h\geq 0$ and $R_h>0$.

Proof. For $\lambda\in[0,1]$ set

$$u_\lambda=(1-\lambda)+\lambda u. \tag{2.5}$$

Then $u_0\equiv 1$ and $u_1\equiv u$. Suppose that $\min_{M^n} u\leq 0$, and define $\lambda_0\in(0,1]$ by

$$\lambda_0=\min\Big\{\lambda\in(0,1]\,:\,\min_{M^n}u_\lambda=0\Big\}. \tag{2.6}$$

Then if $0<\lambda<\lambda_0$ we must have $u_\lambda>0$. Define also the metrics

$$g_\lambda=u_\lambda^{4/(n-4)}g, \tag{2.7}$$

and let Q_λ be the Q-curvature of g_λ. Note also that for $0<\lambda<\lambda_0$, we have $Q_\lambda\geq 0$ and $Q_\lambda>0$ somewhere. This is a consequence of the following formula

$$\begin{aligned} Q_\lambda&=\frac{2}{n-4}u_\lambda^{-\frac{n+4}{n-4}}P_g u_\lambda=\frac{2}{n-4}u_\lambda^{-\frac{n+4}{n-4}}\Big\{P_g\big((1-\lambda)+\lambda u\big)\Big\}\\ &=\frac{2}{n-4}u_\lambda^{-\frac{n+4}{n-4}}\Big\{(1-\lambda)P_g(1)+\lambda P_g u\Big\}=\frac{2}{n-4}u_\lambda^{-\frac{n+4}{n-4}}\Big\{(1-\lambda)\frac{n-4}{2}Q_g+\lambda P_g u\Big\}\\ &\geq(1-\lambda)Q_g u_\lambda^{-\frac{n+4}{n-4}}. \end{aligned} \tag{2.8}$$

Since $\lambda<\lambda_0\leq 1$, by the assumptions on Q_g, we have that $Q_\lambda\geq 0$ and it is not identically zero.

Let now R_λ denote the scalar curvature of g_λ. We will show that if $0\leq\lambda<\lambda_0$, then $R_\lambda>0$. This is indeed true if $\lambda=0$. Assume now by contradiction that there exists $\lambda_1\in(0,\lambda_0)$ with $\min R_{\lambda_1}=0$: this would however contradict Lemma 2.1.

By (1.1) it follows that

$$R_\lambda=u_\lambda^{-\frac{n}{n-4}}\Big\{-\frac{4(n-1)}{(n-4)}\Delta_g u_\lambda-\frac{8(n-1)}{(n-4)^2}\frac{|\nabla_g u_\lambda|^2}{u_\lambda}+R_g u_\lambda\Big\}. \tag{2.9}$$

Since $R_\lambda>0$, the conformal factor u_λ satisfies the differential inequality

$$\Delta_g u_\lambda\leq\frac{(n-4)}{4(n-1)}R_g u_\lambda. \tag{2.10}$$

By continuity, this must also hold when $\lambda=\lambda_0$. By the strong maximum principle, (2.6) and (2.10), it follows that $u_{\lambda_0}\equiv 0$. If $\lambda_0=1$, the proof is over. Assuming instead that $\lambda_0\in(0,1)$, it follows from (2.5) that $u=-\frac{(1-\lambda_0)}{\lambda_0}$, and therefore

$$P_g u=-\Big(\frac{n-4}{2}\Big)\frac{(1-\lambda_0)}{\lambda_0}Q_g.$$

Since Q_g is somewhere positive, this contradicts the fact that $P_g u\geq 0$. We conclude that either $u\equiv 0$ or $u>0$.

If $u>0$, the tensor field $h=u^{\frac{4}{n-4}}g$ is a Riemannian metric with non-negative Q-curvature. Once more, consider the functions $\{u_\lambda\}$ as in (2.5) and the metrics g_λ as in (2.7). Then R_{g_λ} satisfies (2.9) and by the strong maximum principle we must have that either $R_\lambda>0$ or $R_\lambda\equiv 0$. Recall by the previous lemma one has $R_g>0$. Therefore, it cannot be $R_\lambda\equiv 0$ since we are in positive conformal class. It then follows $R_\lambda>0$ for all $\lambda\in[0,1]$, as desired. □

We next state a result on the positivity of the Paneitz operator, proved in [19]. We only report the proof of the last statement, for brevity reasons and limit ourselves to mention that the arguments missing here rely on a Bochner formula and a subtle integration by parts.

Proposition 2.1. Under the assumptions of Theorem 2.1 the first eigenvalue of the Paneitz operator is positive and hence P_g is invertible. As a consequence, we also have the following inequality for a Sobolev-type quotient

$$q_0(g):=\inf_{\phi\in W^{2,2}(M,g)\setminus\{0\}}\frac{\int_M \phi P_g\phi\, d\mu_g}{\big(\int_M|\phi|^{\frac{2n}{n-4}}\, d\mu_g\big)^{\frac{n-4}{n}}}>0. \tag{2.11}$$

Moreover, if G_P denotes the Green's function of the Paneitz operator with pole at $p\in M^n$, then $G_P>0$ on $M^n\setminus\{p\}$.

Proof. We just prove the latter assertion. Let (f_j) be a sequence of non-negative functions, whose supports shrink to $\{p\}$, $p\in M$ and such that $\int_M f_j d\mu_g=1$ for all j. Then $f_j\rightharpoonup\delta_p$ in the sense of distributions. By the property on eigenvalues of P_g, we have a unique solution G_j to $P_gG_j=f_j$. By standard regularity theory one has that

$$G_j\to G_P \quad \text{in} \quad C^4_{loc}(M^n\setminus\{p\}).$$

By Theorem 2.1 it follows that G_j is positive on M, and hence $G_P(p,\cdot)\geq 0$ on $M^n\setminus\{p\}$.

Suppose by contradiction there exists $x_0\neq p$ such that $G_P(x_0)=0$ and let $g_j=G_j^{\frac{4}{n-4}}g$. By construction, $P_gG_j\geq 0$, hence by Theorem 2.1 the metrics g_j satisfy $R_{g_j}>0$, $Q_{g_j}\geq 0$ and not identically zero. Hence the scalar curvature of g_j satisfies

$$\frac{1}{2(n-1)}\Delta_{g_j}R_{g_j}\leq c_1(n)R_{g_j}^2. \tag{2.12}$$

From the proof of Lemma 2.1 (see (2.10)), it follows that G_j satisfies

$$\Delta_g G_j \le \frac{(n-4)}{4(n-1)} R_g G_j \quad \text{on} \quad M^n. \tag{2.13}$$

Therefore we have that

$$\Delta_g G_P \le \frac{(n-4)}{4(n-1)} R_g G_P \quad \text{on} \quad M^n \setminus \{p\}.$$

By the strong maximum principle, $G_P(p,x_0)=0$ implies $G_P \equiv 0$, a contradiction. □

We next study in more detail the regularity of the Green's function G_P with pole at p. Recall the construction of *conformal normal coordinates* constructed in [34]: first, a suitable choice of conformal factor is made and then standard normal coordinates are used for this deformed metric. The following result is proved via the classical method of parametrix, with a careful expansion of the lower-order terms in the Paneitz operator.

Proposition 2.2. Let (M^n,g) be as in Theorem 2.1. Suppose also that either $n=5,6$, or 7, or that $n \ge 5$ and that (M^n,g) is locally conformally flat. For $p \in M$, consider the with conformal metric $\tilde{g}$ as in the construction of conformal normal coordinates. Then there exists a constant α such that in those coordinates one has

$$G_P(x) = \frac{c_n}{d_{\tilde{g}}(x,p)^{n-4}} + \alpha + \mathcal{O}^{(4)}(r), \tag{2.14}$$

where $c_n = \frac{1}{(n-2)(n-4)\omega_{n-1}}$, $\omega_{n-1} = |S^{n-1}|$ and $f = \mathcal{O}^{(k)}(r^m)$ denotes quantities that satisfy

$$|\nabla^j f(x)| \le C_j r^{m-j}$$

for $1 \le j \le k$, with $r = |x| = d_{\tilde{g}}(x,p)$.

We next prove a positive-mass theorem for the Paneitz operator, which will be useful for finding conformal metrics with constant Q-curvature. The proof is an adaptation of the argument in [26], where the result is obtained in the locally conformally flat case.

Theorem 2.2. *Suppose the conditions of Theorem 2.1 hold true and let α be the constant given in Proposition 2.2. Then one has $\alpha \ge 0$, with $\alpha = 0$ if and only if (M^n,g) is (globally) conformally equivalent to the round sphere.*

Proof. Denote by $G_{L,p}$ the Green's function of the conformal laplacian $L_g = -\Delta + \frac{(n-2)}{4(n-1)} R_g$ with pole at p. Similarly to [26], consider the conformal metric

$$\hat{g} = G_{L,p}^{\frac{4}{n-2}} g. \tag{2.15}$$

The open manifold $(X^n,\hat{g})$, $X^n = M^n\setminus\{p\}$, is asymptotically flat and scalar-flat. Define also

$$\Phi = G_{L,p}^{-\frac{n-4}{n-2}} G_P. \tag{2.16}$$

By (1.4) we have

$$P_{\hat{g}}\Phi = P_{G_{L,p}^{\frac{4}{n-2}} g}\big(G_{L,p}^{-\frac{n-4}{n-2}} G_P\big) = G_{L,p}^{-\frac{n+4}{n-2}} P_g(G_P) = 0.$$

As $R_{\hat{g}} \equiv 0$, we have that

$$Q_{\hat{g}} = -2|A_{\hat{g}}|^2, \tag{2.17}$$

where

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g\right)$$

is the *Schouten tensor*. Recalling the definition of the Paneitz operator, we also have

$$0 = P_{\hat{g}}\Phi = \Delta_{\hat{g}}^2\Phi + \mathrm{div}_{\hat{g}}\big\{\big(4A_{\hat{g}}(\nabla\Phi,\cdot)\big\} - (n-4)|A_{\hat{g}}|^2\Phi. \tag{2.18}$$

For a small $\delta > 0$, let B_δ be the geodesic ball centered at p of radius $\delta > 0$, with respect to the metric g. Integrating (2.18) on $M^n\setminus B_\delta$ and applying the Green's formula we get

$$\begin{aligned} 0 &= \int_{M^n\setminus B_\delta} P_{\hat{g}}\Phi \, d\mu_{\hat{g}} \\ &= \int_{M^n\setminus B_\delta} \Big\{\Delta_{\hat{g}}^2\Phi + \mathrm{div}_{\hat{g}}\big\{\big(4A_{\hat{g}}(\nabla\Phi,\cdot)\big\} - (n-4)|A_{\hat{g}}|^2\Phi\Big\}\, d\mu_{\hat{g}} \\ &= \oint_{\partial B_\delta}\Big\{\frac{\partial}{\partial\nu}\big(\Delta_{\hat{g}}\Phi\big) + 4A_{\hat{g}}(\nabla\Phi,\nu)\Big\} dS_{\hat{g}} - (n-4)\int_{M^n\setminus B_\delta} |A_{\hat{g}}|^2\Phi\, d\mu_{\hat{g}}, \end{aligned} \tag{2.19}$$

where ν is the outer unit normal to ∂B_δ with respect to the metric $\hat{g}$.

Since $\hat{g}$ is scalar-flat, we find

$$\frac{\partial}{\partial\nu}\big(\Delta_{\hat{g}}\Phi\big) = -\frac{\partial}{\partial\nu}\big(L_{\hat{g}}\Phi\big). \tag{2.20}$$

Using (1.2) we find also

$$L_{\hat{g}}\Phi = G_{L,p}^{-\frac{n+2}{n-2}} L_g\big(G_{L,p}^{\frac{2}{n-2}} G_P\big). \tag{2.21}$$

Let $r(x) = d_g(x,p)$ denote the distance from p with respect to the metric g. By Lemma 6.4 of [34], one has the expansion

$$G_{L,p}^{\frac{2}{n-2}} = \begin{cases} r^{-2} + \mathcal{O}(r), & \text{if } n=5, \\ r^{-2} + \mathcal{O}(r^2\log r), & \text{if } n=6, \\ r^{-2} + \mathcal{O}(r^2), & \text{if } n=7. \end{cases} \tag{2.22}$$

Together with Proposition 2.2, we then have

$$G_{L,p}^{\frac{2}{n-2}} G_P = c_n r^{2-n} + \alpha r^{-2} + \mathcal{O}(r^{-1}). \tag{2.23}$$

From the asymptotics of the Green's functions and the fact that, in conformal normal coordinates, $R_g = \mathcal{O}(r^2)$ we find

$$\begin{aligned} L_g\big(G_{L,p}^{\frac{2}{n-2}} G_P\big) &= -\Delta_g\big(G_{L,p}^{\frac{2}{n-2}} G_P\big) + \frac{(n-2)}{4(n-1)} R_g G_{L,p}^{\frac{2}{n-2}} G_P \\ &= 2(n-4)\alpha r^{-4} + \mathcal{O}(r^{4-n}) \\ &= 2(n-4)\alpha r^{-4} + \mathcal{O}(r^{-3}), \quad \text{if} \quad 5 \le n \le 7. \end{aligned} \tag{2.24}$$

By (2.22),

$$G_{L,p}^{-\frac{n+2}{n-2}} = r^{n+2} + \mathcal{O}(r^{n+3}), \tag{2.25}$$

and therefore

$$L_{\hat{g}}\Phi = G_{L,p}^{-\frac{n+2}{n-2}} L_g\big(G_{L,p}^{\frac{2}{n-2}} G_P\big) = 2(n-4)\alpha r^{n-2} + \mathcal{O}(r^{n-1}). \tag{2.26}$$

It is easy to check that

$$\frac{\partial}{\partial \nu} = -G_{L,p}^{-\frac{2}{n-2}} \frac{\partial}{\partial r}, \tag{2.27}$$

hence from (2.23) and (2.26) we deduce

$$\frac{\partial}{\partial \nu}\big(L_{\hat{g}}\Phi\big)\big|_{\partial B_\delta} = -2(n-2)(n-4)\alpha\delta^{n-1} + \mathcal{O}(\delta^n). \tag{2.28}$$

Concerning the surface measure, we have the transformation law

$$\oint_{\partial B_\delta} dS_{\hat{g}} = \oint_{\partial B_\delta} G_{L,p}^{\frac{2(n-1)}{(n-2)}} dS_g = \omega_{n-1}\delta^{1-n} + \mathcal{O}(\delta^{2-n}). \tag{2.29}$$

Therefore, the first boundary term in formula (2.19) becomes

$$\oint_{\partial B_\delta} \frac{\partial}{\partial \nu}\big(\Delta_{\hat{g}}\Phi\big)\, dS_{\hat{g}} = 2(n-2)(n-4)\omega_{n-1}\alpha + o(1), \tag{2.30}$$

while the second one satisfies

$$\oint_{\partial B_\delta} 4A_{\hat{g}}(\nabla\Phi,\nu) dS_{\hat{g}} = o(1). \tag{2.31}$$

These imply

$$2(n-2)(n-4)\omega_{n-1}\alpha = (n-4)\int_{M^n\setminus B_\delta} |A_{\hat{g}}|^2 \Phi\, d\mu_{\hat{g}} + o(1). \tag{2.32}$$

As a consequence, we have $\alpha \ge 0$. If $\alpha = 0$, then $\hat{g}$ has vanishing Ricci curvature and hence $(X^n, \hat{g})$ is isometric to the Euclidean space (see [39], page 492). The proof is thereby concluded. $\square$

3 A conformal flow

Throughout this section we will always assume that (M,g_0) is a compact manifold satisfying the assumptions of Theorem 2.1. By Proposition 2.1, the Paneitz operator P_{g_0} is invertible and hence it makes sense to consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = -u + \mu P_{g_0}^{-1}\big(|u|^{\frac{n+4}{n-4}}\big), \\ u(\cdot,0)=1, \end{cases} \qquad \mu = \frac{\int_M u P_{g_0} u \, d\mu_{g_0}}{\int_M |u|^{\frac{2n}{n-4}} d\mu_{g_0}}. \tag{3.1}$$

It is rather standard, via a fixed point argument, to show the following result.

Lemma 3.1. *There exists $T \in (0,+\infty]$ such that the flow* (3.1) *has a smooth solution for $0 \le t < T$.*

We first show that the positivity of the conformal factor is preserved.

Proposition 3.1. For every $0 \le t < T$ one has the inequality

$$u(t,x) > 0. \tag{3.2}$$

Proof. Formula (3.1) implies

$$\frac{\partial}{\partial t} P_{g_0} u = P_{g_0}\Big(\frac{\partial}{\partial t} u\Big) = -P_{g_0} u + \mu |u|^{\frac{n+4}{n-4}}, \tag{3.3}$$

which in turn leads to

$$\frac{\partial}{\partial t} P_{g_0} u \ge -P_{g_0} u. \tag{3.4}$$

Integrating this inequality we obtain

$$P_{g_0} u(t,x) \ge e^{-t} P_{g_0} u(0,x) = e^{-t} P_{g_0}(1) = \frac{n-4}{2} e^{-t} Q_{g_0}(x). \tag{3.5}$$

This implies $P_{g_0} u \ge 0$ with $P_{g_0} u > 0$ somewhere. The strong maximum principle in Theorem 2.1 implies $u > 0$ for $t \in [0,T)$, giving us the desired assertion. □

Remark 3.1. As a consequence of the latter proof one has that $Q_g > 0$ for all $t \in (0,T)$. Since u is positive, (3.3) implies that

$$\frac{\partial}{\partial t} P_{g_0} u \ge -P_{g_0} u + \mu u^{\frac{n+4}{n-4}},$$

from which we deduce that $P_{g_0} u > 0$ for every $t \in (0,T)$.

Since $u > 0$ for all existence times, (3.1) can also be rewritten as

$$\frac{\partial}{\partial t} u = -u + \mu P_{g_0}^{-1}\big(u^{\frac{n+4}{n-4}}\big), \quad \mu = \frac{\int_M u P_{g_0} u \, d\mu_{g_0}}{\int_M u^{\frac{2n}{n-4}} \, d\mu_{g_0}}. \tag{3.6}$$

We show next that along the flow we have a conserved and a monotonic quantity.

Lemma 3.2. *If u solves* (3.6)*, then for $0 \le t < T$ we have that*

$$\frac{d}{dt}\int_M uP_{g_0}u \, d\mu_{g_0} = 0, \tag{3.7}$$

and for the total volume the inequality

$$\frac{d}{dt}V = \frac{d}{dt}\int_M u^{\frac{2n}{n-4}} \, d\mu_{g_0} = \frac{2n}{n-4}\frac{1}{\mu}\int_M fP_{g_0}f \, d\mu_{g_0} \ge 0, \tag{3.8}$$

where

$$f = -u + \mu P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right). \tag{3.9}$$

In particular, we have that

$$\frac{d}{dt}\mu = \frac{d}{dt}\left(\frac{\int_M uP_{g_0}u \, d\mu_{g_0}}{V}\right) \le 0, \quad \frac{d}{dt}\mathcal{F}_{g_0}[u] = \frac{d}{dt}\left(\frac{\int_M uP_{g_0}u \, d\mu_{g_0}}{V^{\frac{n-4}{n}}}\right) \le 0. \tag{3.10}$$

Finally, we have the following upper bound

$$V \le C_0(g_0). \tag{3.11}$$

Proof. (3.6) and the definition of μ imply that

$$\begin{aligned}
&\frac{d}{dt}\int_M uP_{g_0}u \, d\mu_{g_0} \\
=&\int_M \left\{\left(\frac{\partial u}{\partial t}\right)P_{g_0}u + uP_{g_0}\left(\frac{\partial u}{\partial t}\right)\right\} d\mu_{g_0} = 2\int_M uP_{g_0}\left(\frac{\partial u}{\partial t}\right) d\mu_{g_0} \\
=&2\int_M uP_{g_0}\left(-u + \mu P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right) d\mu_{g_0} = 2\int_M \left(-uP_{g_0}u + \mu uP_{g_0}P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right) d\mu_{g_0} \\
=&\int_M \left\{-2uP_{g_0}u + 2\mu u^{\frac{2n}{n-4}}\right\} d\mu_{g_0} = -2\int_M uP_{g_0}u \, d\mu_{g_0} + 2\left(\frac{\int_M uP_{g_0}u \, d\mu_{g_0}}{\int_M u^{\frac{2n}{n-4}} \, d\mu_{g_0}}\right)\int_M u^{\frac{2n}{n-4}} \, d\mu_{g_0} = 0,
\end{aligned}$$

which yields the first statement. To prove the second we notice that

$$\begin{aligned}
\frac{d}{dt}\int_M u^{\frac{2n}{n-4}} \, d\mu_{g_0} &= \frac{2n}{n-4}\int_M u^{\frac{n+4}{n-4}}\frac{\partial u}{\partial t} \, d\mu_{g_0} = \frac{2n}{n-4}\int_M u^{\frac{n+4}{n-4}}\left\{-u + \mu P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right\} d\mu_{g_0} \\
&= \frac{2n}{n-4}\int_M \left\{-u^{\frac{2n}{n-4}} + \mu u^{\frac{n+4}{n-4}}P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right\} d\mu_{g_0}.
\end{aligned} \tag{3.12}$$

We also have

$$\begin{aligned}
\int_M fP_{g_0}f \, d\mu_{g_0} &= \int_M \left\{-u + \mu P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right\}\left\{-P_{g_0}u + \mu u^{\frac{n+4}{n-4}}\right\} d\mu_{g_0} \\
&= \int_M \left\{uP_{g_0}u - \mu u^{\frac{2n}{n-4}} - \mu P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)P_{g_0}u + \mu^2 u^{\frac{n+4}{n-4}}P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right\} d\mu_{g_0} \\
&= \int_M \left\{-\mu u^{\frac{2n}{n-4}} + \mu^2 u^{\frac{n+4}{n-4}}P_{g_0}^{-1}\left(u^{\frac{n+4}{n-4}}\right)\right\} d\mu_{g_0}.
\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we deduce (3.8).

The upper bound on the conformal volume follows from the fact that the Paneitz-Sobolev constant is positive:

$$
\begin{aligned}
0<q_0\leq \mathcal{F}_{g_0}[u] &= V^{-\frac{n-4}{n}}\int_M uP_{g_0}u\, d\mu_{g_0}\\
&= V^{-\frac{n-4}{n}}\int_M u_0P_{g_0}u_0\, d\mu_{g_0}=\frac{n-4}{2}V^{-\frac{n-4}{n}}\int_M Q_{g_0}\, d\mu_{g_0},
\end{aligned}
$$

so $V\leq C(g_0)$. This concludes the proof. □

Corollary 3.1. One has the estimate

$$
\int_0^T \|f\|_{W^{2,2}}\, dt\leq C_1(g_0), \quad \int_0^T \Big(\int_M |f|^{\frac{2n}{n-4}}\, d\mu_{g_0}\Big)^{\frac{n-4}{n}}\, dt\leq C_2(g_0). \tag{3.14}
$$

Proof. From the upper bound on volume it follows that

$$
\int_0^T \Big(\int_{M^n} fP_{g_0}f\, d\mu_{g_0}\Big)\, dt\leq C_1(g_0). \tag{3.15}
$$

On the other hand, from the positivity of P_{g_0} one finds

$$
\|\phi\|_{W^{2,2}}\approx \int_M \phi P_{g_0}\phi\, d\mu_{g_0},
$$

so the first inequality in (3.14) follows. The second one is a consequence of the lower bound on the Paneitz-Sobolev quotient. □

We next check the long-time existence of the above flow.

Proposition 3.2. The flow (3.1) admits a global in time solution. Moreover, given an initial datum g_0 there exist positive constants C, C' such that

$$
u\leq C'e^{Ct}. \tag{3.16}
$$

Proof. Fix a number $s>1$: since $u>0$ and $P_{g_0}u>0$ for $t\in[0,T)$, by (3.3) one has

$$
\begin{aligned}
\frac{d}{dt}\int_M (P_{g_0}u)^s\, d\mu_{g_0} &= s\int_M (P_{g_0}u)^{s-1}\frac{\partial}{\partial t}(P_{g_0}u)\, d\mu_{g_0}\\
&= s\int_M (P_{g_0}u)^{s-1}\{-P_{g_0}u+\mu u^{\frac{n+4}{n-4}}\}\, d\mu_{g_0}\\
&= -s\int_M (P_{g_0}u)^s d\mu_{g_0}+s\mu\int_M (P_{g_0}u)^{s-1}u^{\frac{n+4}{n-4}}\, d\mu_{g_0}.
\end{aligned} \tag{3.17}
$$

For the second integral in the last line, by Hölder's inequality

$$\int_M (P_{g_0}u)^{s-1} u^{\frac{n+4}{n-4}}\, d\mu_{g_0} \le \Big(\int_M (P_{g_0}u)^s\, d\mu_{g_0}\Big)^{\frac{s-1}{s}} \Big(\int_M u^{\frac{n+4}{n-4}s}\, d\mu_{g_0}\Big)^{\frac{1}{s}}. \tag{3.18}$$

Choose now s such that

$$\frac{2n}{n+4} < s < \frac{n}{4}. \tag{3.19}$$

One can apply Hölder's inequality once more to find

$$\Big(\int_M u^{\frac{n+4}{n-4}s}\, d\mu_{g_0}\Big)^{\frac{1}{s}} \le \Big(\int_M u^{\frac{ns}{n-4s}}\, d\mu_{g_0}\Big)^{\frac{n-4s}{ns}} \Big(\int_M u^{\frac{2n}{n-4}}\, d\mu_{g_0}\Big)^{\frac{4}{n}}. \tag{3.20}$$

By the Sobolev embedding theorem and by the fact that $P_{g_0} > 0$ we have

$$\|u\|_{W^{s,4}} \approx \|P_{g_0}u\|_{L^s}.$$

It follows that

$$\Big(\int_M u^{\frac{ns}{n-4s}}\, d\mu_{g_0}\Big)^{\frac{n-4s}{ns}} \le C_s \Big(\int_M (P_{g_0}u)^s\, d\mu_{g_0}\Big)^{\frac{1}{s}} \tag{3.21}$$

for s as in (3.19). Substituting this into (3.20) and using the conformal volume bound of Lemma 3.2 we have

$$\int_M (P_{g_0}u)^{s-1} u^{\frac{n+4}{n-4}}\, d\mu_{g_0} \le C_s \int_M (P_{g_0}u)^s\, d\mu_{g_0}. \tag{3.22}$$

Inserting this estimate into (3.17) gives

$$\frac{d}{dt}\int_M (P_{g_0}u)^s\, d\mu_{g_0} \le C_s \int_M (P_{g_0}u)^s\, d\mu_{g_0}, \qquad \frac{2n}{n+4} < s < \frac{n}{4}. \tag{3.23}$$

Integrating in time one finds

$$\int_M (P_{g_0}u)^s\, d\mu_{g_0} \le C_0 e^{C_s t}, \quad 0 \le t < T. \tag{3.24}$$

By the Sobolev embedding, one has also

$$\|u\|_{L^{\frac{ns}{n-4s}}} \le C_1 e^{C'_s t}. \tag{3.25}$$

If then s is sufficiently close to $n/4$, we deduce that

$$\|u\|_{L^p} \le C_3 e^{C_p t}, \tag{3.26}$$

for any $p > 1$.

Take now $s=n/4+1$. Returning to (3.18), we have that

$$\begin{aligned}&\int_M (P_{g_0}u)^{s-1}u^{\frac{n+4}{n-4}}\,d\mu_{g_0}\\ \le&\Big(\int_M (P_{g_0}u\,d\mu_{g_0})^s\Big)^{\frac{s-1}{s}}\Big(\int_M u^{\frac{n+4}{n-4}s}\,d\mu_{g_0}\Big)^{\frac{1}{s}}\le\Big(\int_M (P_{g_0}u)^s\,d\mu_{g_0}\Big)^{\frac{s-1}{s}}\big(C_3e^{C_nt}\big)^{\frac{1}{s}}\\ \le& C_4e^{C_5t}\Big(\int_M (P_{g_0}u)^s\,d\mu_{g_0}\Big)^{\frac{s-1}{s}}\le\int_M (P_{g_0}u)^s\,d\mu_{g_0}+C_6e^{C_7t}.\end{aligned}\tag{3.27}$$

Inserting this inequality into (3.17) gives

$$\frac{d}{dt}\int_M (P_{g_0}u)^s\,d\mu_{g_0}\le C'e^{Ct},\quad s=\frac{n}{4}+1.$$

Integrating in time and using Sobolev's embeddings we conclude that

$$\|u\|_{C^\alpha}\le C'e^{Ct},\tag{3.28}$$

for some $\alpha\in(0,1)$. This implies (3.16). By (3.3), one has also that the C^α-norm of $P_{g_0}u$ grows at most exponentially fast. Therefore the $C^{4,\alpha}$-norm of u also grows at most exponentially fast, so we must have global existence in time. □

4 Convergence of the flow

In this section we describe the converge of the above flow once suitable initial data are chosen. We want in particular to obtain initial conformal data so that they satisfy the assumptions of Theorem 2.1 and so that the Paneitz-Sobolev quotient is low enough. The estimate of the latter one depends on the dimension and on the local conformal flatness and is stated separately in the next two propositions.

Proposition 4.1. Let $(M^n,\bar{g})$ be a compact manifold of dimension $n\ge 8$. Suppose that $Q_{\bar{g}}\ge 0$, $Q_{\bar{g}}\not\equiv 0$, that $R_{\bar{g}}\ge 0$ and that $(M^n,\bar{g})$ is not locally conformally flat.

If the Weyl tensor $W(x_0)$ at $x_0\in M$ does not vanish, then for $\varepsilon>0$ small there exists a positive function $\psi_\varepsilon\in C^\infty$ such that

$$\mathcal{F}_{\bar{g}}(\psi_\varepsilon)\le S_n-c_n\varepsilon^4|\log\varepsilon||W(x_0)|^2,\quad \text{if}\quad n=8,$$

and

$$\mathcal{F}_{\bar{g}}(\psi_\varepsilon)\le S_n-c_n\varepsilon^4|W(x_0)|^2,\quad \text{if}\quad n\ge 9,$$

where c_n is a dimensional constant and where S_n is defined by

$$S_n=\inf_{\varphi\in C_0^\infty(\mathbb{R}^n)}\frac{\int_{\mathbb{R}^n}(\Delta\varphi)^2\,dx}{\Big(\int_{\mathbb{R}^n}|\varphi|^{\frac{2n}{n-4}}\,dx\Big)^{\frac{n-4}{n}}}.\tag{4.1}$$

Moreover, the conformal metric $h=\psi_\varepsilon^{\frac{4}{n-4}}\bar{g}$ satisfies: $Q_h\geq 0$, $Q_h\not\equiv 0$, $R_h>0$ and

$$\mathcal{F}_h(1)\leq S_n-c_n\varepsilon^4|\log\varepsilon||W(x_0)|^2, \quad \text{if} \quad n=8, \tag{4.2a}$$

$$\mathcal{F}_h(1)\leq S_n-c_n\varepsilon^4|W(x_0)|^2, \quad \text{if} \quad n\geq 9. \tag{4.2b}$$

We give next some ideas of the proof. Esposito and Robert considered in [13] the following test function,

$$\tilde{u}_\varepsilon(x)=\frac{\eta(x)}{\left(\varepsilon^2+d_{\bar{g}}(x,x_0)^2\right)^{\frac{n-4}{2}}},$$

where $\eta(x)$ is a cut-off supported in $B_{2\delta}(x_0)$ and identically equal to 1 in $B_\delta(x_0)$. In the same paper it was shown that for $\varepsilon>0$ small one has the inequalities

$$\mathcal{F}_{\bar{g}}(\tilde{u}_\varepsilon)\leq S_n-C(n)\varepsilon^4|\log\varepsilon||W(x_0)|^2, \quad \text{if} \quad n=8,$$

and

$$\mathcal{F}_{\bar{g}}(\tilde{u}_\varepsilon)\leq S_n-C(n)\varepsilon^4|W(x_0)|^2, \quad \text{if} \quad n\geq 9.$$

These allowed to show that the infimum of $\mathcal{F}$ is achieved if $n\geq 8$ and M is not locally conformally flat. However the positivity of a minimizing conformal factor was not guaranteed and therefore it could give rise to a solution not geometrically admissible.

In [19] the latter test function was modified in order to achieve some sign condition on the conformal factor, the scalar curvature and the Q-curvature. Recalling the invertibility of P_g from Proposition 2.1 (and its conformal covariance), the function $\hat{u}_\varepsilon$ was then defined by

$$P_{\bar{g}}\hat{u}_\varepsilon=\eta(x)\frac{b_n\varepsilon^4}{(\varepsilon^2+|x|^2)^{\frac{n+4}{2}}}. \tag{4.3}$$

By Theorem 2.1, the conformal metric induced by $\hat{u}_\varepsilon$ has the desired sign properties, including the curvatures. In [19] the difference between $\tilde{u}_\varepsilon$ and $\hat{u}_\varepsilon$ was also estimated and it was shown that

$$\mathcal{F}_{\bar{g}}(\hat{u}_\varepsilon)=\mathcal{F}_{\bar{g}}(u_\varepsilon)+o(\varepsilon^4|\log\varepsilon|) \quad \text{for} \quad n=8,$$
$$\mathcal{F}_{\bar{g}}(\hat{u}_\varepsilon)=\mathcal{F}_{\bar{g}}(u_\varepsilon)+o(\varepsilon^4) \quad \text{for} \quad n\geq 9,$$

giving the desired result.

We have then a related result for low dimensions or in the locally conformally flat case.

Proposition 4.2. Let $(M^n,\bar{g})$ be a compact manifold of dimension n, with $n=5,6$, or 7; or let $(M^n,\bar{g})$ be locally conformally flat of dimension $n\geq 5$. Suppose that $Q_{\bar{g}}\geq 0$, $Q_{\bar{g}}\not\equiv 0$ and that $R_{\bar{g}}\geq 0$. If $(M^n,\bar{g})$ is not conformally equivalent to the round sphere, then for $\epsilon>0$ small and $x_0\in M$, there exists a positive $\psi_\varepsilon\in C^\infty$ and a constant $c_{x_0}>0$ such that

$$\mathcal{F}_{\bar{g}}(\psi_\varepsilon)\leq S_n-c_{x_0}\varepsilon^{n-4}. \tag{4.4}$$

Moreover, the conformal metric $h=\psi_\varepsilon^{\frac{4}{n-4}}\bar{g}$ satisfies $Q_h\geq 0, Q_h\not\equiv 0$, $R_h>0$ and the inequality

$$\mathcal{F}_h(1)\leq S_n - c_{x_0}\varepsilon^{n-4}.$$

To prove the result, consider a cut-off function $\tilde{\chi}$ equal to 1 in B_1 and equal to zero outside B_2. Define then $\tilde{\chi}_{\tilde{\delta}}(x)=\tilde{\chi}(x/\tilde{\delta})$ and the function $\check{u}_\varepsilon$ defined by

$$\check{u}_\varepsilon := \tilde{\chi}_{\tilde{\delta}}(u_\varepsilon+\beta)+(1-\tilde{\chi}_{\tilde{\delta}})\overline{G}_{x_0}.$$

Here $\beta=\beta_{x_0}=\frac{1}{c_n}\alpha_{x_0}>0$, α_{x_0} is the zero-th order term in the expansion of G_{x_0} in (2.14) and $\overline{G}_{x_0}=\frac{1}{c_n}G_{x_0}$ with $\tilde{\delta}\ll\delta$. By the positivity of the Green's function, the function $\check{u}_\varepsilon$ is positive on M. This test function is a modification of a similar one constructed in [39]: notice that adding the constant β to the first term makes the Green's function for the Paneitz operator fit with the *bubble* u_ϵ. The function $\check{u}_\varepsilon$ represents a good approximation for the function $\hat{u}_\varepsilon$, which is defined implicitly, in the sense specified by the next lemma.

Lemma 4.1. *The following estimate holds, for some constant $C>0$:*

$$|\hat{u}_\varepsilon-\check{u}_\varepsilon|\leq o(1)+C\tilde{\delta}^{n-3}\min\left\{|x|^{4-n},\delta^{4-n}\right\}=o(1),\quad \tilde{\delta}\to 0.$$

The functional $\mathcal{F}$ can be well estimated on $\check{u}_\varepsilon$ via an integration by parts, since the Paneitz operator vanishes identically on the Green's function, away from the point x_0. Using this fact and Lemma 4.1 one can show the following estimate.

$$\mathcal{F}_{\bar{g}}(\hat{u}_\varepsilon)=S_n\left(1-\beta(1+o(1))\frac{\int_M u_\varepsilon^{\frac{n+4}{n-4}}d\mu_{\bar{g}}}{\int_M u_\varepsilon^{\frac{2n}{n-4}}d\mu_{\bar{g}}}\right).$$

This one, with a scaling argument used to estimate the latter integrals, leads to the inequality (4.4).

We next show convergence of the flow, under the assumptions of Proposition 4.1 or 4.2, to a solution of the constant Q-curvature equation.

Theorem 4.1. *Let $(M^n,\bar{g})$ be a compact manifold of dimension $n\geq 5$ not conformally equivalent to the standard sphere. Suppose also that $Q_{\bar{g}}\geq 0$, $Q_{\bar{g}\not\equiv 0}$ and that $R_{\bar{g}}\geq 0$.*

Let $g_0=h$, where h is the metric constructed in Proposition 4.2 or Proposition 4.1. Then the flow (3.1) has a solution for all time and satisfies

$$\int_M u^2\, d\mu_{g_0}\geq C_0 \tag{4.5}$$

for some constant $C_0>0$. Moreover, there exists $t_j\nearrow\infty$ such that $u_j=u_j(t_j,\cdot)$ weakly converges in $W^{2,2}(M^n,g)$ to a solution $u>0$ of

$$P_{g_0}u=\bar{\mu}u^{\frac{n+4}{n-4}}, \tag{4.6}$$

with $\bar{\mu}>0$. Hence, $g_\infty=u^{\frac{4}{n-4}}g_0$ gives rise to a metric with positive scalar curvature and constant positive Q-curvature.

Proof. Take as initial metric g_0 be the one given by Proposition 4.1 or 4.2. Proposition 3.2 then implies that (3.1) is defined for all times. Moreover, we have that

$$\mathcal{F}_{g_0}[u_0] \leq S_n - \epsilon_0, \tag{4.7}$$

with $u_0 \equiv 1$ and where ϵ_0 is some positive constant. Lemma 3.2 then implies

$$\mathcal{F}_{g_0}[u] = \frac{\int_M u(P_{g_0}u)\, d\mu_{g_0}}{\left(\int_M u^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}}} \leq S_n - \epsilon_0 \tag{4.8}$$

for all times.

Recalling (4.1), on (M,g_0), given $\delta > 0$, we can use a standard localization argument to prove

$$\left(\int_M |\varphi|^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}} \leq \left(S_n^{-1} + \delta\right) \int_M (\Delta_{g_0}\varphi)^2\, d\mu_{g_0} + C_\delta \int_M \varphi^2\, d\mu_{g_0},$$

which in turn implies

$$\left(\int_M |\varphi|^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}} \leq \left(S_n^{-1} + 2\delta\right) \int_M \varphi(P_{g_0}\varphi)\, d\mu_{g_0} + C'_\delta \int_M \varphi^2\, d\mu_{g_0}. \tag{4.9}$$

Inserting (4.8) into (4.9) gives

$$\begin{aligned}\left(\int_M u^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}} &\leq \left(S_n^{-1} + 2\delta\right) \int_M u(P_{g_0}u)\, d\mu_{g_0} + C'_\delta \int_M u^2\, d\mu_{g_0} \\ &\leq \left(S_n^{-1} + 2\delta\right)\left(S_n - \epsilon_0\right)\left(\int_M u^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}} + C'_\delta \int_M u^2\, d\mu_{g_0}. \end{aligned} \tag{4.10}$$

Taking $\delta = \epsilon_0/10$, the first term on the right-hand side can be brought to the left-hand side, so we deduce

$$\left(\int_M u^{\frac{2n}{n-4}}\, d\mu_{g_0}\right)^{\frac{n-4}{n}} \leq C(\epsilon_0) \int_M u^2\, d\mu_{g_0}. \tag{4.11}$$

Notice that the left-hand side is a power of the conformal volume, so we conclude

$$\int_M u^2\, d\mu_{g_0} \geq C_0 > 0$$

for all times, as desired.

By Lemma 3.2 and Corollary 3.1 there exist $t_j \nearrow \infty$ such that $u_j = u(t_j,\cdot)$ and $\mu_j = \mu(t_j)$ satisfy

$$\mu_j \nearrow \bar{\mu}, \tag{4.12a}$$

$$u_j \rightharpoonup u \text{ weakly in } W^{2,2}(M^n), \tag{4.12b}$$

$$u_j \to u \text{ strongly in } L^2(M^n), \tag{4.12c}$$

$$f_j = -u_j + \mu_j P_{g_0}^{-1}\left(u_j^{\frac{n+4}{n-4}}\right) \to 0 \text{ strongly in } W^{2,2}(M^n). \tag{4.12d}$$

Therefore $u \geq 0$ satisfies

$$u = \bar{\mu} P_{g_0}^{-1}\big(u^{\frac{n+4}{n-4}}\big), \tag{4.13}$$

hence u is a strong solution of

$$P_{g_0} u = \bar{\mu} u^{\frac{n+4}{n-4}}.$$

By Theorem 2.1, it follows that $u > 0$, concluding the proof. □

5 Conformally invariant extensions

We next mention some results from [27,28] and [17], where some conformally-invariant extensions of the above results were found.

We start with some results in [28] (discussed here only for $n \geq 5$), where the pointwise positivity of the scalar curvature is replaced by the one of the Yamabe invariant.

Proposition 5.1 (see [28]). Let (M^n, g) be compact, with $n \geq 5$ and $Y(g) > 0$. Let $p \in M$ and let $G_{L,p}$ denote the Green's function of L_g with singularity at p. Then $G_{L,p}^{\frac{n-4}{n-2}} |Ric_{G_{L,p}^{\frac{4}{n-2}}}|^2 \in L^1(M,g)$ and in distributional sense one has

$$P(G_{L,p}^{\frac{n-4}{n-2}}) = c_n \delta_p - \frac{n-4}{(n-2)^2} G_{L,p}^{\frac{n-4}{n-2}} \Big| Ric_{G_{L,p}^{\frac{4}{n-2}} g} \Big|^2, \tag{5.1}$$

where $c_n = 2^{-\frac{n-6}{n-2}} n^{\frac{2}{n-2}} (n-1)^{-\frac{n-4}{n-2}} (n-2)(n-4) \omega_n^{\frac{2}{n-2}}$.

The proof of this result relies on (1.4), which implies

$$\begin{aligned} P_g(G_{L,p}^{\frac{n-4}{n-2}}) &= G_{L,p}^{\frac{n+4}{n-2}} P_{G_{L,p}^{\frac{4}{n-2}} g} 1 = \frac{n-4}{2} G_{L,p}^{\frac{n+4}{n-2}} Q_{G_{L,p}^{\frac{4}{n-2}} g} \\ &= -\frac{n-4}{(n-2)^2} G_{L,p}^{\frac{n-4}{n-2}} \Big| Ric_{G_{L,p}^{\frac{4}{n-2}} g} \Big|^2 \quad \text{on} \quad M \setminus \{p\}. \end{aligned}$$

In the latter equality the fact that $R_{G_{L,p}^{\frac{4}{n-2}} g} = 0$ on $M \setminus \{p\}$ was used. The distributional equality in the proposition, including the point p, follows from a careful asymptotic analysis of the singular metrics (dilated by the two Green's functions) and the use of conformal normal coordinates. The next lemma relaxes the pointwise conditions in Theorem 2.1.

Lemma 5.1 (see [28]). *Let (M^n, g) be as in Proposition 5.1 and let $u: M \to \mathbb{R}$ be a smooth function such that $u \geq 0$ and $P_g u \geq 0$. If u vanishes somewhere on M, then $u \equiv 0$.*

Proof. By (5.1) we have that

$$\int_M G_{L,p}^{\frac{n-4}{n-2}} P_g u d\mu_g = -\frac{n-4}{(n-2)^2} \int_M G_{L,p}^{\frac{n-4}{n-2}} \left| Ric_{G_{L,p}^{\frac{4}{n-2}} g} \right|^2 u d\mu_g.$$

By the sign conditions on u and $P_g u$ it follows that $P_g u \equiv 0$ and that $|Ric_{G_{L,p}^{\frac{4}{n-2}} g}|^2 u \equiv 0$. If u is not identically zero, then by unique continuation $\{u \neq 0\}$ is dense, so it must be $|Ric_{G_{L,p}^{\frac{4}{n-2}} g}|^2 \equiv 0$. By asymptotic flatness of $(M \setminus \{p\}, G_{L,p}^{\frac{4}{n-2}} g)$ and by volume comparison it follows that the latter manifold must be isometric to the flat $\mathbb{R}^n$. Therefore (M,g), which is locally conformally flat and simply connected, must be globally conformally equivalent to S^n, by a result in [30]. However this implies that the kernel of P_g is trivial and therefore $u \equiv 0$.

The same argument gives the conclusion assuming only that u is of class L^1, smooth near p and that the inequality $P_g u \geq 0$ is satisfied in the distributional sense. □

The above result yields global positivity of the conformal factor.

Proposition 5.2 (see [28])**.** Let (M^n,g) be compact, with $n \geq 5$, $Y(g) > 0$ and $Q_g \geq 0$. Suppose that $u: M \to \mathbb{R}$ is smooth, satisfies $P_g u \geq 0$ and it is not identically constant. Then $u > 0$ on M.

Proof. Arguing by contradiction, suppose that $-\lambda = u(p) = \min_M u \leq 0$. This implies that $u + \lambda \geq 0$, $u(p) + \lambda = 0$ and that $P_g(u+\lambda) \geq \lambda Q_g \geq 0$. Lemma 5.1 implies that $u \equiv -\lambda$, which is impossible. □

An easy consequence of this fact is that if $Y(g) > 0$ and $Q \geq 0$, then the kernel of the Paneitz operator only consists of constant functions and that this kernel is identically zero if Q is also not identically zero.

Lemma 5.2. *Suppose (M^n,g) is such that $n \geq 5$, $Y(g) > 0$, $Q_g \geq 0$ and it is not identically zero. Then $\ker P_g = 0$ and $G_P(p,q) > 0$ for all $q \neq p$.*

Proof. We only need to prove the latter property. Given a smooth function f on M, there exists a unique solution u of $P_g u = f$, which can be written as

$$u(p) = \int_M f(q) G_P(p,q) d\mu_g.$$

If f is non-negative, from Proposition 5.2 it follows that $u \geq 0$ and therefore $G_P(p,q) \geq 0$. If $G_P(p,q)$ vanishes at some point q, the observation at the end of the proof of Lemma 5.1 implies $G_P(p,\cdot) \equiv 0$, which is a contradiction. □

We can now state one of the main results in [28].

Theorem 5.1 (see [28]). *Suppose (M^n,g) satisfies $n\geq 5$ and $Y(g)>0$. Then the following are equivalent*

(a) *there exists a smooth positive function ρ on M such that $Q_{\rho^2 g}>0$,*

(b) *$kerP_g=0$ and $G_P(p,q)>0$ for all $p\neq q$,*

(c) *$kerP_g=0$ and there exists $p\in M$ such that $G_P(p,q)>0$ for all $q\neq p$.*

Proof. The first property implies the second by Lemma 5.2, the conformal invariance of $kerP_g$ and the fact that

$$G_{P_{\rho^{\frac{4}{n-4}}g}}(p,q)=\rho(p)^{-1}\rho(q)^{-1}G_{P_g}(p,q).$$

The second property implies the first by the Krein-Rutman theorem.

Clearly, the second property implies the third one. For the opposite implication, suppose there exists p_0 such that $G_P(p_0,q)>0$ for all $q\neq p_0$. Setting

$$\Theta(p)=\min_{q\neq p}G_P(p,q),$$

we have $\Theta(p_0)>0$. We must also have $\Theta(p)\neq 0$ for all $p\in M$, from the latter comments in the proof of Lemma 5.2. By connectedness of M, it must be $\Theta(p)>0$ for all $p\in M$, concluding the proof. □

The above results allow to prove a conformally-invariant counterpart of the existence result in Theorem 4.1. In [27] the following quantity was introduced, motivated by a duality approach.

$$\Theta_4(g)=\sup_{f\in L^{\frac{2n}{n+4}}(M,g)\setminus\{0\}}\frac{\int_M fG_Pf d\mu_g}{\|f\|^2_{L^{\frac{2n}{n+4}}(M,g)}}, \tag{5.2}$$

where G_Pf stands for the convolution

$$(G_Pf)(p)=\int_M G_P(p,q)f(q)d\mu_g(q).$$

With arguments somehow related to those for the proof of Proposition 4.1, the following result was proved.

Proposition 5.3. Suppose (M^n,g) is a compact manifold with $n\geq 5$, $Y(g)>0$ and $Q\geq 0$, $Q\not\equiv 0$. Then $\Theta_4(g)\geq\Theta_4(g_{S^n})$, with equality satisfied if and only if (M^n,g) is conformally equivalent to the round S^n.

We do not report the proof here for reasons of brevity. The latter result allows to prove convergence of maximizing sequences, by showing that these cannot develop blow-up points.

Proposition 5.4. Suppose (M^n,g) is a compact manifold with $n\geq 5$ and $kerP_g=\{0\}$. If $\Theta_4(g)>\Theta_4(g_{S^n})$ and a sequence $(f_i)_i\subseteq L^{\frac{2n}{n+4}}(M,g)$ satisfies $\|f_i\|_{L^{\frac{2n}{n+4}}(M,g)}=1$ for all i, $\int_M f_iG_Pf_idv_g\to\Theta_4(g)$, then there exists $f\in L^{\frac{2n}{n+4}}(M,g)$ with, up to a subsequence, $f_i\to f$ in $L^{\frac{2n}{n+4}}(M,g)$. Moreover $\|f\|_{L^{\frac{2n}{n+4}}(M,g)}=1$ and f attains $\Theta_4(g)$.

Proof. One has clearly $f_i\rightharpoonup f$ weakly in $L^{\frac{2n}{n+4}}(M,g)$ for some f. Define u_i and u as the unique solutions to $P_gu_i=f_i$ and $P_gu=f$. Then one has the convergences

$$u_i\rightharpoonup u \text{ in } W^{4,\frac{2n}{n+4}}(M,g),\quad u_i\to u \quad\text{in}\quad W^{3,\frac{2n}{n+4}}(M,g),\quad u_i\to u\quad\text{in}\quad W^{1,2}(M,g).$$

The following weak convergences also hold true

$$|f_i|^{\frac{2n}{n+4}}d\mu_g\rightharpoonup d\sigma,\quad (\Delta u_i)^2d\mu_g\rightharpoonup d\nu\quad\text{in}\quad \mathcal{M}(M,g).$$

It is possible to show that

$$\sigma\geq|f|^{\frac{2n}{n+4}}d\mu+\sum_i\sigma(\{p_i\})\delta_{p_i},\quad \nu=(\Delta u)^2d\mu+\sum_i\nu(\{p_i\})\delta_{p_i},$$

with $\nu(\{p_i\})\leq\Theta_4(S^n)\sigma(\{p_i\})^{\frac{n+4}{n}}$. This implies that $\sigma(M)=1$ and that

$$\begin{aligned}&\int_M f_iG_Pf_id\mu=\int_M u_iP_gu_id\mu=E(u_i)\\ :=&\int_M\left((\Delta_gu_i)^2-4A_g(\nabla u_i,\nabla u_i)+(n-2)J_g|\nabla u_i|^2+\frac{n-4}{2}Q_gu_i^2\right)d\mu\to E(u)+\sum_i\nu(\{p_i\}),\end{aligned}$$

where

$$J_g=\frac{R_g}{2(n-1)}.$$

Therefore we obtain

$$\begin{aligned}\Theta_4(g)=&E(u)+\sum_i\nu(\{p_i\})\leq\Theta_4(g)\|f\|^2_{L^{\frac{2n}{n+4}}(M,g)}+\Theta_4(S^n)\sum_i\sigma(\{p_i\})^{\frac{n+4}{n}}\\ \leq&\Theta_4(g)\left[\|f\|^2_{L^{\frac{2n}{n+4}}(M,g)}+\sum_i\sigma(\{p_i\})^{\frac{n+4}{n}}\right]\leq\Theta_4(g)\left(\|f\|^{\frac{2n}{n+4}}_{L^{\frac{2n}{n+4}}(M,g)}+\sum_i\sigma(\{p_i\})\right)^{\frac{n+4}{n}}\\ \leq&\Theta_4(g).\end{aligned}$$

This then implies $\sigma(\{p_i\})=0$ and $\nu(\{p_i\})=0$ for all i and $\|f\|_{L^{\frac{2n}{n+4}}(M,g)}=1$. Therefore we obtain strong convergence of f_i to f in $L^{\frac{2n}{n+4}}$ and $E(u)=\int_M fG_Pfd\mu_g=\Theta_4(g)$, concluding the proof. □

We have now the following result.

Theorem 5.2. *Suppose (M^n,g) is a compact manifold with $n\geq 5$, $Y(g)>0$ and $Q\geq 0$, $Q\not\equiv 0$. Then one has the following properties.*

(a) $\Theta_4(g)\geq\Theta_4(S^n)$, with equality if and only if (M^n,g) is conformally equivalent to the round sphere S^n.

(b) $\Theta_4(g)$ is achieved. Maximizers are smooth and have non-negative sign. If they are positive, they satisfy $G_P f=\frac{2}{n-4}f^{\frac{n-4}{n+4}}$, i.e., $Q_{f^{\frac{4}{n+4}}g}\equiv 1$.

Proof. If (M,g) is conformally equivalent to the round sphere, maximizers are completely classified, see [32]: we can then assume from now on the contrary. By Proposition 5.3 we have that $\Theta_4(g)>\Theta_4(S^n)$ and by Theorem 5.1 we have that $kerP_g=\{0\}$ and that $G_P>0$. By Proposition 5.4 the set of functions f that are normalized in $L^{\frac{2n}{n+4}}(M,g)$ and satisfy $\int_M fG_P f d\mu=\Theta_4(g)$ are non-empty and compact in $L^{\frac{2n}{n+4}}(M,g)$. If f is such a function with non-zero positive part, we claim that its negative part f^- must be identically zero. In fact, it must be

$$\begin{aligned}\Theta_4(g)=&\int_M fG_P f d\mu=\int_M \left(f^+G_P f^+-2f^+G_P f^-+f^-G_P f^-\right)d\mu\\ \leq&\int_M |f|G_P|f|d\mu\leq\Theta_4(g).\end{aligned}$$

This implies that $\int_M f^-G_P f^+d\mu=0$. Since $G_P>0$, this implies that $f^-=0$. It is easy then to show that f is of class C^∞ and that $f>0$. $\square$

It is also possible to prove that if (M^n,g) is not conformally equivalent to the round S^n the set of maximizers for $\Theta_4(g)$, normalized in $L^{\frac{2n}{n+4}}(M,g)$ is compact in any C^k sense.

We next describe some results from [17], in which the sign assumption on the Q-curvature is replaced by a conformally covariant condition. We introduce the following three quantities

$$\begin{aligned}Y_4(g)=&\inf_{u\in W^{2,2}(M,g)\setminus\{0\}}\frac{\int_M uP_g u d\mu_g}{\left(\int_M |u|^{\frac{2n}{n-4}}d\mu_g\right)^{\frac{n-4}{n}}},\\ Y_4^+(g)=&\inf_{u\in C^\infty(M,g),u>0}\frac{\int_M uP_g u d\mu_g}{\left(\int_M |u|^{\frac{2n}{n-4}}d\mu_g\right)^{\frac{n-4}{n}}}=\frac{n-4}{2}\inf_{\tilde g\in[g]}\frac{\int_M Q_{\tilde g}d\mu_{\tilde g}}{Vol_{\tilde g}(M)^{\frac{n-4}{n}}},\\ Y_4^*(g)=&\frac{n-4}{2}\inf_{\tilde g\in[g],R_{\tilde g}>0}\frac{\int_M Q_{\tilde g}d\mu_{\tilde g}}{Vol_{\tilde g}(M)^{\frac{n-4}{n}}}.\end{aligned}$$

We clearly have the inequalities

$$Y_4(g) \le Y_4^+(g) \le Y_4^*(g).$$

The following result was proved in [17].

Theorem 5.3. *Let (M^n,g) be a compact Riemannian manifold with $n \ge 6$, $Y(g) > 0$ and $Y_4^*(g) > 0$. Then there exists a metric $\tilde{g} \in [g]$ such that $R_{\tilde{g}>0}$ and $Q_{\tilde{g}>0}$.*

Here are two consequences of the above result.

Corollary 5.1. Let (M^n,g) be a compact Riemannian manifold with $n \ge 6$. Then the following are equivalent.

i) $Y(g) > 0$ and $P_g > 0$,

ii) $Y(g) > 0$ and $Y_4^*(g) > 0$,

iii) there exists $\tilde{g} \in [g]$ with $R_{\tilde{g}} > 0$ and $Q_{\tilde{g}>0}$.

Corollary 5.2. Let (M^n,g) be a compact Riemannian manifold with $n \ge 6$. If $Y(g) > 0$ and $Y_4^*(g) > 0$ then $P_g > 0$ and G_P is positive. Moreover $Y_4(g)$ is achieved by some positive conformal factor for which the scalar curvature is positive and the Q-curvature is a positive constant. Moreover one has the relations

$$Y_4(g) = Y_4^+(g) = Y_4^*(g).$$

We do not give a complete proof of Theorem 5.3 here for reasons of brevity: we just limit ourselves to describe the main strategy used in [17], which relies on a homotopy argument.

Recall that the Schouten tensor is defined as

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g \right),$$

and its σ_2-*curvature* as

$$\sigma_2(A) = \frac{1}{2}\left(\frac{R_g^2}{4(n-1)^2} - |A|^2 \right).$$

The relation to Q-curvature is expressed via the integral formula, where $J_g = \frac{R_g}{2(n-1)}$

$$\int_M Q_g d\mu_g = \frac{n-4}{2} \int_M J_g^2 d\mu_g + 4 \int_M \sigma_2 d\mu_g.$$

Given $t \ge 1$, the authors in [17] considered the following functional:

$$\frac{n-4}{2} t \int_M J_g^2 d\mu_g + 4 \int_M \sigma_2(A) d\mu_g.$$

Critical points of this functional, restricted to metrics with unit volume, are solutions of the problem:

$$t\left(-\Delta J_g+\frac{n-4}{2}J_g^2\right)+4\sigma_2(A)\equiv constant.$$

It is possible to show that for $t_0\gg 1$ there exists a conformal metric $g_0=u_0^{\frac{4}{n-4}}g\in[g]$ such that

$$t_0\left(-\Delta_{g_0}J_{g_0}+\frac{n-4}{2}J_{g_0}^2\right)+4\sigma_2(A_0)>0,\quad R_{g_0}>0.$$

Define then the function f by

$$t_0\left(-\Delta_{g_0}J_{g_0}+\frac{n-4}{2}J_{g_0}^2\right)+4\sigma_2(A_0)=fu_0^{-\frac{n+4}{n-4}}.$$

Fixing this function f, one can then consider the following problem

$$t\left(-\Delta_{\tilde{g}}J_{\tilde{g}}+\frac{n-4}{2}J_{\tilde{g}}^2\right)+4\sigma_2(A_{\tilde{g}})=fu^{-\frac{n+4}{n-4}},\quad \tilde{g}=u^{\frac{4}{n-4}}g. \tag{5.3}$$

Define then the set

$$S=\left\{t\in[1,t_0]:\ \text{there exists a solution } u \text{ to (5.3) with } \ R_{\tilde{g}}>0\right\}.$$

Clearly $t_0\in S$: it is then proved via an implicit function theorem that S is relatively open in $[1,t_0]$ and via some a-priori estimates that it is also relatively closed, showing that also $1\in S$ and giving the desired conclusion.

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References

[1] Thierry Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl., 55 (1976), 269–296.

[2] Thomas P. Branson, Differential operators canonically associated to a conformal structure, Math. Scand., 57 (1985), 295–345.

[3] Simon Brendle, Global existence and convergence for a higher order flow in conformal geometry, Ann. Math., 158 (2003), 323–343.

[4] Hichem Chtioui, Aymen Bensouf and Mohammed Ali Al-Ghamdi, Q-curvature problem on S^n under flatness condition: the case $\beta=n$, J. Inequal. Appl., (2015), 384.

[5] Yaiza Canzani, Rod Gover, Dmitry Jakobson and Raphaël Ponge, Conformal invariants from nodal sets. I. Negative eigenvalues and curvature prescription, Int. Math. Res. Not. IMRN (2014), 2356–2400, With an appendix by Gover and Andrea Malchiodi.

[6] Hichem Chtioui and Afef Rigane, On the prescribed Q-curvature problem on S^n, J. Funct. Anal., 261 (2011), 2999–3043.

[7] Xuezhang Chen and Xingwang Xu, Q-curvature flow on the standard sphere of even dimension, J. Funct. Anal., 261 (2011), 934–980.

[8] Sun-Yung A. Chang and Paul C. Yang, Extremal metrics of zeta function determinants on 4-manifolds, Ann. Math., 142 (1995), 171–212.

[9] Zindine Djadli, Emmanuel Hebey and Michel Ledoux, Paneitz-type operators and applications, Duke Math. J., 104 (2000), 129–169.

[10] Zindine Djadli and Andrea Malchiodi, Existence of conformal metrics with constant Q-curvature, Ann. Math., 168 (2008), 813–858.

[11] Zindine Djadli, Andrea Malchiodi and Mohameden Ould Ahmedou, Prescribing a fourth order conformal invariant on the standard sphere. I. A perturbation result, Commun. Contemp. Math., 4 (2002), 375–408.

[12] Zindine Djadli, Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications, Ann. Sc. Norm. Super. Pisa Cl. Sci., 1 (2002), 387–434.

[13] Pierpaolo Esposito and Frédéric Robert, Mountain pass critical points for Paneitz-Branson operators, Calc. Var. Partial Differential Equations, 15 (2002), 493–517.

[14] Charles Fefferman and C. Robin Graham, Juhl's formulae for GJMS operators and Q-curvatures, J. Amer. Math. Soc., 26 (2013), 1191–1207.

[15] Ali Fardoun and Rachid Regbaoui, Compactness properties for geometric fourth order elliptic equations with application to the Q-curvature flow, J. Reine Angew. Math., 734 (2018), 229–264.

[16] Sylvestre Gallot, Dominique Hulin and Jacques Lafontaine, Riemannian Geometry, third edition, Universitext, Springer-Verlag, Berlin, 2004.

[17] Matthew J. Gursky, Fengbo Hang and Yueh-Ju Lin, Riemannian manifolds with positive Yamabe invariant and Paneitz operator, Int. Math. Res. Not. IMRN, (2016), 1348–1367.

[18] C. Robin Graham, Ralph Jenne, Lionel J. Mason and George A. J. Sparling, *Conformally invariant powers of the Laplacian. I. Existence*, J. London Math. Soc., 46 (1992), 557–565.

[19] Matthew J. Gursky and Andrea Malchiodi, A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature, J. Euro. Math. Soc. (JEMS), 17 (2015), 2137–2173.

[20] Hans-Christoph Grunau, Mohameden Ould Ahmedou and Wolfgang Reichel, The Paneitz equation in hyperbolic space, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), 847–864.

[21] A. Rod Gover, Q-curvature prescription; forbidden functions and the GJMS null space, Proc. Amer. Math. Soc., 138 (2010), 1453–1459.

[22] Matthew J. Gursky, The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, Commun. Math. Phys., 207 (1999), 131–143.

[23] Ali Hyder and Luca Martinazzi, Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature, prescribed volume and asymptotic behavior, Discrete Contin. Dyn. Syst., 35 (2015), 283–299.

[24] Pak Tung Ho, Prescribed Q-curvature flow on S^n, J. Geom. Phys. 62 (2012), 1233–1261.

[25] Emmanuel Hebey and Frédéric Robert, Compactness and global estimates for the geometric Paneitz equation in high dimensions, Electron. Res. Announc. Amer. Math. Soc., 10 (2004), 135–141.

[26] Emmanuel Humbert and Simon Raulot, Positive mass theorem for the Paneitz-Branson op-

erator, Calc. Var. Partial Differential Equations, 36 (2009), 525–531.

[27] Hang, Fengbo and Yang, Paul C, Q-curvature on a class of manifolds with dimension at least 5, Commun. Pure Appl. Math., 69(8) (2016), 1452–1491.

[28] Hang, Fengbo and Yang, Paul C, Sign of Green's function of Paneitz operators and the Q curvature, Int. Math. Res. Not., 19 (2015), 9775–9791.

[29] Andreas Juhl, Families of Conformally Covariant Differential Operators, Q-Curvature and Holography, Progress in Mathematics, vol. 275, Birkhäuser Verlag, Basel, 2009.

[30] N. H. Kuiper, On conformally-flat spaces in the large, Ann. Math., 50 (1949), 916–924.

[31] Gang Li, Constant Q-curvature metrics near the hyperbolic metric, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), 591–614.

[32] Chang-Shou Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$, Comment. Math. Helv., 73 (1998), 206–231.

[33] Jiayu Li, Yuxiang Li and Pan Liu, The Q-curvature on a 4-dimensional Riemannian manifold (M,g) with $\int_M Q dV_g = 8\pi^2$, Adv. Math. 231 (2012), 2194–2223.

[34] John M. Lee and Thomas H. Parker, The Yamabe problem, Bull. Amer. Math. Soc. (N.S.), 17 (1987), 37–91.

[35] Li, Yanyan and Xiong, Jingang, Compactness of conformal metrics with constant Q-curvature, I. Adv. Math., 345 (2019), 116–160.

[36] Andrea Malchiodi, Compactness of solutions to some geometric fourth-order equations, J. Reine Angew. Math., 594 (2006), 137–174.

[37] Cheikh Birahim Ndiaye, Algebraic topological methods for the supercritical Q-curvature problem, Adv. Math., 277 (2015), 56–99.

[38] Jie Qing and David Raske, On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds, Int. Math. Res. Not., Art. ID 94172, (2006).

[39] Richard Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geom., 20 (1984), 479–495.

[40] R. Schoen and S.-T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math., 92 (1988), 47–71.

[41] Neil S. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa, 22 (1968), 265–274.

[42] R. C. A. M. Van der Vorst, Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$, Differential Integral Equations, 6 (1993), 259–276.

[43] Juncheng Wei and Chunyi Zhao, Non-compactness of the prescribed Q-curvature problem in large dimensions, Calc. Var. Partial Differential Equations, 46 (2013), 123–164.

[44] Hidehiko Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J., 12 (1960), 21–37.