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Global Regularity of 2-D Density Patches for Viscous Inhomogeneous Incompressible Flow with General Density: High Regularity Case

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Abstract. This paper is a continuation work of [26] and studies the propagation of the high-order boundary regularities of the two-dimensional density patch for viscous inhomogeneous incompressible flow. We assume the initial density $\rho_0=\eta_1\mathbf{1}_{\Omega_0}+\eta_2\mathbf{1}_{\Omega_0^c}$, where (η_1,η_2) is any pair of positive constants and Ω_0 is a bounded, simply connected domain with $W^{k+2,p}(\mathbb{R}^2)$ boundary regularity. We prove that for any positive time t, the density function $\rho(t)=\eta_1\mathbf{1}_{\Omega(t)}+\eta_2\mathbf{1}_{\Omega(t)^c}$, and the domain $\Omega(t)$ preserves the $W^{k+2,p}$-boundary regularity.

Key Words: Inhomogeneous incompressible Navier-Stokes equations, density patch, striated distributions, Littlewood-Paley theory.

AMS Subject Classifications: 35Q30, 76D03

1 Introduction

We consider the two-dimensional density-dependent incompressible Navier-Stokes system:

$$\begin{cases} \partial_t\rho+\mathrm{div}(\rho v)=0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \partial_t(\rho v)+\mathrm{div}(\rho v\otimes v)-\Delta v+\nabla\pi=0, & \\ \mathrm{div}\, v=0, & \\ (\rho,v)|_{t=0}=(\rho_0,v_0). & \end{cases} \tag{1.1}$$

Here the unknowns $(\rho,v)\in\mathbb{R}^+\times\mathbb{R}^2$ represent the density and the velocity field of the two-dimensional fluid at time t and point x respectively, and π designates the unknown pressure which ensures the incompressibility of the fluid.

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This system (1.1) can describe the dynamics of a viscous fluid which is incompressible but with variable density, e.g., mixtures of incompressible and non-reactant components, fluids containing a melted substance. In the simple case when $\rho_0\equiv 1$, the system (1.1) reduces to the classical incompressible Navier-Stokes system:

$$\partial_t v+\mathrm{div}(v\otimes v)-\Delta v+\nabla\pi=0,\quad \mathrm{div}v=0,\quad v|_{t=0}=v_0.$$

There is a substantial amount of literature devoted to the study of the well-posedness issue of the system (1.1), e.g., [28,32] in the weak solution framework, [1–3,12,24,29] in the strong solution framework. If the density function has a jump across some hypersurface which is of interest in this paper, [13,14,16,22,30] have also established some well-posedness results (see [26] for more detailed introduction to these references).

We are interested in the propagation of regularities of the interface between fluids with different densities, for which we take the assumptions as follows. Let Ω_0 be a simply connected bounded domain with $W^{k+2,p}(\mathbb{R}^2)$-boundary regularity, $k\ge 1$, $p\in(2,4)$, that is, we can parametrize $\partial\Omega_0$ as

$$\begin{aligned}&\gamma:\mathbb{S}^1\mapsto\partial\Omega_0 \text{ via } s\mapsto\gamma(s) \quad\text{with}\quad \gamma\in(W^{k+2,p}(\mathbb{S}^1))^2\\ &\text{and}\quad \partial_s\gamma(s)=X_0(\gamma(s)),\quad s\in\mathbb{S}^1.\end{aligned}\tag{1.2}$$

Here $X_0(\cdot)\in\mathbb{R}^2$ is a vector field defined on $\mathbb{R}^2$ which is tangential to $\partial\Omega_0$: if $\partial\Omega_0=f_0^{-1}(0)$ is the level set then $X_0=(-\partial_2,\partial_1)^T f_0$. We denote by $\partial_{X_0}u=X_0\cdot\nabla u$, the directional derivative of u along X_0. Then we easily calculate

$$\partial_s^2\gamma(s)=\partial_s(X_0(\gamma(s)))=X_0(\gamma(s))\cdot\nabla X_0(\gamma(s))=(\partial_{X_0}X_0)(\gamma(s)),$$

and repeating this calculation gives $\partial_s^\ell\gamma(s)=(\partial_{X_0}^{\ell-1}X_0)(\gamma(s))$. Hence the boundary regularity assumption is equivalent to the following assumption on X_0:

$$\partial_{X_0}^{\ell-1}X_0\in(W^{2,p}(\mathbb{R}^2))^2,\quad \ell=1,\cdots,k,\quad \mathrm{div}\,X_0=0.\tag{1.3}$$

For any $\eta_1,\eta_2>0$, we take the initial density ρ_0 and the initial velocity v_0 as

$$\rho_0=\eta_1\mathbf{1}_{\Omega_0}+\eta_2\mathbf{1}_{\Omega_0^c},\quad v_0\in(L^2\cap\dot B^{s_0}_{2,1}(\mathbb{R}^2))^2\quad\text{and}\quad \partial^\ell_{X_0}v_0\in(L^2\cap\dot B^{s_\ell}_{2,1}(\mathbb{R}^2))^2,\tag{1.4}$$

for some

$$s_0\in(0,1),\quad s_\ell=s_0-\theta_0\ell \text{ with some fixed } \theta_0\in(0,s_0/k),\quad \ell=1,\cdots,k,\quad p\in(2,2/(1-s_k)).$$

Here we have taken $v_0\in(L^2(\mathbb{R}^2))^2$ with finite $(\dot B^{s_0}_{2,1}(\mathbb{R}^2))^2$-Besov norm defined as follows (see e.g., [5]):

Definition 1.1. Consider a smooth radial function φ on $\mathbb{R}$, supported in $[3/4,8/3]$ such that $\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1$ for any $\tau>0$. We denote

$$\Delta_j a=\mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}(\xi)),\quad j\in\mathbb{Z}.$$