Analysis in Theory and Applications
Anal. Theory Appl., Vol. **35**, No. 2 (2019), pp. 192-204

DOI: 10.4208/ata.OA-0005

A Differential Harnack Inequality for the Newell-Whitehead-Segel Equation

Derek Booth[1], Jack Burkart[2], Xiaodong Cao[3,*], Max Hallgren[3], Zachary Munro[4], Jason Snyder[5] and Tom Stone[6]

[1] *Department of Mathematics, Harvard University, Cambridge, MA 02138, USA*
[2] *Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA*
[3] *Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA*
[4] *Department of Mathematics and Statistics, McGill University, Montreal, Canada*
[5] *Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095, USA*
[6] *Department of Mathematics, Brown Univeristy, Providence, RI 02912, USA*

Received 14 September 2017; Accepted (in revised version) 24 September 2018

Abstract. This paper will develop a Li-Yau-Hamilton type differential Harnack estimate for positive solutions to the Newell-Whitehead-Segel equation on $\mathbb{R}^n$. We then use our LYH-differential Harnack inequality to prove several properties about positive solutions to the equation, including deriving a classical Harnack inequality and characterizing standing solutions and traveling wave solutions.

Key Words: Newell-Whitehead-Segel equation, Harnack estimate, Harnack inequality, wave solutions.

AMS Subject Classifications: 35C07, 35K10, 35K55

1 Introduction

Consider any positive solution $f:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ to the Newell-Whitehead-Segel Equation,

$$f_t=\Delta f+af-bf^3, \tag{1.1}$$

here, we assume $a>0$, $b>0$. This equation was first introduced by A. C. Newell and J. A. Whitehead in 1969 [6] and shortly after was studied by L. Segel [9]. Exact solutions to the equation were computed using the Homotopy Perturbation method by S.

*Corresponding author. *Email addresses:* `derekbooth@college.harvard.edu` (D. Booth), `jack.burkart@stonybrook.edu` (J. Burkart), `cao@math.cornell.edu` (X. Cao), `meh249@cornell.edu` (M. Hallgren), `zachary.munro@mail.mcgill.ca` (Z. Munro), `snyder@math.ucla.edu` (J. Snyder), `tds@math.brown.edu` (T. Stone)

Nourazar, M. Soori and A. Nazari-Golshan in 2011 [8], while some approximate solutions were computed in 2015 by J. Patade and S. Bhalekar [7]. The equation is an example of a reaction-diffusion equation, as it is used to model the change of concentration of a substance, given any chemical reactions that the substance may be undergoing (modeled by the $af-bf^3$ term) and any diffusion causing the chemical to spread throughout the medium (modeled by the Δf term). More specifically, the Newell-Whitehead-Segel equation models Rayleigh-Bénard convection, a reaction-diffusion phenomenon that occurs when a fluid is heated from below.

In this paper, we are just concerned with positive solutions on $\mathbb{R}^n$. For further discussion about working with functions on closed manifolds or complete non-compact manifolds, see [3]. Our main theorem, Theorem 1.1, will outline a Li-Yau-Hamilton type differential Harnack estimate (2) that we will prove based on computing time-evolutions of the relevant quantities, see Hamilton [4]. In the following, Harnack inequality or Harnack estimate refers to an LYH-type differential Harnack inequality. As an application, we will integrate our estimate (2) along a space time curve to obtain a classical Harnack inequality (16), see Corollary 4.1. Then we will use our Harnack estimate to characterize both traveling wave solutions and standing solutions to the Newell-Whitehead-Segel equation.

Theorem 1.1. *With $f>0$ a solution to* (1.1)*, define $l=\log f$. Then:*

$$H=\alpha\Delta l+\beta|\nabla l|^2+\gamma e^{2l}+\varphi(t)\geq 0, \tag{1.2}$$

provided the following three inequalities hold:

$$\begin{aligned}&(a)\ \ \alpha>\beta\geq 0,\\&(b)\ \ \gamma\leq\frac{-nb\alpha^2(2\alpha+\beta)}{3n\alpha^2-2(\alpha-\beta)\beta}<0,\\&(c)\ \ 4\gamma(\alpha-\beta)+n\alpha^2 b<0,\end{aligned}$$

with

$$\varphi(t)=\left(\frac{a\alpha}{1-e^{2at}}\right)\left(\frac{\gamma}{\alpha b}e^{2at}-\frac{\alpha\gamma n}{4\gamma(\alpha-\beta)+\alpha^2 bn}\right).$$

If, instead of inequality (c), we have:

$$(d)\ \ 4\gamma(\alpha-\beta)+n\alpha^2 b\geq 0,$$

then:

$$H=\alpha\Delta l+\beta|\nabla l|^2+\gamma e^{2l}+\psi(t)\geq 0, \tag{1.3}$$

for:

$$\psi(t)=\begin{cases}\dfrac{n\alpha^2}{2(\alpha-\beta)t}, & t\leq T:=\dfrac{n\alpha^2}{2(\alpha-\beta)(-a\gamma)}\left(2\left(\dfrac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right),\\[2ex] \dfrac{-an\alpha^2\gamma\left(e^{2a(t-T)}+1\right)}{n\alpha^2 b\left(e^{2a(t-T)}+1\right)+4\gamma(\alpha-\beta)}, & t>T.\end{cases}$$

Remark 1.1. Condition (a) of Theorem 1.1 says that we are allowed to choose $\beta=0$. While our proof of this theorem will require $\beta>0$, we can take $\beta\to 0$ at the end.

Remark 1.2. The quantity H defined in (2) and (3) is referred to as a (LYH-differential) Harnack quantity.

Remark 1.3. Inequalities (2) and (3) are called differential Harnack inequalities because they involve derivatives of f and integration along space-time paths leads to a comparison of the function f at different points in space and time.

2 Li-Yau-Hamilton type differential Harnack inequality

We begin by calculating the evolution of the Harnack quantity H. For notational convenience, we introduce the box operator $\Box g(x,t):=g_t-\Delta g$. Our first lemma is the following:

Lemma 2.1. *With H defined as in* (1.2)*, we have:*

$$\begin{aligned}\Box H=&2\nabla l\cdot\nabla H+2(\alpha-\beta)|\nabla\nabla l|^2+\varphi_t-\Delta\varphi-2\nabla l\cdot\nabla\varphi\\&-2be^{2l}\left[(H-\varphi)+2\alpha|\nabla l|^2+\beta|\nabla l|^2-\gamma\frac{a}{b}+3\frac{\gamma}{b}|\nabla l|^2\right].\end{aligned}\tag{2.1}$$

Proof. We begin by calculating the evolution quantities of the components of H.

$$\Box(\Delta l)=\Delta|\nabla l|^2-2b\Delta le^{2l}-4b|\nabla l|^2e^{2l},\tag{2.2a}$$
$$\Box(|\nabla l|^2)=2\nabla l\cdot\nabla(\Delta l)+2\nabla l\cdot\nabla(|\nabla l|^2)-4b|\nabla l|^2e^{2l}-\Delta(|\nabla l|^2),\tag{2.2b}$$
$$\Box(e^{2l})=2\nabla l\cdot\nabla(e^{2l})+2ae^{2l}-2be^{4l}-6|\nabla l|^2e^{2l}.\tag{2.2c}$$

So, using these in the evolution equation for H:

$$\begin{aligned}\Box H=&\alpha\Box(\Delta l)+\beta\Box(|\nabla l|^2)+\gamma\Box(e^{2l})+\Box\varphi\\=&\alpha\Big(\Delta|\nabla l|^2-2b\Delta le^{2l}-4b|\nabla l|^2e^{2l}\Big)\\&+\beta\Big(2\nabla l\cdot\nabla(\Delta l)+2\nabla l\cdot\nabla(|\nabla l|^2)-4b|\nabla l|^2e^{2l}-\Delta(|\nabla l|^2)\Big)\\&+\gamma\Big(2\nabla l\cdot\nabla(e^{2l})+2ae^{2l}-2be^{4l}-6|\nabla l|^2e^{2l}\Big)+\varphi_t-\Delta\varphi.\end{aligned}$$

Now, by using the Weitzenbock-Bochner formula for $\mathbb{R}^n$:

$$\Delta(|\nabla l|^2)=2\nabla l\cdot\nabla(\Delta l)+2|\nabla\nabla l|^2\tag{2.3}$$

the expression can be simplified and the lemma follows. $\Box$

Using the Cauchy-Schwarz inequality, we can show that the Harnack quantity satisfies the following inequality.

Lemma 2.2. *The following inequality holds:*

$$\begin{aligned}\Box H \geq & 2\nabla l\cdot\nabla H + H\left[2\left(\frac{\alpha-\beta}{n\alpha^2}\right)(H-2\beta|\nabla l|^2-2\gamma e^{2l}-2\varphi)-2be^{2l}\right]\\ &+\varphi_t-\Delta\varphi-2\nabla l\cdot\nabla\varphi+2|\nabla l|^2e^{2l}\left[2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\gamma-2\alpha b-\beta b-3\gamma\right]\\ &+|\nabla l|^2\varphi\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\right]+e^{2l}\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma\varphi+2b\varphi+2a\gamma\right]\\ &+2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\beta^2|\nabla l|^4+\gamma^2e^{4l}+\varphi^2\right]. \end{aligned}\tag{2.4}$$

Proof. We will achieve our result by applying the Cauchy-Schwarz inequality in the form of $|\nabla\nabla l|^2\geq\frac{1}{n}(\Delta l)^2$ and also by substituting $\Delta l=\frac{1}{\alpha}(H-\beta|\nabla l|^2-\gamma e^{2l}-\varphi)$. Upon doing this, we receive the following:

$$\begin{aligned}\Box H \geq & 2\nabla l\cdot\nabla H+2\frac{(\alpha-\beta)}{n\alpha^2}(H-\beta|\nabla l|^2-\gamma e^{2l}-\varphi)^2+\varphi_t-\Delta\varphi-2\nabla l\cdot\nabla\varphi\\ &-2be^{2l}\left[(H-\varphi)+2\alpha|\nabla l|^2+\beta|\nabla l|^2-\gamma\frac{a}{b}+3\frac{\gamma}{b}|\nabla l|^2\right]\\ =& 2\nabla l\cdot\nabla H+H\left[2\left(\frac{\alpha-\beta}{n\alpha^2}\right)(H-2\beta|\nabla l|^2-2\gamma e^{2l}-2\varphi)-2be^{2l}\right]\\ &+2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\beta^2|\nabla l|^4+\gamma^2e^{4l}+\varphi^2\right]+2|\nabla l|^2e^{2l}\left[2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\gamma-2\alpha b-\beta b-3\gamma\right]\\ &+|\nabla l|^2\varphi\left[4\frac{\alpha-\beta}{n\alpha^2}\beta\right]+e^{2l}\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma\varphi+2b\varphi+2a\gamma\right]+\varphi_t-\Delta\varphi-2\nabla l\cdot\nabla\varphi.\end{aligned}$$

This yields the desired inequality. □

3 Proof of the main theorem

We now proceed to prove our main theorem. We apply the parabolic maximum principle by assuming for the sake of contradiction that there exists a first point (z,t_0), $t_0\neq 0$ at which $H(z,t_0)=0$, which we show must occur in some compact region away from the origin. At such a first time, the time derivative $H_t\leq 0$, the Laplacian $\Delta H\geq 0$ and the gradient $\nabla H=0$ (vector). Our method of proof will be working with the time evolution of the right hand quantities to construct a contradiction of the form $0\geq H_t(z,t_0)\geq A(z,t_0)>0$ for some quantity A. As a consequence of this contradiction, the quantity H must be nonnegative for all space and time.

Assume we are at the first point (z,t_0) where $H=0$, at which ∇H is the 0 vector. Therefore, by simplifying (2.2c), we have:

$$\begin{aligned}\Box H \geq & 2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\beta^2|\nabla l|^4\right]+2|\nabla l|^2 e^{2l}\left[2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\gamma-2\alpha b-\beta b-3\gamma\right]\\ &+2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\gamma^2 e^{4l}+\varphi^2\right]+|\nabla l|^2\varphi\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\right]\\ &+e^{2l}\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma\varphi+2b\varphi+2a\gamma\right]+\varphi_t-\Delta\varphi-2\nabla l\cdot\nabla\varphi.\end{aligned}$$

From conditions (a) and (b), the first two terms are both nonnegative. Thus:

$$\begin{aligned}\Box H \geq & 2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\gamma^2 e^{4l}+\varphi^2\right]+\varphi_t-\Delta\varphi-2\nabla l\cdot\nabla\varphi\\ &+|\nabla l|^2\varphi\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\right]+e^{2l}\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma\varphi+2b\varphi+2a\gamma\right].\end{aligned}\tag{3.1}$$

Via application of the Cauchy-Schwarz inequality $a^2-2ab\geq -b^2$, we get:

$$|\nabla l|^2\varphi\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\beta\right]-2\nabla l\cdot\nabla\varphi\geq-\frac{n\alpha^2|\nabla\varphi|^2}{4\beta(\alpha-\beta)\varphi}.$$

Hence,

$$\Box H\geq 2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\gamma^2 e^{4l}+\varphi^2\right]+\varphi_t-\Delta\varphi-\frac{n\alpha^2|\nabla\varphi|^2}{4\beta(\alpha-\beta)\varphi}+e^{2l}\left[4\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma\varphi+2b\varphi+2a\gamma\right].$$

We will now use Cauchy-Schwarz again:

$$\begin{aligned}&2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma^2 e^{4l}+2e^{2l}\left[2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma\varphi+b\varphi+a\gamma\right]\\ \geq&-\frac{n\alpha^2}{2(\alpha-\beta)\gamma^2}\left[\left(2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right)\varphi+a\gamma\right]^2.\end{aligned}$$

Therefore, we arrive at the following:

$$\begin{aligned}\Box H\geq & 2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\varphi^2+\varphi_t-\Delta\varphi\\ &-\frac{n\alpha^2|\nabla\varphi|^2}{4\beta(\alpha-\beta)\varphi}-\frac{n\alpha^2}{2(\alpha-\beta)\gamma^2}\left[\left(2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right)\varphi+a\gamma\right]^2.\end{aligned}$$

To simply our notation for our differential equation, we define the following constants:

$$\omega=\frac{1}{\alpha}\sqrt{\frac{2(\alpha-\beta)}{n}},$$
$$\mu=a\alpha\sqrt{\frac{n}{2(\alpha-\beta)}},$$
$$\nu=\frac{1}{\alpha}\sqrt{\frac{2(\alpha-\beta)}{n}}+\frac{\alpha b}{\gamma}\sqrt{\frac{n}{2(\alpha-\beta)}}.$$

Then, the above inequality becomes

$$\square H\geq(\omega\varphi)^2-(\mu+\nu\varphi)^2+\varphi_t-\left(\Delta\varphi+\frac{1}{2\beta\omega^2}\frac{|\nabla\varphi|^2}{\varphi}\right). \tag{3.2}$$

In the same fashion as [2], we can define our test function to have a spatially-dependent portion that is the sum of rational functions of the form:

$$\sum_{k=1}^{n}\left(\frac{c}{(x_k-p_k)^2}+\frac{c}{(q_k-x_k)^2}\right).$$

In doing so, we accomplish the dual task of causing this differential term to be 0 by choosing an appropriate c as well as ensuring that the test function blows up towards positive infinity at the boundary of the n-rectangle $R=\prod[p_i,q_i]$. Therefore, we can ensure that any point at which $H(x,t)=0$ occurs in a spatially compact region. Then, we can take each $p_k\to-\infty$ and $q_k\to\infty$ to retrieve only the time-dependent part in the limiting case. Therefore, we can choose our function φ to be time dependent only and focus on solving the following differential inequality:

$$(\omega\varphi)^2-(\mu+\nu\varphi)^2+\varphi_t>0. \tag{3.3}$$

In order to solve this differential equation, we first assume that inequality (c) holds. Then, we show that $\varphi(t)$ is a valid solution to this differential equation which possesses the properties we desire. For:

$$\varphi(t)=\left(\frac{a\alpha}{1-e^{2at}}\right)\left(\frac{\gamma}{\alpha b}e^{2at}-\frac{\alpha\gamma n}{4\gamma(\alpha-\beta)+\alpha^2bn}\right)=\frac{\mu}{1-e^{2\mu\omega t}}\left(\frac{1}{\nu-\omega}e^{2\mu\omega t}-\frac{1}{\nu+\omega}\right).$$

By Lemma 4 of [3], we know that in the constant form this is a valid solution to this differential equation. The only other behavior we desire is that $\varphi(t)>0$ for all time and that $\varphi(t)$ diverges towards positive infinity as $t\to0$, so that we can ensure that $H(x,t)$ starts off positive and therefore its first zero must be a negative time derivative.

For any $t>0$, we have:

$$\operatorname{sign}(\varphi(t))=\operatorname{sign}\left[\left(\frac{a\alpha}{1-e^{2at}}\right)\right]\operatorname{sign}\left[\left(\frac{\gamma}{\alpha b}e^{2at}-\frac{\alpha\gamma n}{4\gamma(\alpha-\beta)+\alpha^2bn}\right)\right].$$

The sign of the first term is certainly negative, as both $\alpha>0$ and $a>0$. Furthermore, by application of inequality (c), we see that the second term is negative at time $t=0$ and since $\gamma<0$, for any $t>0$ this term is also negative. Thus, the overall sign is positive and $\varphi(t)>0\forall t$.

We can observe further that the limit behavior of the function as $t\to 0$ can be broken down into two terms as well. Thus, we observe:

$$\lim_{t\to 0}\left(\frac{a\alpha}{1-e^{2at}}\right)=-\infty.$$

Similarly, by applying inequality (c) we can observe:

$$\lim_{t\to 0}\left(\frac{\gamma}{\alpha b}e^{2at}-\frac{\alpha\gamma n}{4\gamma(\alpha-\beta)+\alpha^2 bn}\right)=\left(\frac{\gamma}{\alpha b}-\frac{\alpha\gamma n}{4\gamma(\alpha-\beta)+\alpha^2 bn}\right)<0.$$

Therefore, the limit of the entire function $\varphi(t)$ as $t\to 0$ must be positive infinity and the function exhibits the behavior we desire. Thus, we have the contradiction

$$0\geq\Box H\geq(\omega\varphi)^2-(\mu+\nu\varphi)^2+\varphi_t>0.$$

This proves our theorem in the case that inequalities (a), (b) and (c) hold.

Now, we assume that inequality (c) does not hold. In this case, we refer back to (3.2):

$$\begin{aligned}\Box H\geq&2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\left[\psi^2\right]+\psi_t-\Delta\psi-\frac{n\alpha^2|\nabla\psi|^2}{4\beta(\alpha-\beta)\psi}\\&+2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma^2 e^{4l}+2e^{2l}\left[\left(2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right)\psi+a\gamma\right].\end{aligned}$$

If (c) does not hold, that means $2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b>0$ and thus for a sufficiently well-chosen $\psi(t)$ and for small t, we can ensure that both of the last two terms are positive and therefore ignore them both in our calculations. So, we choose:

$$\psi_1(t)=\frac{n\alpha^2}{2(\alpha-\beta)t},\quad t\leq\frac{n\alpha^2}{2(\alpha-\beta)(-a\gamma)}\left(2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right)=T.$$

If this is the case, we claim that for any $t\leq T$, the last two terms are both non-negative. Since $\psi(t)$ is a decreasing function, it suffices to check at $t=T$:

$$\begin{aligned}&2e^{2l}\left[\left(2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right)\psi(T)+a\gamma\right]\\=&2e^{2l}\left[\left(\frac{n\alpha^2}{2(\alpha-\beta)}\right)\left(\frac{2(\alpha-\beta)(-a\gamma)}{n\alpha^2}\right)+a\gamma\right]\\=&2e^{2l}\left[-a\gamma+a\gamma\right]=0.\end{aligned}$$

Thus, we can ignore these last two terms, as well as ignoring the spatial terms once again and solve the ordinary differential equation:

$$2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\psi^2+\psi_t>0, \tag{3.4}$$

whose solution is given by $\psi(t)$ as desired. This function is also positive for all time and approaches positive infinity as $t\to 0$.

In the case that $t>T$, we cannot ignore the last two terms and we must carry out the Cauchy-Schwarz approximation as was done in the last case, from which we obtain:

$$(\omega\psi)^2-(\mu+\nu\psi)^2+\psi_t>0 \tag{3.5}$$

with the same constants ν,μ,ω as defined earlier. However, we must solve this time to be continuous and differentiable with $\psi(t)$ at $t=T$. Thus, we get:

$$\psi_2(t)=\frac{-\mu(1+e^{2\mu\omega(t-T)})}{(\nu-\omega)e^{2\mu\omega(t-T)}+(\nu+\omega)}=an\alpha^2\left(\frac{-\gamma\left(e^{2a(t-T)}+1\right)}{n\alpha^2b\left(e^{2a(t-T)}+1\right)+4\gamma(\alpha-\beta)}\right).$$

This function is positive for all time $t>T$, as the numerator is positive since $\gamma<0$ and the denominator is positive because inequality (c) does not hold. Furthermore:

$$\begin{aligned}
\psi_1(T)&=-a\gamma\left(\frac{1}{2\left(\frac{\alpha-\beta}{n\alpha^2}\right)\gamma+b}\right),\\
\psi_2(T)&=\frac{-2an\alpha^2\gamma}{2n\alpha^2b+4\gamma(\alpha-\beta)}=\frac{-a\gamma}{2\left(\frac{(\alpha-\beta)}{n\alpha^2}\right)\gamma+b},\\
\psi_1'(T)&=\frac{-n\alpha^2}{2(\alpha-\beta)T^2}=\frac{-2n(\alpha-\beta)a^2\alpha^2\gamma^2}{(2(\alpha-\beta)\gamma+bn\alpha^2)^2},\\
\psi_2'(T)&=an\alpha^2\left(\frac{-\gamma\left(e^{2a(T-T)}+1\right)}{n\alpha^2b\left(e^{2a(T-T)}+1\right)+4\gamma(\alpha-\beta)}\right)=\frac{-2n(\alpha-\beta)a^2\alpha^2\gamma^2}{(n\alpha^2b+2\gamma(\alpha-\beta))^2}.
\end{aligned}$$

Thus, $\psi(t)$ is continuous, differentiable and positive everywhere and we have the contradiction

$$0\geq\Box H\geq(\omega\psi)^2-(\mu+\nu\psi)^2+\psi_t>0.$$

This proves our theorem in the case that inequalities (a), (b) and (d) hold.

Remark 3.1. $\psi(t)$ turns out to be exactly twice differentiable.

Remark 3.2. It is worth noting that

$$\lim_{t\to\infty}\varphi(t)=\lim_{t\to\infty}\psi(t)=\frac{a}{b}|\gamma|=-\frac{a}{b}\gamma.$$

When estimating quantities using the Harnack, it is often useful to consider just the limiting case $t\to\infty$, allowing us to replace all occurrences of $\phi(t)$ and $\psi(t)$ with $-\frac{a}{b}\gamma$.

Remark 3.3. There are situations in which we can obtain a simpler Harnack by choosing specific values of α, β, or γ. If we choose $\gamma=-2nb$, we get that

$$H=\alpha\Delta l+\beta|\nabla l|^2-2nbe^{2l}+\frac{n\alpha^2}{2(\alpha-\beta)t}\geq 0.$$

4 Applications

In this section we give several applications of our differential Harnack estimate. First, we integrate our Harnack along a space-time curve to derive a classical Harnack inequality. Then, we characterize traveling wave solutions and standing solutions to the Newell-Whitehead-Segel equation.

4.1 Classical Harnack

Here we use our differential Harnack estimate to prove a classical Harnack inequality, comparing values of a positive solutions at different points.

Corollary 4.1. Let f be a positive solution to (1.1). Pick two points $(x_1,t_1),(x_2,t_2)\in\mathbb{R}^n\times[0,\infty)$ with $0<t_1<t_2$. Then we have

$$\frac{f(x_2,t_2)}{f(x_1,t_1)}\geq\exp\left\{\frac{-(x_2-x_1)^2}{4(t_2-t_1)}\right\}\cdot\exp\left\{a\left(1+\frac{n}{3}\right)(t_2-t_1)\right\}\cdot\left(\frac{1-e^{2at_1}}{1-e^{2at_2}}\right)^{2n/3}. \tag{4.1}$$

Proof. Let Γ be any space-time curve connecting (x_1,t_1) and (x_2,t_2) and define $l=\log f$ as before. Then we have

$$l(x_2,t_2)-l(x_1,t_1)=\int_\Gamma\left[l_t+\nabla l\cdot\frac{dx}{dt}\right]dt.$$

Using the fact that $l_t=\Delta l+|\nabla l|^2+a-be^{2l}$, we get

$$l(x_2,t_2)-l(x_1,t_1)=\int_\Gamma\left[\Delta l+|\nabla l|^2+a-be^{2l}+\nabla l\cdot\frac{dx}{dt}\right]dt.$$

By the choice of our α, β and γ (see below), it follows from our differential Harnack estimate that

$$\Delta l\geq\frac{-\beta}{\alpha}|\nabla l|^2+\frac{-\gamma}{\alpha}e^{2l}+\frac{-1}{\alpha}\varphi(t).$$

Thus, we get

$$l(x_2,t_2)-l(x_1,t_1)\geq\int_\Gamma\left[|\nabla l|^2\left(1-\frac{\beta}{\alpha}\right)-e^{2l}\left(b+\frac{\gamma}{\alpha}\right)+a-\frac{\varphi}{\alpha}+\nabla l\cdot\frac{dx}{dt}\right]dt.$$

Applying the Cauchy-Schwarz Inequality $a^2+2ab\geq -b^2$ to the ∇l terms, we see that

$$|\nabla l|^2\left(1-\frac{\beta}{\alpha}\right)+\nabla l\cdot\frac{dx}{dt}\geq\frac{-1}{4}\left(\frac{\alpha}{\alpha-\beta}\right)\left(\frac{dx}{dt}\right)^2.$$

Thus

$$l(x_2,t_2)-l(x_1,t_1)\geq\int_\Gamma\left[-\frac{1}{4}\left(\frac{\alpha}{\alpha-\beta}\right)\left(\frac{dx}{dt}\right)^2-e^{2l}\left(b+\frac{\gamma}{\alpha}\right)+a-\frac{\varphi}{\alpha}\right]dt.$$

At this point, we may choose $\beta=0$ and $\gamma=-nb\alpha$, which implies $b+\frac{\gamma}{\alpha}\leq 0$ and $4\gamma(\alpha-\beta)+n\alpha^2\beta=-3n\alpha^2 b<0$, thus we can simplify the above inequality to

$$l(x_2,t_2)-l(x_1,t_1)\geq\int_\Gamma\left[-\frac{1}{4}\left(\frac{dx}{dt}\right)^2+a-\frac{\varphi}{\alpha}\right]dt. \tag{4.2}$$

Because Γ is any space-time curve connecting (x_1,t_1) and (x_2,t_2), we can take the infimum over all such space-time paths to get

$$\int_\Gamma\left(\frac{dx}{dt}\right)^2 dt=\frac{(x_2-x_1)^2}{t_2-t_1},$$

and

$$\int_\Gamma\frac{\varphi}{\alpha}dt=\int_{t_1}^{t_2}\left(\frac{an}{e^{2at}-1}\right)\left(e^{2at}+\frac{1}{3}\right)dt=\frac{1}{3}an(t_1-t_2)+\frac{2}{3}n\log\frac{1-e^{2at_2}}{1-e^{2at_1}}.$$

Thus we get:

$$l(x_2,t_2)-l(x_1,t_1)\geq\frac{-1}{4}\frac{(x_2-x_1)^2}{t_2-t_1}+\left(a+\frac{1}{3}an\right)(t_2-t_1)+\frac{2}{3}n\log\frac{1-e^{2at_1}}{1-e^{2at_2}}.$$

Exponentiate both sides to arrive at Corollary 4.1. □

4.2 Traveling wave solutions

We call f a traveling wave solution of (1.1) if it is of the form

$$f(x,t)=f(x_1,x_2,\cdots,x_n,t)=v(x_1,x_2,\cdots,x_n+\eta t),$$

for some function $v:\mathbb{R}^n\to\mathbb{R}$ (see [1]). Traveling wave solutions to the Newell-Whitehead-Segel equation are used to model traveling wave convection in binary fluids and other forms of oscillatory instability (see [5]). We use our differential Harnack to derive a lower bound for η, the wavespeed, of a positive traveling wave solution.

Corollary 4.2. Let $f(x,t)=v(x_1,x_2,\ldots,x_n+\eta t)$ be a positive traveling wave solution. Suppose that $v(z)\to 0$ for some z such that $|z|\to\infty$. Then we have

$$\eta^2\geq\frac{4}{3}a.$$

Proof. We start by rewriting our Harnack quantity so that it is in terms of f, instead of l. From our original estimate, we have

$$\alpha\Delta l+\beta|\nabla l|^2+\gamma e^{2l}+\varphi(t)\geq 0.$$

Recalling that $l=\log f$ and $\Delta f=f_t-af+bf^3$, we get that

$$\alpha\frac{f_t}{f}-\alpha a+(\beta-\alpha)\frac{|\nabla f|^2}{f^2}+(\gamma+\alpha b)f^2+\varphi(t)\geq 0. \tag{4.3}$$

This is our revised Harnack estimate. In the case that $f(x,t)=v(x_1,x_2,\cdots,x_n+\eta t)$ is a traveling wave solution, we get that

$$\alpha\eta\frac{v_{x_n}}{v}-\alpha a+\alpha bv^2+(\beta-\alpha)\frac{|\nabla v|^2}{v^2}+\gamma v^2+\varphi(t)\geq 0.$$

Notice that $|v_{x_n}|\leq|\nabla v|$. Applying Cauchy-Schwarz again then yields that

$$(\alpha-\beta)\frac{|\nabla v|^2}{v^2}-\alpha\eta\frac{|\nabla v|}{v}\geq-\frac{(\alpha\eta)^2}{4(\alpha-\beta)}.$$

Thus our inequality becomes

$$\frac{(\alpha\eta)^2}{4(\alpha-\beta)}\geq(a\alpha-\varphi)-(b\alpha+\gamma)v^2.$$

Because this inequality holds for any t, we can simplify by considering just the limiting cases, where $v\to 0$ and , by Remark 3.2,

$$\lim_{t\to\infty}\varphi(t)=\lim_{t\to\infty}\psi(t)=-\frac{a}{b}\gamma.$$

Rearranging then gives us our bound on η:

$$\eta^2\geq a\left(\alpha+\frac{\gamma}{b}\right)\frac{4(\alpha-\beta)}{\alpha^2}.$$

To maximize the right hand side, we choose $\beta=0$ and $\gamma=-\frac{2}{3}b\alpha$, giving us Corollary 4.2. $\square$

We now use our differential Harnack estimate to prove a gradient estimate for the traveling wave solutions to the Newell-Whitehead-Segel equation.

Corollary 4.3. Let $f(x,t)=v(x_1,x_2,\ldots,x_n+\eta t)$ be a positive traveling wave solution. Then we have

$$|\nabla v|\leq v\eta.$$

Proof. We start with (4.3), our Harnack in terms of f:

$$\alpha\frac{f_t}{f}-\alpha a+(\beta-\alpha)\frac{|\nabla f|^2}{f^2}+(\gamma+\alpha b)f^2+\varphi(t)\geq 0.$$

Now, from $f(x,t)=v(x_1,x_2,\ldots,x_n+\eta t)$, we use that $f_t=\eta v_{x_n}\leq\eta|\nabla v|$ and again take the limiting case $\varphi,\psi\to-\frac{a}{b}\gamma$ to get

$$(\gamma+\alpha b)\left(v^2-\frac{a}{b}\right)\geq(\alpha-\beta)\frac{|\nabla v|^2}{v^2}-\alpha\eta\frac{|\nabla v|}{v}.$$

By choosing $\beta=0$ and $\gamma=-b\alpha$, we reduce the expression to

$$0\geq\frac{|\nabla v|^2}{v^2}-\eta\frac{|\nabla v|}{v}.$$

Simplification yields Corollary 4.3. □

4.3 Standing solutions

We call a solution f a *standing solution* if $f_t=0$.

Corollary 4.4. All positive standing solutions are constant.

Proof. We begin with (4.3)

$$\alpha\frac{f_t}{f}-\alpha a+(\beta-\alpha)\frac{|\nabla f|^2}{f^2}+(\gamma+\alpha b)f^2+\varphi(t)\geq 0.$$

At this point, we assume $f_t=0$. We also again take the limiting case where $\varphi,\psi\to-\frac{a}{b}\gamma$. Thus we have

$$-\alpha a+(\beta-\alpha)\frac{|\nabla f|^2}{f^2}+(\gamma+b\alpha)f^2-\frac{a}{b}\gamma\geq 0.$$

At this point we rearrange and factor to get

$$|\nabla f|^2\leq\frac{f^2}{\alpha-\beta}(b\alpha+\gamma)\left(f^2-\frac{a}{b}\right).$$

Choosing $\gamma=-b\alpha$, the right hand side becomes 0, giving us $|\nabla f|=0$. Because we have $|\nabla f|=f_t=0$, we conclude that f is constant. □

Acknowledgements

D. Booth and J. Burkart's research were supported by NSF through the Research Experience for Undergraduates Program at Cornell University, grant-1156350. Z. Munro and J. Snyder's research were supported by Cornell University Summer Program for Undergraduate Research. X. Cao's research was partially supported by a grant from the Simons Foundation (#280161). The authors would like to thank Professor Robert Strichartz for his encouragement.

References

[1] Mihai Băileşteanu, A Harnack inequality for the parabolic Allen-Cahn equation, ArXiv e-prints, October 2015.

[2] Xiaodong Cao, Mark Cerenzia and Demetre Kazaras, Harnack estimate for the endangered species equation, Proc. Amer. Math. Soc., 143(10) (2015), 4537–4545.

[3] Xiaodong Cao, Bowei Liu, Ian Pendleton and Abigail Ward, Differential Harnack Estimates for Fisher's Equation, ArXiv e-prints, October 2015.

[4] Richard Hamilton, Li-Yau estimates and their Harnack inequalities, in Geometry and Analysis. No. 1, volume 17 of Adv. Lect. Math. (ALM), pages 329–362. Int. Press, Somerville, MA, 2011.

[5] Boris Malomed, The Newell-Whitehead-Segel Equation for Traveling Waves, May 1996.

[6] Alan Newell and John Whitehead, Finite bandwidth, finite amplitude convection, J. Fluid Mech., 38(2) (1969), 279–303.

[7] Javvant Patade and Sachin Bhalekar, Approximate analytical solutions of newell-whitehead-segel equation using a new iterative method, World J. Model. Simulation, 2 (2015), 94–103.

[8] Salman Nourazar, Mohsen Soori and Akbar Nazari-Golshan, On The Exact Solution of Newell-Whitehead-Segel Equation Using the Homotopy Perturbation Method, ArXiv e-prints, January 2015.

[9] Lee Segel, Distant side-walls cause slow amplitude modulation of cellular convection, J. Fluid Mech., 38(1) (1969), 203–224.