Analysis in Theory and Applications
Anal. Theory Appl., Vol. **35**, No. 2 (2019), pp. 192-204

DOI: 10.4208/ata.OA-0005

A Differential Harnack Inequality for the Newell-Whitehead-Segel Equation

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Received 14 September 2017; Accepted (in revised version) 24 September 2018

Abstract. This paper will develop a Li-Yau-Hamilton type differential Harnack estimate for positive solutions to the Newell-Whitehead-Segel equation on $\mathbb{R}^n$. We then use our LYH-differential Harnack inequality to prove several properties about positive solutions to the equation, including deriving a classical Harnack inequality and characterizing standing solutions and traveling wave solutions.

Key Words: Newell-Whitehead-Segel equation, Harnack estimate, Harnack inequality, wave solutions.

AMS Subject Classifications: 35C07, 35K10, 35K55

1 Introduction

Consider any positive solution $f:\mathbb{R}^n\times[0,\infty)\rightarrow\mathbb{R}$ to the Newell-Whitehead-Segel Equation,

$$f_t=\Delta f+af-bf^3, \tag{1.1}$$

here, we assume $a>0$, $b>0$. This equation was first introduced by A. C. Newell and J. A. Whitehead in 1969 [6] and shortly after was studied by L. Segel [9]. Exact solutions to the equation were computed using the Homotopy Perturbation method by S.

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Nourazar, M. Soori and A. Nazari-Golshan in 2011 [8], while some approximate solutions were computed in 2015 by J. Patade and S. Bhalekar [7]. The equation is an example of a reaction-diffusion equation, as it is used to model the change of concentration of a substance, given any chemical reactions that the substance may be undergoing (modeled by the $af-bf^3$ term) and any diffusion causing the chemical to spread throughout the medium (modeled by the Δf term). More specifically, the Newell-Whitehead-Segel equation models Rayleigh-Bénard convection, a reaction-diffusion phenomenon that occurs when a fluid is heated from below.

In this paper, we are just concerned with positive solutions on $\mathbb{R}^n$. For further discussion about working with functions on closed manifolds or complete non-compact manifolds, see [3]. Our main theorem, Theorem 1.1, will outline a Li-Yau-Hamilton type differential Harnack estimate (2) that we will prove based on computing time-evolutions of the relevant quantities, see Hamilton [4]. In the following, Harnack inequality or Harnack estimate refers to an LYH-type differential Harnack inequality. As an application, we will integrate our estimate (2) along a space time curve to obtain a classical Harnack inequality (16), see Corollary 4.1. Then we will use our Harnack estimate to characterize both traveling wave solutions and standing solutions to the Newell-Whitehead-Segel equation.

Theorem 1.1. *With $f>0$ a solution to* (1.1)*, define $l=\log f$. Then:*

$$H=\alpha\Delta l+\beta|\nabla l|^2+\gamma e^{2l}+\varphi(t)\geq 0, \tag{1.2}$$

provided the following three inequalities hold:

$$\begin{aligned} &(a)\ \ \alpha>\beta\geq 0,\\ &(b)\ \ \gamma\leq\frac{-nb\alpha^2(2\alpha+\beta)}{3n\alpha^2-2(\alpha-\beta)\beta}<0,\\ &(c)\ \ 4\gamma(\alpha-\beta)+n\alpha^2 b<0, \end{aligned}$$

with

$$\varphi(t)=\left(\frac{a\alpha}{1-e^{2at}}\right)\left(\frac{\gamma}{\alpha b}e^{2at}-\frac{\alpha\gamma n}{4\gamma(\alpha-\beta)+\alpha^2bn}\right).$$

If, instead of inequality (c), we have:

$$(d)\ \ 4\gamma(\alpha-\beta)+n\alpha^2 b\geq 0,$$

then:

$$H=\alpha\Delta l+\beta|\nabla l|^2+\gamma e^{2l}+\psi(t)\geq 0, \tag{1.3}$$

for:

$$\psi(t)=\begin{cases}\dfrac{n\alpha^2}{2(\alpha-\beta)t}, & t\leq T:=\dfrac{n\alpha^2}{2(\alpha-\beta)(-a\gamma)}\left(2\left(\dfrac{\alpha-\beta}{n\alpha^2}\right)\gamma+b\right),\\[2ex] \dfrac{-an\alpha^2\gamma\left(e^{2a(t-T)}+1\right)}{n\alpha^2b\left(e^{2a(t-T)}+1\right)+4\gamma(\alpha-\beta)}, & t>T.\end{cases}$$