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The Interctitical Defocusing Nonlinear Schrödinger Equations with Radial Initial Data in Dimensions Four and Higher

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Abstract. In this paper, we consider the defocusing nonlinear Schrödinger equation in space dimensions $d\geq 4$. We prove that if u is a radial solution which is *priori* bounded in the critical Sobolev space, that is, $u\in L_t^\infty \dot{H}_x^{s_c}$, then u is global and scatters. In practise, we use weighted Strichartz space adapted for our setting which ultimately helps us solve the problems in cases $d\geq 4$ and $0<s_c<\frac{1}{2}$. The results in this paper extend the work of [27, Commun. PDEs, 40 (2015), 265–308] to higher dimensions.

Key Words: Nonlinear Schrödinger equation, scattering, frequency-localized Morawetz estimae, weighted Strichartz space.

AMS Subject Classifications: 35P25, 35Q55, 47J35

1 Introduction

We consider the Cauchy problem for the nonlinear Schrödinger equation (NLS) in $\mathbb{R}_t\times\mathbb{R}_x^d$ with $d\geq 4$:

$$\begin{cases} (i\partial_t+\Delta)u=\mu|u|^p u, \\ u(0,x)=u_0(x). \end{cases} \tag{1.1}$$

In particular, we call the Eq. (1.1) defocusing when $\mu=1$, and focusing when $\mu=-1$. In this paper, we are dedicated to dealing with the defocusing case.

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The solutions of Eq. (1.1) are left invariant by the scaling transformation

$$u(t,x)\mapsto\lambda^{\frac{2}{p}}u(\lambda^2 t,\lambda x) \tag{1.2}$$

for $\lambda>0$. This scaling invariance defines a notion of criticality. To be more specified, a direct computation shows that the only homogeneous L_x^2-based Sobolev space that is left invariant by (1.2) is $\dot{H}_x^{s_c}$, where the critical regularity s_c is given by $s_c:=\frac{d}{2}-\frac{2}{p}$. We call the problem mass-critical for $s_c=0$, energy-critical for $s_c=1$ and intercritical for $0<s_c<1$. With $s_c=\frac{d}{2}-\frac{2}{p}$ in mind, we will transfer from s_c to p freely.

We proceed by make the notion of solution precise.

Definition 1.1 (Strong solution)**.** A function $u:I\times\mathbb{R}^d\to\mathbb{C}$ on a non-empty time interval $0\in I$ is a strong solution to (1.1) if it belongs to $C_t\dot{H}_x^{s_c}(K\times\mathbb{R}^d)\cap L_{t,x}^{\frac{d+2}{2}p}(K\times\mathbb{R}^d)$ for any compact interval $K\subset I$ and obeys the Duhamel formula

$$u(t)=e^{it\Delta}u_0-i\int_0^t e^{i(t-s)\Delta}(|u|^p u)(s)ds \tag{1.3}$$

for each $t\in I$. We call I the lifespan of u. We say that u is a maximal-lifespan solution if it cannot be extended to any strictly larger interval. We say u is a global solution if $I=\mathbb{R}$.

Let u be a maximal-lifespan solution to the problem (1.1), a standard technique shows that the $\|u\|_{L_{t,x}^{\frac{d+2}{2}p}(I\times\mathbb{R}^d)}<\infty$ implies scattering. That is $I=\infty$ and there exists $u_\pm\in\dot{H}_x^{s_c}(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty}\|u(t)-e^{it\Delta}u_\pm\|_{\dot{H}_x^{s_c}(\mathbb{R}\times\mathbb{R}^d)}=0.$$

The above fact promotes us to define the notion of scattering size and blow up as follows:

Definition 1.2 (Scattering size and blow up)**.** We define the scattering size of a solution $u:I\times\mathbb{R}^d\to\mathbb{C}$ to (1.1) by

$$S_I(u):=\iint_{I\times\mathbb{R}^d}|u(t,x)|^{\frac{d+2}{2}p}dxdt.$$

If there exists $t_0\in I$ so that $S_{[t_0,\sup I)}(u)=\infty$, then we say u *blows up* forward in time, correspondingly if there exists $t_0\in I$ so that $S_{(\inf I,t_0]}(u)=\infty$, then we say u *blows up* backward in time.

The problem which we concern in this paper can be subsumed into the following conjecture.