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KAM Theory for Partial Differential Equations

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Abstract. In the last years much progress has been achieved in KAM theory concerning bifurcation of quasi-periodic solutions of Hamiltonian or reversible partial differential equations. We provide an overview of the state of the art in this field.

Key Words: KAM for PDEs, quasi-periodic solutions, small divisors, infinite dimensional Hamiltonian and reversible systems, water waves, nonlinear wave and Schrödinger equations, KdV.

AMS Subject Classifications: 37K55, 37J40, 37C55, 76B15, 35S05

1 Introduction

Many partial differential equations (PDEs) arising in Physics are infinite dimensional dynamical systems

$$u_t = X(u), \quad u \in E, \tag{1.1}$$

defined on an infinite dimensional phase space E of functions $u := u(x)$, whose vector field X (in general unbounded) is Hamiltonian or Reversible. A vector field X is Hamiltonian if

$$X(u) = J\nabla H(u),$$

where J is a non-degenerate antisymmetric linear operator, the function $H : E \to \mathbb{R}$ is the Hamiltonian and ∇ denotes the L^2-gradient. We refer to [106] for a general introduction to Hamiltonian PDEs. A vector field X is reversible if there exists an involution S of the phase space, i.e., a linear operator of E satisfying $S^2 = \mathrm{Id}$, see e.g., (1.21), such that

$$X \circ S = -S \circ X.$$

Such symmetries have important consequences on the dynamics of (1.1), as we describe below. Classical examples are

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1. **Nonlinear wave (NLW)/ Klein-Gordon.**

$$y_{tt}-\Delta y+V(x)y=f(x,y),\quad x\in\mathbb{T}^d:=(\mathbb{R}/2\pi\mathbb{Z})^d,\quad y\in\mathbb{R},\tag{1.2}$$

with a real valued potential $V(x)\in\mathbb{R}$. If $V(x)=m$ is constant, (1.2) is also called a nonlinear Klein-Gordon equation. Eq. (1.2) can be written as the first order Hamiltonian system

$$\frac{d}{dt}\begin{pmatrix} y \\ p \end{pmatrix}=\begin{pmatrix} p \\ \Delta y-V(x)y+f(x,y) \end{pmatrix}=\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} \nabla_y H(y,p) \\ \nabla_p H(y,p) \end{pmatrix},$$

where $\nabla_y H$, $\nabla_p H$ denote the $L^2(\mathbb{T}^d_x)$-gradient of the Hamiltonian

$$H(y,p):=\int_{\mathbb{T}^d}\frac{p^2}{2}+\frac{1}{2}\big((\nabla_x y)^2+V(x)y^2\big)+F(x,y)dx\tag{1.3}$$

with potential $F(x,y):=-\int_0^y f(x,z)dz$ and $\nabla_x y:=(\partial_{x_1}y,\cdots,\partial_{x_d}y)$.

Considering in (1.3) an Hamiltonian density $F(x,y,\nabla_x y)$, which depends also on the first order derivatives $\nabla_x y$, the corresponding Hamiltonian PDE is a quasi-linear wave equation with a nonlinearity which depends (linearly) with respect to the second order derivatives $\partial^2_{x_i x_j}y$.

If the nonlinearity $f(x,y,\nabla_x y)$ in (1.2) depends on first order derivatives, the equation, called derivative nonlinear wave equation (DNLW), is no more Hamiltonian (at least with the usual symplectic structure) but it can admit a reversible symmetry, see e.g., [20].

2. **Nonlinear Schrödinger (NLS).**

$$\mathrm{i}u_t-\Delta u+V(x)u=\partial_{\bar{u}}F(x,u),\quad x\in\mathbb{T}^d,\quad u\in\mathbb{C},\tag{1.4}$$

where $F(x,u)\in\mathbb{R}$, $\forall u\in\mathbb{C}$, and, for $u=a+\mathrm{i}b$, $a,b\in\mathbb{R}$, we define the operator $\partial_{\bar{u}}:=\frac{1}{2}(\partial_a+\mathrm{i}\partial_b)$. The NLS equation (1.4) can be written as the infinite dimensional complex Hamiltonian equation

$$u_t=\mathrm{i}\nabla_{\bar{u}}H(u),\quad H(u):=\int_{\mathbb{T}^d}|\nabla u|^2+V(x)|u|^2-F(x,u)dx.$$

A simpler pseudo-differential model equation which is often considered is (1.4) with the multiplicative potential replaced by a convolution potential $V*u$, defined as the Fourier multiplier

$$u(x)=\sum_{j\in\mathbb{Z}^d}u_j e^{\mathrm{i}j\cdot x}\mapsto V*u:=\sum_{j\in\mathbb{Z}^d}V_j u_j e^{\mathrm{i}j\cdot x}.$$

If the nonlinearity in the right hand side in (1.4) depends also on first and second order derivatives, we have, respectively, derivative NLS (DNLS) and fully-non-linear (or quasi-linear) Schrödinger type equations. According to the nonlinearity it may admit an Hamiltonian or reversible structure.

3. **Korteweg de Vries (KdV)**

$$u_t+u_{xxx}+\partial_x u^2=0, \quad x\in\mathbb{T}, \quad u\in\mathbb{R}, \tag{1.5}$$

and its perturbations

$$u_t+u_{xxx}+\partial_x u^2+\mathcal{N}(x,u,u_x,u_{xx},u_{xxx})=0, \tag{1.6}$$

where

$$\mathcal{N}(x,u,u_x,u_{xx},u_{xxx}):=-\partial_x\big[(\partial_u f)(x,u,u_x)-\partial_x((\partial_{u_x} f)(x,u,u_x))\big] \tag{1.7}$$

is the most general Hamiltonian (local) quasi-linear nonlinearity. In this case (1.6) is the Hamiltonian PDE

$$u_t=\partial_x\nabla_u H(u), \quad H(u)=\int_{\mathbb{T}}\frac{u_x^2}{2}-\frac{u^3}{3}+f(x,u,u_x)dx, \tag{1.8}$$

where $\nabla_u H$ denotes the $L^2(\mathbb{T}_x)$ gradient. If the Hamiltonian density $f(x,u)$ does not depend on the first order derivative u_x, the nonlinearity in (1.7) reduces to $\mathcal{N}=-\partial_x[(\partial_u f)(x,u)]$ and (1.6) is a semilinear PDE, i.e., the order of the derivatives in the nonlinearity is strictly less than the order of the constant coefficient differential operator ∂_{xxx}.

We refer to [17] for more details about the Hamiltonian and reversible structure of these PDEs.

4. **Water waves equations.** We present in Section 2.3 the Euler equations for an irrotational and incompressible fluid occupying the time dependent domain

$$\mathcal{D}_\eta:=\big\{(x,y)\in\mathbb{T}\times\mathbb{R}:-h<y<\eta(t,x)\big\}, \quad \mathbb{T}:=\mathbb{T}_x:=\mathbb{R}/2\pi\mathbb{Z}, \tag{1.9}$$

under the action of gravity and possibly capillary forces at the free interface. The irrotational velocity field $\nabla_{x,y}\Phi(x,y,t)$ is described by the velocity potential $\Phi(x,y,t)$ and, setting $\psi(x,t)=\Phi(x,y,t)$, it evolves according to the system

$$\begin{cases}\partial_t\eta=G(\eta)\psi,\\ \partial_t\psi=-g\eta-\dfrac{\psi_x^2}{2}+\dfrac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi+\eta_x\psi_x\big)^2+\kappa\partial_x\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big),\end{cases} \tag{1.10}$$

where $G(\eta)$ is the Dirichlet-Neumann operator defined in (2.34) (it is a pseudo-differential operator of order 1), g is the gravitational constant and κ the surface tension.

From the water-waves equations it is possible to derive, in suitable regimes, several approximate model PDEs used in fluid mechanics as the KdV, NLS, Benjamin-Ono, Boussinesq, Benjamin-Bona-Mahoni equations, $\cdots$, see e.g., [54].

The Hamiltonian or the reversible structure of such PDEs eliminates dissipative phenomena and, when the variable x belongs to a bounded domain like a compact interval $x \in [0,\pi]$, or $x \in \mathbb{T}^d$ (periodic boundary conditions), or, more in general, x belongs to a compact manifold, their dynamics is expected to have a "recurrent" behaviour in time, with many periodic and quasi-periodic solutions.

Definition 1.1 (Quasi-periodic solution)**.** A quasi-periodic solution of the Eq. (1.1) with n frequencies is a smooth solution (differentiable, C^∞, Gevrey, analytic), defined for all times, of the form

$$u(t) = U(\omega t) \in E, \quad \text{where } \mathbb{T}^n \ni \varphi \mapsto U(\varphi) \in E, \tag{1.11}$$

is 2π-periodic in the angular variables $\varphi := (\varphi_1, \cdots, \varphi_n)$ and the frequency vector $\omega \in \mathbb{R}^n$ is irrational, namely $\omega \cdot \ell \neq 0, \forall \ell \in \mathbb{Z}^n \setminus \{0\}$. When $n=1$ the solution $u(t)$ is periodic in time, with period $2\pi/\omega$.

If $U(\omega t)$ is a quasi-periodic solution then, since the orbit $\{\omega t\}_{t\in\mathbb{R}}$ is dense on $\mathbb{T}^n$, the torus-manifold

$$\mathcal{T} := U(\mathbb{T}^n) \subset E,$$

is invariant under the flow Φ^t of (1.1). Denoting by $\Psi^t_\omega : \mathbb{T}^n \to \mathbb{T}^n$ the linear flow

$$\Psi^t_\omega(\varphi) := \varphi + \omega t, \quad \varphi \in \mathbb{T}^n,$$

the search of a quasi-periodic solution amounts to look for U such that

$$\Phi^t \circ U = U \circ \Psi^t_\omega. \tag{1.12}$$

Note that (1.12) only requires that the flow Φ^t is defined and smooth on the compact manifold $\mathcal{T} := U(\mathbb{T}^n)$. This remark is important because, for some of the PDEs considered above, the flow could be ill-posed in a whole neighborhood of the torus $\mathcal{T}$. Eq. (1.12) is equivalent to

$$\omega \cdot \partial_\varphi U(\varphi) - X(U(\varphi)) = 0, \quad \forall \varphi \in \mathbb{T}^n.$$

We may expect the existence of many periodic, quasi-periodic, or almost-periodic solutions of the PDEs (1.2), (1.4), (1.6), (1.10) described above because such are the solutions of the linear PDEs obtained neglecting the nonlinearity. For example, the solutions of the linear wave equation (see (1.2) with $f=0$)

$$y_{tt} - \Delta y + V(x) y = 0, \quad x \in \mathbb{T}^d, \tag{1.13}$$

are the linear superpositions of the "normal mode oscillations". The self-adjoint operator $-\Delta + V(x)$ possesses a complete L^2-orthonormal basis of eigenfunctions $\psi_j(x)$, $j \in \mathbb{N}$, with eigenvalues $\mu_j \to +\infty$, i.e.,

$$(-\Delta + V(x)) \psi_j(x) = \mu_j \psi_j(x), \quad j \in \mathbb{N}. \tag{1.14}$$

Supposing, for simplicity, that $-\Delta+V(x)>0$ so that any $\mu_j>0$, all the solutions of (1.13) are

$$y(t,x)=\sum_{j\in\mathbb{N}}a_j\cos(\sqrt{\mu_j}t+\theta_j)\psi_j(x),\quad a_j,\,\theta_j\in\mathbb{R}, \tag{1.15}$$

which, according to the irrationality properties of the linear frequencies $\sqrt{\mu_j}$, are periodic, quasi-periodic, or almost-periodic in time, i.e., quasi-periodic with infinitely many frequencies.

Remark 1.1. On the other hand, if $x\in\mathbb{R}^d$ the solutions of the linear Klein-Gordon equation (1.13) disperse to zero because of the wave propagation of different plane waves with different speeds. This phenomenon holds also for the linear Schrödinger, Airy (i.e., $u_t+u_{xxx}=0$), and Water waves equations on $\mathbb{R}^d$. This is why the existence of quasi-periodic solutions for nonlinear PDEs on $\mathbb{R}^d$ is rare.

A natural question is to know what happens of these solutions under the effect of the nonlinearity. There exist special nonlinear equations for which all the solutions are still periodic, quasi-periodic or almost-periodic in time, for example the nonlinear KdV equation (1.5). Actually (1.5) possesses infinitely many analytic prime integrals in involution, i.e., pairwise commuting, and it is completely integrable in the strongest possible sense: it possesses global analytic action-angle variables, in the form of Birkhoff coordinates, see e.g., [100], and the whole infinite dimensional phase space is foliated by quasi-periodic and almost-periodic solutions (the quasi-periodic solutions are called "finite gap" solutions). The Birkhoff coordinates are a cartesian smooth version of the action-angle variables to avoid the singularity when one action component vanishes, i.e., close to the elliptic equilibrium. This situation generalizes what happens for a finite dimensional Hamiltonian system in $\mathbb{R}^{2n}$ which possesses n-independent prime integrals in involution. According to the Liouville-Arnold theorem, in suitable local symplectic angle-action variables $(\theta,I)\in\mathbb{T}^n\times\mathbb{R}^n$, the integrable Hamiltonian $H(I)$ depends only on the actions and the dynamics is described by

$$\dot{\theta}=\partial_I H(I),\quad \dot{I}=0.$$

Thus the phase space is foliated by the invariant tori $\mathbb{T}^n\times\{\xi\}$, $\xi\in\mathbb{R}^n$, filled by the solutions $\theta(t)=\theta_0+\omega(\xi)t$, $I(t)=\xi$, with frequency vector $\omega(\xi)=(\partial_I H)(\xi)$. Other integrable PDEs which possess Birkhoff coordinates are the mKdV equation, see [101],

$$u_t+u_{xxx}+\partial_x u^3=0,\quad x\in\mathbb{T}, \tag{1.16}$$

and the cubic 1-d NLS equation, see [82],

$$\mathrm{i}u_t=-u_{xx}+|u|^2u,\quad x\in\mathbb{T}. \tag{1.17}$$

Other integrable PDEs (for which the existence of Birkhoff coordinates is not known) are the Sine-Gordon, Benjamin-Ono, Camassa-Holm, Degasperis-Procesi equations. Clearly also the linear wave equation (1.13) is completely integrable, since, introducing coordinates by $y=\sum_{j\in\mathbb{N}} y_j\psi_j(x)$, it is described by infinitely many decoupled scalar harmonic oscillators

$$\ddot{y}_j+\mu_j y_j=0,\quad j\in\mathbb{N},$$

with linear frequencies of oscillations $\sqrt{\mu_j}$.

- The central question of KAM theory is: do the periodic, quasi-periodic, almost-periodic solutions of an integrable PDE (linear or nonlinear) persist, just slightly deformed, under the effect of a perturbation?

For a finite dimensional nearly integrable Hamiltonian system

$$\dot{\theta}=\partial_I H_0(I)+\varepsilon\partial_I H_1(\theta,I),\quad \dot{I}=-\varepsilon\partial_\theta H_1(\theta,I), \tag{1.18}$$

with a non-degenerate integrable Hamiltonian $H_0(I)$ (i.e., D^2H_0 invertible) the classical KAM –Kolmogorov-Arnold-Moser–theorem proves the persistence of quasi-periodic solutions of (1.18), for ε small enough, with a diophantine frequency vector $\omega\in\mathbb{R}^n$, i.e., satisfying for some $\gamma>0$ and $\tau>n-1$,

$$|\omega\cdot\ell|\geq\frac{\gamma}{|\ell|^\tau},\quad \forall\ell\in\mathbb{Z}^n\setminus\{0\}. \tag{1.19}$$

Such frequencies form a Cantor set of $\mathbb{R}^n$. This celebrated result was proved by Kolmogorov [103] and Arnold [2] for analytic systems and Moser [110,111] for merely differentiable ones. The set of KAM perturbed invariant Lagrangian tori has positive measure and therefore the KAM theorem has an important dynamical consequence: nearly integrable Hamiltonian systems are generically not-ergodic.

KAM theory has been extended in many directions: for systems satisfying weak non-degeneracy conditions, see e.g., [46,122,124], for frequency vectors satisfying non-resonance conditions weaker than Diophantine (like Brujno condition), for applications to Celestial Mechanics, see e.g., [48,71], for lower dimensional (not Lagrangian) tori, see e.g., [37,47,62,109,112,114,125], for reversible systems, e.g., [3,112,123], or for systems satisfying other algebraic properties, like being volume preserving, see e.g., [34]. Let us explain heuristically why reversibility is a dynamically relevant property compatible with the existence of quasi-periodic solutions. A nearly integrable system in action-angle variables

$$\dot{\theta}=\omega(I)+\varepsilon g(\theta,I),\quad \dot{I}=\varepsilon f(\theta,I), \tag{1.20}$$

which is reversible with respect to the involution

$$S:(\theta,I)\mapsto(-\theta,I) \tag{1.21}$$

has $f(\theta,I)$, which is odd in θ, and $g(\theta,I)$ even in θ. Now, since the angles θ rotate much faster than the actions evolve, we may expect that the evolution of the actions is "effectively" governed, at the first order, by the averaged system

$$\dot{I}=\varepsilon\frac{1}{(2\pi)^n}\int_{\mathbb{T}^n}f(\theta,I)d\theta=0,$$

where the actions do not evolve in time. This "averaging principle" suggests that we may expect the existence of quasi-periodic solutions for a reversible system (1.20). Notice that also for the Hamiltonian system (1.18) the "averaging principle" gives the effective equation

$$\dot{I} = \varepsilon \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \partial_I H_1(\theta, I) d\theta = 0.$$

In this Note we shall concentrate on the extensions, started around 30 years ago, of KAM theory to infinite dimensional dynamical systems, specifically PDEs. Actually, in the late eighties and early nineties the local bifurcation theory of periodic and quasi-periodic solutions–which fill lower dimensional tori in an infinite dimensional phase space- started to be extended to 1-dimensional semilinear wave and Schrödinger equations by Kuksin [104], Wayne [128], Craig [53,57], Bourgain [35], Pöeschel [115], followed by many others.

Important elements which make a substantial difference for KAM theory for PDEs are

1. space dimension $x \in \mathbb{T}^d$. The bigger is the space dimension d, the more complex resonance phenomena appear. In particular, if $d \geq 2$ then:

   (a) the eigenvalues μ_j of $(-\Delta + V(x))\psi_j(x) = \mu_j \psi_j(x)$ (see (1.14)) appear in clusters of unbounded size. Albeit the eigenvalues μ_j are "generically" simple varying the potential $V(x)$, see e.g., [99], they are not separated enough, unlike in $d=1$. For example, if $V(x)=0$, and $x \in \mathbb{T}^d$, they are

   $$\mu_j = |j|^2 = j_1^2 + \cdots + j_d^2, \quad j = (j_1, \cdots, j_d) \in \mathbb{Z}^d.$$

   (b) the eigenfunctions $\psi_j(x)$ may be "not localized" with respect to the exponentials, i.e., roughly speaking, the Fourier coefficients $(\hat{\psi}_j)_i$ do not rapidly converge to zero for $|i-j| \to \infty$. This property always holds if $d=1$ but may fail for $d \geq 2$, see [67]. A localization property implies that the interaction induced by the nonlinearity among the normal modes $\psi_j(x)$ is "short range".

2. derivatives in the nonlinearity. The more derivatives are in the nonlinearity, the stronger is its perturbative effect. A main motivation for studying KAM theory for unbounded perturbations is that most equations arising in Physics are indeed quasi-linear, like the water waves Euler equations, the PDEs arising in elasticity theory, the Einstein equations which are approximated by a system of quasi-linear wave equations, etc..

Before concluding this introduction we mention that another very important research direction concerns the search of almost-periodic solutions, i.e., quasi-periodic solutions with infinitely many frequencies, which correspond to the classical KAM Lagrangian tori. Only a few results are available so far. The first result is due to Pöschel [117] for a regularizing Schrödinger equation proving the existence of almost-periodic solutions with a very fast decay in Fourier space. A breakthrough result was then proved by Bourgain [42] for a semilinear Schrödinger equation and solutions with just an exponential Fourier decay. We refer to [51,72], and references therein, for recent extensions.

2 KAM for PDEs

The progress in KAM theory for PDEs concerns novel results in the spectral analysis of the quasi-periodically time-dependent linear system

$$h_t + DX(u(\omega t))h = 0$$

obtained linearizing the Eq. (1.1) at an approximate quasi-periodic solution $u(\omega t)$. The key step is to prove the invertibility of the corresponding quasi-periodic operator

$$\omega\cdot\partial_\varphi + DX(u(\varphi)), \quad \varphi\in\mathbb{T}^\nu,$$

(or finite dimensional restrictions of it), for most values of some parameter, with estimates for the action of their inverse in high norms.

We describe in Sections 2.1-2.3 the perturbative reducibility approach with applications to PDEs, and, in Section 2.4, the ”Craig-Wayne-Bourgain” approach. The applicability of the first approach is related to the possibility of verifying suitable ”second order Melnikov non-resonance conditions”.

2.1 Reducibility

Transfomation laws. Consider a quasi-periodically time dependent linear system

$$u_t + A(\omega t)u = 0, \quad u\in E, \tag{2.1}$$

with a diophantine frequency vector $\omega\in\mathbb{R}^\nu$, defined on a phase space E, which may be a finite or infinite dimensional Hilbert space. Under a quasi-periodically time dependent transformation

$$v = \Phi(\omega t)[u], \tag{2.2}$$

where $\Phi(\varphi): E\to E$, $\varphi\in\mathbb{T}^\nu$, are invertible linear operators of the phase space, system (2.1) transforms into

$$v_t + B(\omega t)v = 0 \tag{2.3}$$

with the new vector field

$$B(\varphi) = -(\omega\cdot\partial_\varphi\Phi)(\varphi)\Phi^{-1}(\varphi) + \Phi(\varphi)A(\varphi)\Phi^{-1}(\varphi). \tag{2.4}$$

If $A(\omega t)$ is Hamiltonian and $\Phi(\omega t)$ is symplectic, then the new operator $B(\omega t)$ is Hamiltonian as well. Other algebraic structures, with relevant dynamical consequences, can be preserved: for example, if $A(\omega t)$ is reversible and $\Phi(\omega t)$ is reversibility preserving, then $B(\omega t)$ is reversible as well.

Reducibility. If the operator B in (2.3) is a diagonal, time independent operator, i.e., $B(\omega t) = B = \mathrm{diag}_j(b_j)$ in a suitable basis of E, then (2.3) reduces to the decoupled scalar linear ordinary differential equations

$$\dot{v}_j + b_j v_j = 0,$$

where (v_j) denote the coordinates of v in the basis of eigenvectors of B. Then (2.3) is integrated in a straightforward way, and all the solutions of system (2.1) are obtained back via the change of variables (2.2). In this case we say that (2.1) has been reduced to constant coefficients. We shall also say that system (2.1) is reducible if B is reduced to a constant coefficient block-diagonal operator. Notice that, if all the b_j are purely imaginary, then the linear system (2.1) is stable (in the sense of Lyapunov), otherwise, it is unstable.

If $\omega \in \mathbb{R}$ and E is finite dimensional, the classical Floquet theory proves that any time periodic linear system (2.1) is reducible. If $\omega \in \mathbb{R}^\nu$ this is in general not true, see e.g., [34], i.e., there exist non reducible linear systems. In the case that $A(\omega t)$ is a small perturbation of a constant coefficient operator, perturbative algorithms for reducibility can be implemented. For finite dimensional systems this approach is systematically adopted in [112], and we refer to [63] for results of "almost-reducible" quasi-periodic systems. We also mention very interesting non-perturbative reducibility results in case of two frequencies, see e.g., [90] and references therein. We outline below a perturbative reducibility scheme in a simple setting.

Perturbative reducibility. Consider a quasi-periodic operator

$$\omega \cdot \partial_\varphi + A(\varphi), \quad \text{where } A(\varphi) = D + R(\varphi), \quad \varphi \in \mathbb{T}^\nu, \tag{2.5}$$

is a perturbation of a diagonal operator

$$D = \mathrm{diag}(\mathrm{i}d_j)_{j\in\mathbb{Z}} = \mathrm{Op}(\mathrm{i}d_j), \quad d_j \in \mathbb{R}, \tag{2.6}$$

where the eigenvalues $\mathrm{i}d_j$ are simple and constant in φ. We consider the case that the d_j's are real because this is the common situation arising for PDEs (if some $\mathrm{Im}\, d_j \neq 0$ then there are hyperbolic directions which do not create resonance phenomena). We look for a transformation $\Phi(\varphi)$ of the form (2.2) which removes from $R(\varphi)$ the angles φ up to terms of size $\sim \mathcal{O}(\|R\|^2)$. It is convenient to transform (2.1) under the flow $\Phi_F(\varphi,\tau)$ generated by an auxiliary linear equation

$$\partial_\tau \Phi_F(\varphi,\tau) = F(\varphi)\Phi_F(\varphi,\tau), \quad \Phi_F(\varphi,0) = \mathrm{Id}, \tag{2.7}$$

with a linear operator $F(\varphi)$ to be chosen (which could also be τ-dependent). This amounts to compute the Lie derivative of $A(\varphi)$ in the direction of the vector field $F(\varphi)$. Notice that, if E is infinite dimensional we have to guarantee that $F(\varphi)$ generates a flow $\Phi_F(\varphi,\tau)$. A sufficient condition is that $F(\varphi)$ is bounded. If F is unbounded operator, a sufficient condition which implies an L^2 energy estimate is that $F+F^*$ is bounded, see also [108].

Given a linear operator $A_0(\varphi)$, the conjugated operator under the flow $\Phi_F(\varphi,\tau)$ generated by (2.7),

$$A(\varphi,\tau):=\Phi_F(\varphi,\tau)A_0(\varphi)\Phi_F(\varphi,\tau)^{-1},$$

satisfies the Heisenberg equation

$$\partial_\tau A(\varphi,\tau)=[F(\varphi),A(\varphi,\tau)], \quad A(\varphi,\tau)=A_0(\varphi),$$

where $[A,B]:=A\circ B-B\circ A$ denotes the commutator between two linear operators A,B. According to the goals that one has in mind it is convenient to expand $A(\varphi,\tau)$ in different ways. For example we have the Lie expansion

$$A(\varphi,\tau)_{|\tau=1}=A_0(\varphi)+\mathrm{Ad}_F A_0+\frac{1}{2}\mathrm{Ad}_F^2 A_0+\cdots, \tag{2.8}$$

where $\mathrm{Ad}_F[\cdot]:=[F,\cdot]$. This expansion is convergent if F and A_0 are bounded and F is small, because the adjoint action produces, in such a case, bounded operators with smaller and smaller size. Notice also that if F,A are pseudo differential operators and the order of F is strictly less than 1, then (2.8) is an expansion in operators with decreasing orders, by the fundamental property that the commutator of pseudo differential operators of order m_1,m_2 has order m_1+m_2-1.

Remark 2.1. If F has order 1 then (2.8) is not an expansion in operators with decreasing order.

Conjugating (2.5) under the flow generated by (2.7) we obtain

$$\omega\cdot\partial_\varphi+D-\omega\cdot\partial_\varphi F+[F,D]+R+\text{ smaller terms}\cdots. \tag{2.9}$$

We want to choose F in such a way to solve the "homological" equation

$$-\omega\cdot\partial_\varphi F-[D,F]+R=[R], \tag{2.10}$$

where

$$[R]:=\mathrm{diag}_j(R_j^j(0)), \quad R_j^j(0):=\frac{1}{(2\pi)^\nu}\int_{\mathbb{T}^\nu}R_j^j(\varphi)d\varphi, \tag{2.11}$$

is the normal form part of the operator R, independent of φ, that we can not eliminate. Representing $F=(F_k^j(\varphi))_{j,k\in\mathbb{Z}}$ and $R=(R_k^j(\varphi))_{j,k\in\mathbb{Z}}$ as matrices, and computing the commutator with the diagonal operator D in (2.6) we obtain that (2.10) is represented as

$$-\omega\cdot\partial_\varphi F_k^j(\varphi)-\mathrm{i}(d_k-d_j)F_k^j(\varphi)+R_k^j(\varphi)=[R]_k^j,$$

and, performing the Fourier expansion in φ,

$$F_k^j(\varphi)=\sum_{\ell\in\mathbb{Z}^\nu}F_k^j(\ell)e^{\mathrm{i}\ell\cdot\varphi}, \quad R_k^j(\varphi)=\sum_{\ell\in\mathbb{Z}^\nu}R_k^j(\ell)e^{\mathrm{i}\ell\cdot\varphi},$$

it reduces to the infinitely many scalar equations

$$-\mathrm{i}\omega\cdot\ell F_k^j(\ell)-\mathrm{i}(d_k-d_j)F_k^j(\ell)+R_k^j(\ell)=[R]_k^j\delta_{\ell,0},\quad j,k\in\mathbb{Z},\quad \ell\in\mathbb{Z}^\nu, \tag{2.12}$$

where $\delta_{\ell,0}:=1$ if $\ell=0$ and zero otherwise. Assuming the so called second-order Melnikov non-resonance conditions

$$|\omega\cdot\ell+d_k-d_j|\geq\frac{\gamma}{\langle\ell\rangle^\tau},\quad \forall(\ell,j,k)\neq(0,j,j), \tag{2.13}$$

we can define the solution of the homological equations (2.12) (see (2.11))

$$F_k^j(\ell):=\begin{cases}\dfrac{R_k^j(\ell)}{\mathrm{i}(\omega\cdot\ell+d_k-d_j)}, & \forall(\ell,j,k)\neq(0,j,j),\\ 0, & \forall(\ell,j,k)=(0,j,j).\end{cases} \tag{2.14}$$

Therefore the transformed operator (2.9) becomes

$$\omega\cdot\partial_\varphi+D_++\text{smaller terms},\quad \text{where }\ D_+:=D+[R]=(\mathrm{i}d_j+[R]_j^j)_{j\in\mathbb{Z}}, \tag{2.15}$$

is the new diagonal operator constant in φ. Moreover, by (2.14) and (2.13) the operator F has size $F=\mathcal{O}(\|R\|\gamma^{-1})$ and the smaller terms in (2.15) have size $\mathcal{O}(\|R\|^2\gamma^{-1})$. We can iterate the scheme to reduce also these terms, which are φ-dependent, and so on. Notice that, if $R(\varphi)$ is bounded, then $F(\varphi)$ is bounded as well and thus (2.7) defines a flow. On the other hand the loss of time derivatives induced on F by the divisors in (2.13) can be recovered by a smoothing procedure in the angles φ. In order to continue the iteration one also needs to impose non-resonance conditions as in (2.13) at each step and therefore we need information about the perturbed normal form D_+ in (2.15). If these steps work, after an infinite iteration one conjugates the quasi-periodic operator (2.5) to a diagonal, constant in φ, one

$$\omega\cdot\partial_\varphi+\mathrm{diag}_j(\mathrm{i}d_j^\infty),\quad \mathrm{i}d_j^\infty=\mathrm{i}d_j+[R]_j^j+\cdots. \tag{2.16}$$

Imposing the first order Melnikov non-resonance conditions

$$|\omega\cdot\ell+d_j^\infty|\geq\frac{\gamma}{\langle\ell\rangle^\tau},\quad \forall\ell,j,$$

the linear operator (2.16) is invertible and its inverse loses τ time-derivatives. Since the change of coordinates which conjugates (2.5) to (2.16) maps spaces of high regularity in themselves, one proves the existence of an inverse of (2.5) which satisfies tame estimates in high norms (with loss of τ derivatives).

This is the essence of the Nash-Moser-KAM reducibility scheme. The following questions arise naturally:

1. What happens if the eigenvalues d_j are multiple? This is the common situation for $1d$ PDEs with periodic boundary conditions and PDEs in higher space dimensions. It is conceivable to reduce $\omega\cdot\partial_\varphi + A(\varphi)$ to a block-diagonal normal form linear system.

2. What happens if the operator $R(\varphi)$ in (2.5) is unbounded? This is the common situation for PDEs with nonlinearities which contain derivatives.

3. What happens if the Melnikov non-resonance conditions (2.13) also lose space derivatives? This is the common situation when the dispersion relation $d_j \sim j^\alpha$, $\alpha < 1$, has a sublinear growth.

In the next sections we shall see some answer to these questions.

2.2 KAM with second order Melnikov

KAM for $1d$ NLS and NLW with Dirichlet boundary conditions. The iterative scheme outlined in section 2.1 has been effectively implemented by Kuksin [104] and Wayne [128] for proving existence of quasi-periodic solutions of 1-d semilinear wave

$$y_{tt} - y_{xx} + V(x)y + \varepsilon f(x,y) = 0, \quad y(0) = y(\pi) = 0, \tag{2.17}$$

and Schrödinger equations

$$\mathrm{i}u_t = u_{xx} + V(x)u + \varepsilon f(|u|^2)u, \quad u(0) = u(\pi) = 0, \tag{2.18}$$

with Dirichlet boundary conditions. These equations are regarded as a perturbation of the linear PDEs

$$y_{tt} - y_{xx} + V(x)y = 0, \quad \mathrm{i}u_t = u_{xx} + V(x)u, \tag{2.19}$$

which depend on the potential $V(x)$, used as a parameter. Notice that the eigenvalues of $-\partial_{xx} + V(x)$ with Dirichlet boundary conditions are simple, the operator R (as in (2.5)) obtained linearizing the nonlinearity f at a quasi-periodic approximate solution is bounded (f does not contain derivatives) and it is possible to impose second order Melnikov non-resonance conditions as in (2.13) because the unperturbed linear frequencies of (2.19) grow as $\sim j^\alpha$, $\alpha \geq 1$.

Later on these results have been extended in Kuksin-Pöschel [107] to parameter independent Schrödinger equations

$$\mathrm{i}u_t = u_{xx} + f(|u|^2)u, \quad f'(0) \neq 0, \quad u(0) = u(\pi) = 0, \tag{2.20}$$

and in Pöschel [116] to nonlinear Klein-Gordon equations

$$y_{tt} - y_{xx} + my = y^3 + \text{h.o.t}, \quad y(0) = y(\pi) = 0. \tag{2.21}$$

A Birkhoff normal form analysis implies that the frequencies of the expected quasi-periodic solutions vary diffeomorphically with their "amplitudes", allowing to prove that the Melnikov non-resonance conditions are satisfied for most amplitudes. Since the frequency vector ω may satisfy only a diophantine condition (1.19) with a constant γ which tends to 0 as the solution tends to 0, there is a singular perturbation difficulty, see also [18,33,73,95]. These quasi-periodic solutions are small deformations of linear solutions. On the other hand, approximating the PDE with its Birkhoff normal form allows to prove also the existence of other kind of solutions with a purely nonlinear dynamical behavior, which is very different than the linear dynamics: for example the Birkhoff-Lewis long-time-periodic solutions which survive by the destruction of resonant tori [11,32], or quasi-periodic solutions with some small frequency which bifurcate from beating solutions of the quintic NLS, see [89],

$$-\mathrm{i}u_t+u_{xx}=\pm|u|^4u.$$

Periodic boundary conditions $x\in\mathbb{T}$. The above results do not apply for periodic boundary conditions $x\in\mathbb{T}$ because two eigenvalues of $-\partial_{xx}+V(x)$ coincide (or are too close), and thus the second order Melnikov non resonance conditions (2.13) are violated. This is the first instance where the difficulty mentioned in item 1 at the end of Section 2.1 appears.

Historically this difficulty was first solved by Craig-Wayne [57] and Bourgain [35] developing a novel approach that we describe in Section 2.4. On the other hand, the KAM reducibility approach has been extended by Chierchia-You [49] for semi-linear wave equations like (2.17) with periodic boundary conditions. Because of the near resonance between pairs of frequencies (consider an operator as in (2.5)-(2.6) with $d_{-j}\approx d_j$) the linearized equations are reduced to a diagonal system of 2×2 self-adjoint matrices. Since NLW is a second order equation, its Hamiltonian vector field is regularizing of order 1 (it gains one space derivative), and this is sufficient to prove that the perturbed frequencies of (2.17) satisfy an estimate like

$$\sqrt{\mu_j}+\mathcal{O}(\varepsilon|j|^{-1})\sim|j|+\mathcal{O}(|j|^{-1}),$$

as $|j|\to+\infty$, where μ_j denote the eigenvalues of $-\partial_{xx}+V(x)$. This asymptotic decay is sufficient to verify the second order Melnikov non-resonance conditions

$$|\omega\cdot\ell+d_k-d_j|\geq\frac{\gamma}{\langle\ell\rangle^{\tau}},\quad\forall(\ell,k,j)\neq(0,k,\pm k),$$

along the KAM iteration. We also refer to Yuan [129] for results for the completely resonant wave equation

$$u_{tt}-u_{xx}+u^3=0,\quad x\in\mathbb{T}.$$

If the Hamiltonian nonlinearity does not depend on the space variable x the equation is translation invariant and Geng-You [74,76] were able to exploit the corresponding "momentum" conservation, which is preserved along the KAM iteration, to fulfill the non-resonance conditions. Indeed such symmetry enables to prove that many monomials are never present along the KAM iteration and, in particular, it removes the degeneracy of the normal frequencies.

For semilinear Schrödinger equations the nonlinear vector field is not smoothing. The first KAM reducibility result has been proved by Eliasson-Kuksin in [66] as a particular case of a much more general result valid in any space dimension, that we discuss below. For 1-dimensional Schrödinger equations another recent approach to obtain the asymptotics of the perturbed frequencies, which allows to verify the Melnikov non-resonance conditions, is developed in Berti-Kappeler-Montalto [29] for perturbations of any finite gap solutions, via a regularization technique based on pseudo-differential ideas, that we explain below.

PDEs in higher space dimension: reducibility results. For PDEs in higher space dimensions the reducibility approach has been first worked out for semilinear Schrödinger equations

$$-\mathrm{i}u_t=-\Delta u+V*u+\varepsilon\partial_{\bar{u}}F(x,u,\bar{u}),\quad x\in\mathbb{T}^d,$$

with a convolution potential by Eliasson-Kuksin [65, 66]. This is a much more difficult situation than the $1d$-case because the eigenvalues appear in clusters of unbounded size. In such a case the reducibility result that one could look for is to block-diagonalize the linearized operators (with blocks of increasing dimensions). The convolution potential V plays the role of "external parameters". Eliasson-Kuksin introduced in [66] the notion of Töplitz-Lipschitz matrices in order to extract asymptotic information on the eigenvalues, and so verify the second order Melnikov non resonance conditions.

Then for the cubic NLS equation

$$\mathrm{i}u_t=-\Delta u+|u|^2u,\quad x\in\mathbb{T}^2, \tag{2.22}$$

which is parameter independent, Geng-Xu-You [78] proved a KAM result using a Birkhoff normal form analysis. Remark that, unlike the $1d$ cubic NLS equation (1.17), the Birkhoff normal form of (2.22) is not-integrable.

For completely resonant NLS equations in any space dimensions

$$\mathrm{i}u_t=-\Delta u+|u|^{2p}u,\quad p\in\mathbb{N},\quad x\in\mathbb{T}^d, \tag{2.23}$$

Procesi-Procesi [118], realized a systematic study of the Birkhoff normal form and, using the notion of quasi-Töplitz matrices developed in Procesi-Xu [121], proved in [119, 120] existence of reducible quasi-periodic solutions of (2.23).

KAM results have been proved for parameter dependent beam equation by Geng-You [75] and, more recently, in Grébert-Eliasson-Kuksin [64] for multidimensional beam equations

$$u_{tt}+\Delta^2u+mu+\partial_uG(x,u)=0,\quad x\in\mathbb{T}^d,\quad u\in\mathbb{R}.$$

We refer to the work of Grébert-Paturel [83] concerning the Klein-Gordon on the sphere $\mathbb{S}^d$,

$$u_{tt}-\Delta u+mu+\delta M_\rho u+\varepsilon g(x,u)=0,\quad x\in\mathbb{S}^d,\quad u\in\mathbb{R}.$$

No reducibility results are available so far for multidimensional wave equations. We also mention the result by Grébert-Thomann [85] for the 1d harmonic oscillator and Grébert-Paturel [84] in the multidimensional case.

Quasi and fully nonlinear PDEs. Another situation where the reducibility approach outlined in Section 2.1 encounters a serious difficulty is when R is unbounded, this is the difficulty mentioned in item 2 at the end of Section 2.1. In such a case, the auxiliary vector field F (see (2.14)) could not define a flow, and the iterative scheme outlined in Section 2.1 would formally produce remainders which accumulate more and more derivatives.

KAM for semilinear PDEs with derivatives. The first KAM results for PDEs with an unbounded nonlinearity have been proved by Kuksin [105] and, then, Kappeler-Pöschel [100], for perturbations of finite-gap solutions of

$$u_t+u_{xxx}+\partial_x u^2+\varepsilon\partial_x(\partial_u f)(x,u)=0,\quad x\in\mathbb{T}. \tag{2.24}$$

The key idea in [105] is to exploit that the frequencies of KdV grow asymptotically as $\sim j^3$ as $j\to+\infty$, and therefore one can impose second order Melnikov non-resonance conditions like

$$|\omega\cdot\ell+j^3-i^3|\geq(j^2+i^2)/2,\quad i\neq j,$$

which gain 2 space derivatives (outside the diagonal $i=j$), sufficient to compensate the loss of one space derivative produced by the vector field $\varepsilon\partial_x(\partial_u f)(x,u)$. Then Liu-Yuan in [96] proved KAM results for semilinear perturbations of Hamiltonian DNLS and Benjiamin-Ono equations and Zhang-Gao-Yuan [132] for the reversible derivative NLS equation

$$\mathrm{i}u_t+u_{xx}=|u_x|^2u,\quad u(0)=u(\pi)=0.$$

These PDEs are more difficult than KdV because the linear frequencies grow like $\sim j^2$ and not $\sim j^3$, and therefore gain only 1 space derivative when solving the homological equations. More recently Berti-Biasco-Procesi [19,20] proved the existence and the stability of quasi-periodic solutions of autonomous derivative Klein-Gordon equations

$$y_{tt}-y_{xx}+my=g(x,y,y_x,y_t) \tag{2.25}$$

satisfying reversibility conditions which rule out nonlinearities like y_t^3, y_x^3, for which no periodic nor quasi-periodic solutions exist (with these nonlinearities all the solutions dissipate to zero). The key point in [19,20] was to adapt the notion of quasi-Töplitz vector field in [121] to obtain the higher order asymptotic expansion of the perturbed normal elliptic frequencies

$$\mu_j=\mu_j(\varepsilon)=\sqrt{j^2+m}+a_\pm+\mathcal{O}(1/j)\quad\text{as }j\to\pm\infty,$$

for suitable constants $a_\pm$ (of the size $a_\pm=\mathcal{O}(\varepsilon)$ of the solution $y=\mathcal{O}(\varepsilon)$). Thanks to this asymptotics it is sufficient to verify, for each $\ell\in\mathbb{Z}^\nu$, that only finitely many second order

Melnikov non-resonance conditions hold, because infinitely many conditions in (2.13) are already verified by imposing only first order Melnikov conditions like

$$|\omega\cdot\ell+h|\geq\gamma\langle\ell\rangle^{-\tau},\quad |\omega\cdot\ell+(a_+-a_-)+h|\geq\gamma\langle\ell\rangle^{-\tau}, \tag{2.26}$$

for all $(\ell,h)\in\mathbb{Z}^\nu\times\mathbb{Z}$ such that the left hand side functions do not vanish identically. Indeed, for $j>k>\mathcal{O}(|\ell|^\tau\gamma^{-1})$, we get (we consider only the more difficult case with the sign $-$ in (2.13))

$$\begin{aligned}|\omega\cdot\ell+\mu_j-\mu_k|&=|\omega\cdot\ell+j-k+\frac{m(k-j)}{2jk}+\mathcal{O}(1/j)|\\&\overset{(2.26)}{\geq}\gamma\langle\ell\rangle^{-\tau}-\mathcal{O}(\langle\ell\rangle/j^2)-\mathcal{O}(1/j)\geq\frac{\gamma}{2}\langle\ell\rangle^{-\tau},\end{aligned} \tag{2.27}$$

noting that $j-k$ is integer and $|j-k|=\mathcal{O}(\langle\ell\rangle)$ (otherwise no small divisors occur). They key point is that the constant $a_+-a_+=0$ cancels out. If j, k have different sign one uses the second condition in (2.26).

KAM for quasi-linear and fully nonlinear PDEs. All the above results still concern semi-linear perturbations. The search of quasi-periodic solutions for quasi-linear PDEs is a very difficult problem and, for a certain time, it has also been thought that the KAM phenomenon would not in general persist due to the possible formation of singularities for any solution, as in the examples of Lax [94] and Klainerman-Majda [102] for wave equations.

The first existence results of quasi-periodic solutions for quasi-linear PDEs have been proved by Baldi-Berti-Montalto in [5] (see [4] for time periodic solutions of Benjamin-Ono) for fully nonlinear perturbations of the Airy equation

$$u_t+u_{xxx}+\varepsilon f(\omega t,x,u,u_x,u_{xx},u_{xxx})=0,\quad x\in\mathbb{T}, \tag{2.28}$$

with the nonlinearity f satisfying some algebraic property, and, in [7,8] for quasi-linear autonomous Hamiltonian perturbed KdV equations like (1.6)-(1.7).

The key point is to reduce to constant coefficients the linear PDE

$$u_t+(1+a_3(\omega t,x))u_{xxx}+a_2(\omega t,x)u_{xx}+a_1(\omega t,x)u_x+a_0(\omega t,x)u=0, \tag{2.29}$$

which is obtained linearizing (2.28) at an approximate quasi-periodic solution $u(\omega t,x)$. The coefficients $a_i(\omega t,x)=\mathcal{O}(\varepsilon)$, $i=0,\cdots,3$. Instead of trying to diminish the size of the variable-dependent terms in (2.29), as in the scheme outlined in Section 2.1–the big difficulty 2 would appear–, we aim to conjugate (2.29) to a system like

$$u_t+m_3u_{xxx}+m_1u_x+\mathcal{R}_0(\omega t)u=0, \tag{2.30}$$

where $m_3=1+\mathcal{O}(\varepsilon)$, $m_1=\mathcal{O}(\varepsilon)$ are constants and $\mathcal{R}_0(\omega t)$ is a zero order operator. For that we conjugate (2.29) with a time quasi-periodic change of variable (as in (2.2))

$$\Phi(\omega t)[v]=v(x+\beta(\varphi,x)), \tag{2.31}$$

induced by the composition with a diffeomorphism $x \mapsto x+\beta(\varphi,x)$ of $\mathbb{T}_x$ (we require $|\beta_x|<1$). The conjugated system (2.3)-(2.4) is

$$v_t+\Phi^{-1}(\omega t)\big((1+a_3(\varphi,x))(1+\beta_x(\varphi,x))^3\big)v_{xxx}(\varphi,x)+\text{lower order terms}=0$$

and therefore we choose a periodic function $\beta(\varphi,x)$ such that $(1+a_3(\varphi,x))(1+\beta_x(\varphi,x))^3=m_3(\varphi)$ is independent of x. Since $\beta_x(\varphi,x)$ has zero space average, this is possible with

$$m_3(\varphi)=\Big(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{dx}{(1+a_3(\varphi,x))^{\frac{1}{3}}}\Big)^{-3}.$$

We can also eliminate the φ dependence of $m_3(\varphi)$ at the highest order using a time quasi-periodic reparametrization and, using other pseudo-differential transformations, we can reduce also the lower order terms to constant coefficients obtaining (2.30). Notice that we do not use a Lie expansion as in (2.8), because it is not an expansion in decreasing orders, see Remark 2.1 (the composition operator (2.31) could be seen as the time one flow of a transport equation with a vector field of order 1).

This is the main device for tackling the quasi-linear nature of the PDE (2.28). The reduction (2.30) implies the accurate asymptotic expansion of the perturbed frequencies $\mu_j=-\mathrm{i}m_3j^3+\mathrm{i}m_1j+\mathcal{O}(\varepsilon)$ and therefore it is possible to verify the second order Melnikov non-resonance conditions required by a KAM reducibility scheme, which completes the diagonalization of (2.30).

For autonomous parameter independent perturbed KdV equations like (1.6)-(1.7) KAM results have been proved in [7,8], and [80], by a weak Birkhoff normal form analysis.

These techniques have been then employed by Feola-Procesi [69] for quasi-linear forced perturbations of Schrödinger equations and in [52, 70] for the search of analytic solutions of autonomous PDEs. We also mention the recent reducibility result [68] for quasi-linear perturbations of the DeGasperis-Procesi equation, which has an asymptotically linear dispersion relation. These kind of ideas have been also successfully generalized for unbounded perturbations of harmonic oscillators by Bambusi [9,10], see also [15] for a reducibility result in higher space dimension. This approach of reducing a linear system to constant coefficients up to regularizing operators enables also to prove upper bounds for the growth of the Sobolev norms of linear Schrödinger equations, see [16,108].

Since KdV and NLS are differential equations, the pseudo-differential tools which are required in [5, 69] are essentially commutators of multiplication operators and Fourier multipliers. On the other hand, for the water waves equations, that we now present, the theory of pseudo-differential operators has to be used in full strength.

2.3 Water waves equations

The water waves equations for a perfect, incompressible, inviscid, irrotational fluid occupying the time dependent region $\mathcal{D}_\eta$ in (1.9), under the action of gravity, and possible capillary forces at the free surface, are the Euler equations of hydrodynamics combined with conditions at the boundary of the fluid:

$$\begin{cases} \partial_t\Phi+\frac{1}{2}|\nabla\Phi|^2+g\eta=\kappa\partial_x\Big(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\Big) & \text{at } y=\eta(t,x),\\ \Delta\Phi=0 & \text{in } \mathcal{D}_\eta,\\ \partial_y\Phi=0 & \text{at } y=-h,\\ \partial_t\eta=\partial_y\Phi-\partial_x\eta\partial_x\Phi_x & \text{at } y=\eta(t,x), \end{cases} \tag{2.32}$$

where g is the acceleration of gravity and κ is the surface tension coefficient. The unknowns of the problem (2.32) are the free surface $y=\eta(t,x)$ and the velocity potential $\Phi:\mathcal{D}_\eta\to\mathbb{R}$, i.e., the irrotational velocity field of the fluid $v=\nabla_{x,y}\Phi$. The first equation in (2.32) is the Bernoulli condition according to which the jump of pressure across the free surface is proportional to the mean curvature. The second equation in (2.32) is the incompressibility property $\mathrm{div} v=0$. The third equation expresses the impermeability of the bottom of the ocean. The last condition in (2.32) means that the fluid particles on the free surface $y=\eta(x,t)$ remain forever on it along the fluid evolution.

Following Zakharov [130] and Craig-Sulem [55], the evolution problem (2.32) may be written as an infinite-dimensional Hamiltonian system in the unknowns $(\eta(t,x),\psi(t,x))$ where $\psi(t,x)=\Phi(t,x,\eta(t,x))$ is, at each instant t, the trace at the free boundary of the velocity potential. Given $\eta(t,x)$ and $\psi(t,x)$ there is a unique solution $\Phi(t,x,y)$ of the elliptic problem

$$\begin{cases} \Delta\Phi=0 & \text{in } \{-h<y<\eta(t,x)\},\\ \partial_y\Phi=0 & \text{on } y=-h,\\ \Phi=\psi & \text{on } \{y=\eta(t,x)\}. \end{cases}$$

System (2.32) is then equivalent to the Zakharov-Craig-Sulem system

$$\begin{cases} \partial_t\eta=G(\eta)\psi,\\ \partial_t\psi=-g\eta-\frac{\psi_x^2}{2}+\frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi+\eta_x\psi_x\big)^2+\kappa\partial_x\Big(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\Big), \end{cases} \tag{2.33}$$

where $G(\eta):=G(\eta;h)$ is the Dirichlet-Neumann operator defined as

$$G(\eta)\psi:=\big(\Phi_y-\eta_x\Phi_x\big)_{|y=\eta(t,x)}, \tag{2.34}$$

which maps the Dirichlet datum ψ into the (normalized) normal derivative $G(\eta)\psi$ at the top boundary. The operator $G(\eta)$ is linear in ψ, self-adjoint with respect to the L^2 scalar product, positive-semidefinite, and its kernel contains only the constant functions. The Dirichlet-Neumann operator depends smoothly with respect to the wave profile η, and it is a pseudo-differential operator with principal symbol $D\tanh(hD)$.

Furthermore the Eqs. (2.33) are the Hamiltonian system

$$\partial_t\eta=\nabla_\psi H(\eta,\psi),\quad \partial_t\psi=-\nabla_\eta H(\eta,\psi), \tag{2.35}$$

where ∇ denotes the L^2-gradient, and the Hamiltonian

$$H(\eta,\psi)=\frac{1}{2}\int_{\mathbb{T}}\psi G(\eta,h)\psi dx+\frac{g}{2}\int_{\mathbb{T}}\eta^2 dx+\kappa\int_{\mathbb{T}}\sqrt{1+\eta_x^2}dx \tag{2.36}$$

is the sum of the kinetic, potential and capillary energies expressed in terms of the variables (η,ψ).

The water waves system (2.33)-(2.35) exhibits several symmetries. First of all, the mass $\int_{\mathbb{T}}\eta(x)dx$ is a first integral of (2.33). In addition, the subspace of functions that are even in x,

$$\eta(x)=\eta(-x),\quad \psi(x)=\psi(-x), \tag{2.37}$$

is invariant under (2.33). In this case also the velocity potential $\Phi(x,y)$ is even and 2π-periodic in x and so the x-component of the velocity field $v=(\Phi_x,\Phi_y)$ vanishes at $x=k\pi$, for all $k\in\mathbb{Z}$. Hence there is no flow of fluid through the lines $x=k\pi$, $k\in\mathbb{Z}$, and a solution of (2.33) satisfying (2.37) describes the motion of a liquid confined between two vertical walls.

Moreover the water waves system (2.33)-(2.35) is reversible with respect to the involution $S:(\eta,\psi)\mapsto(\eta,-\psi)$, i.e., the Hamiltonian H in (2.36) is even in ψ. As a consequence it is natural to look for solutions of (2.33) satisfying

$$u(-t)=Su(t),\quad i.e.,\ \eta(-t,x)=\eta(t,x),\quad \psi(-t,x)=-\psi(t,x),\quad \forall t,x\in\mathbb{R}. \tag{2.38}$$

Solutions of the water waves equations (2.33) satisfying (2.37) and (2.38) are called gravity standing water waves.

The phase space of (2.33) is (a subspace) of

$$(\eta,\psi)\in H_0^1(\mathbb{T})\times\dot{H}^1(\mathbb{T}),\quad \text{where } \dot{H}^1(\mathbb{T}):=H^1(\mathbb{T})/_\sim$$

is the homogeneous space obtained by the equivalence relation $\psi_1(x)\sim\psi_2(x)$ if and only if $\psi_1(x)-\psi_2(x)=c$ is a constant, and $H_0^1(\mathbb{T})$ is the subspace of $H^1(\mathbb{T})$ of zero average functions. For simplicity of notation we denote the equivalence class $[\psi]$ by ψ. Note that the second equation in (2.33) is in $\dot{H}^1(\mathbb{T})$, as it is natural because only the gradient of the velocity potential has a physical meaning.

Linearizing (2.33) at the equilibrium $(\eta,\psi)=(0,0)$, we get

$$\begin{cases}\partial_t\eta=G(0)\psi,\\ \partial_t\psi=-g\eta+\kappa\eta_{xx},\end{cases} \tag{2.39}$$

where $G(0)=D\tanh(hD)=\mathrm{Op}\big(j\tanh(hj)\big)$ is the Dirichlet-Neumann operator at the flat surface $\eta=0$. The linear frequencies of oscillations of (2.39) are

$$\omega_j=\sqrt{j\tanh(hj)(g+\kappa j^2)},\quad j\in\mathbb{Z}\setminus\{0\}. \tag{2.40}$$

Note that, in the phase space of even functions (2.37), the linear frequencies ω_j are simple. Also notice that

$$\text{if } \kappa>0 \text{ (capillary}-\text{gravity waves)} \Longrightarrow \omega_j \sim |j|^{3/2} \quad \text{as } |j|\to\infty,$$
$$\text{if } \kappa=0 \quad \text{(pure gravity waves)} \Longrightarrow \omega_j \sim |j|^{1/2} \quad \text{as } |j|\to\infty.$$

KAM for water waves. The existence of standing water waves is a small divisor problem, which is particularly difficult because (2.33) is a system of quasi-linear PDEs. The existence of small amplitude time-periodic gravity standing wave solutions for bi-dimensional fluids has been first proved by Plotinkov and Toland [113] in finite depth and by Iooss, Plotnikov and Toland in [93] in infinite depth, see also [91], and [92] for 3 dimensional traveling waves. More recently, the existence of time periodic gravity-capillary standing wave solutions in infinite depth has been proved by Alazard-Baldi [1]. Below we present the recent results by Berti-Montalto [30] and Baldi-Berti-Haus-Montalto [6] concerning the existence and the linear stability of time quasi-periodic gravity-capillary [30] and pure gravity [6] standing wave solutions.

Gravity-capillary water waves. Consider the gravity-capillary water waves equations (2.33) with infinite depth $h=+\infty$ (the case with finite depth is similar) and gravity $g=1$. Then the Dirichlet-Neumann operator $G(0)=|D_x|$ and the standing wave solutions of the linearized system (2.39) are

$$\eta(t,x)=\sum\nolimits_{j\geq 1} a_j \cos(\omega_j t)\cos(jx), \tag{2.41a}$$
$$\psi(t,x)=-\sum\nolimits_{j\geq 1} a_j j^{-1}\omega_j \sin(\omega_j t)\cos(jx), \tag{2.41b}$$

with linear frequencies of oscillations (see (2.40))

$$\omega_j := \omega_j(\kappa) := \sqrt{j(1+\kappa j^2)}, \quad j\geq 1. \tag{2.42}$$

The main result in [30] proves that most of the standing wave solutions (2.41) of the linear system (2.39) can be continued to standing wave solutions of the nonlinear water-waves system (2.33) for most values of the surface tension parameter $\kappa\in[\kappa_1,\kappa_2]$.

Theorem 2.1 (KAM for capillary-gravity water waves, see [30]). *For every choice of finitely many tangential sites $\mathbb{S}^+\subset\mathbb{N}^+$ with cardinality $\nu:=|\mathbb{S}^+|$, there exists $\bar{s}>(\nu+1)/2$, $\varepsilon_0\in(0,1)$ such that for every $|\xi|\leq\varepsilon_0^2$, $\xi:=(\xi_j)_{j\in\mathbb{S}^+}$, there exists a Cantor like set $\mathcal{G}\subset[\kappa_1,\kappa_2]$ with asymptotically full measure as $\xi\to 0$, i.e.,*

$$\lim_{\xi\to 0}|\mathcal{G}|=\kappa_2-\kappa_1,$$

such that, for any surface tension coefficient $\kappa\in\mathcal{G}$, the capillary-gravity system (2.33) *has a time quasi-periodic standing wave solution $u(\tilde{\omega}t,x)=(\eta(\tilde{\omega}t,x),\psi(\tilde{\omega}t,x))$, with $(\eta,\psi)(\varphi,x)\in H^{\bar{s}}(\mathbb{T}^\nu\times\mathbb{T},\mathbb{R}^2)$, of the form*

$$\eta(\tilde{\omega}t,x)=\sum\nolimits_{j\in\mathbb{S}^+}\sqrt{\xi_j}\cos(\tilde{\omega}_j t)\cos(jx)+o(\sqrt{|\xi|}), \tag{2.43a}$$
$$\psi(\tilde{\omega},x)=-\sum\nolimits_{j\in\mathbb{S}^+}\sqrt{\xi_j}j^{-1}\omega_j\sin(\tilde{\omega}_j t)\cos(jx)+o(\sqrt{|\xi|}), \tag{2.43b}$$

with a diophantine frequency vector $\tilde{\omega} := \tilde{\omega}(\kappa,\xi) \in \mathbb{R}^\nu$ *satisfying* $\tilde{\omega}_j - \omega_j(\kappa) \to 0$, $j \in \mathbb{S}^+$, *as* $\xi \to 0$. *The terms* $o(\sqrt{|\xi|})$ *are small in the Sobolev space* $H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$. *In addition these quasi-periodic solutions are linearly stable.*

Theorem 2.1 is proved by a Nash-Moser implicit function theorem. A key step is the reducibility to constant coefficients of the quasi-periodic system obtained linearizing (2.33) at a quasi-periodic approximate solution, i.e.,

$$\mathcal{L}_\omega = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V + G(\eta) B & -G(\eta) \\ (1 + BV_x) + BG(\eta) B & V \partial_x - BG(\eta) \end{pmatrix},$$

where $(V,B) = (\Phi_x, \Phi_y)$ is the velocity field evaluated at the free surface $y = \eta(\varphi, x)$. After the introduction of a linearized Alinhac good unknown and using in a systematic way pseudo-differential calculus it is possible to transform $\mathcal{L}_\omega$ into a complex quasi-periodic linear operator of the form

$$(h,\bar{h}) \mapsto \big(\omega \cdot \partial_\varphi + \mathrm{i}\mathtt{m}_3 |D|^{\frac{1}{2}} (1 - \kappa \partial_{xx})^{\frac{1}{2}} + \mathrm{i}\mathtt{m}_1 |D|^{\frac{1}{2}}\big) h + \mathcal{R}(\varphi) h + \mathcal{Q}(\varphi) \bar{h}, \quad h \in \mathbb{C},$$

where $\mathtt{m}_3, \mathtt{m}_1 \in \mathbb{R}$ are constants satisfying $\mathtt{m}_3 \approx 1$, $\mathtt{m}_1 \approx 0$, and the remainders $\mathcal{R}(\varphi)$, $\mathcal{Q}(\varphi)$ are small bounded operators. Then a KAM reducibility scheme completes the diagonalization of the linearized operator $\mathcal{L}_\omega$. The required second order Melnikov non-resonance conditions are fulfilled for most values of the surface tension parameter κ generalizing ideas of degenerate KAM theory for PDEs [12], exploiting that the linear frequencies $\omega_j(\kappa)$ defined in (2.42) are analytic and non degenerate in κ, and the sharp asymptotic expansion of the perturbed frequencies obtained by the regularization procedure.

A complementary result to the KAM Theorem 2.1 is the following almost global existence result in Berti-Delort [28] which proves long time existence of any solution of the gravity-capillary water waves equations with initial data small and smooth enough, as soon as the gravity-capillary parameters (g,κ) avoid a subset of zero measure.

Theorem 2.2 (Almost global existence of capillary-gravity water waves, see [28]). *There is a zero measure subset* $\mathcal{N}$ *in* $[0,+\infty]^2$ *such that, for any* (g,κ) *in* $[0,+\infty]^2 - \mathcal{N}$, *for any* N *in* $\mathbb{N}$, *there is* $s_0 > 0$ *and, for any* $s \geq s_0$, *there are* $\epsilon_0 > 0$, $c > 0$, $C > 0$ *such that, for any* $\epsilon \in [0,\epsilon_0]$, *any even function* (η_0, ψ_0) *in* $H_0^{s+\frac{1}{4}}(\mathbb{T}^1,\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})$ *with*

$$\|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \epsilon,$$

system (2.2) *has a unique classical solution* (η,ψ) *defined on* $[-T_\epsilon, T_\epsilon] \times \mathbb{T}^1$ *with* $T_\epsilon \geq c\epsilon^{-N}$, *belonging to the space*

$$C^0\big([-T_\epsilon, T_\epsilon], H_0^{s+\frac{1}{4}}(\mathbb{T}^1,\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})\big)$$

satisfying the initial condition $\eta|_{t=0} = \eta_0$, $\psi|_{t=0} = \psi_0$. *Moreover, this solution is even in space and it stays at any time in the ball of center 0 and radius* $C\epsilon$ *of* $H_0^{s+\frac{1}{4}}(\mathbb{T}^1,\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}^1,\mathbb{R})$.

The proof is based on a normal form procedure. The normal form idea, which for finite dimensional dynamical systems goes back to Poincaré and Birkhoff, is to make suitable changes of variables which diminish iteratively the size of the nonlinearity, when possible, and to prove that the "resonant" terms which are left in the normal form do not contribute to the growth of the solutions. Here the Hamiltonian or reversible structure of the PDE plays a role. A systematic approach for Hamiltonian semilinear PDEs has been realized in [13, 14, 61], see [87] for reversible PDEs. This approach fails in case of quasi-linear PDEs because the Birkhoff transformations could not be well defined (it is the similar difficulty 2 which appears in the reducibility scheme). For quasi-linear Hamiltonian wave equations long time existence results have been proved in Delort [59,60]. The strategy in [28] is to conjugate (2.33) to a constant coefficient paradifferential nonlinear system

$$u_t = \mathrm{i}(m_3(u)|D|^{\frac{1}{2}}(1-\kappa\partial_{xx})^{\frac{1}{2}} + m_1(u)|D|^{1/2} + r_0(u;D))u + R(u)u,$$

where $m_3(u)$, $m_1(u)$ are real valued functions independent of x (they are nonlinear functions in u), r_0 is a symbol of order zero, constant in x, and $R(u)$ is smoothing nonlinear remainder. Then we may start a semi-linear normal form procedure to reduce the size of the nonlinearity. The loss of derivatives produced by the small divisors is compensated by the paradifferential structure and the smoothing nature of the remainders. The reversible nature of the water waves equations–which is preserved by the paradifferential transformations- allows to prove that the resonant terms do not contribute to the growth of the Sobolev norms of the solutions.

The assumption that the parameters (g,κ) avoid a subset of zero measure is essential: for some values of (g,κ) resonances may occur yet at the level of the quadratic part of the non-linearity (the so called "Wilton ripples) or for higher order terms in the Taylor expansion of the non-linearity, causing dynamically unstable motions, see e.g., Craig and Sulem [56].

What happens to a solution which does not start on a KAM torus proved in Theorem 2.1 for times longer that those covered by the almost global Theorem 2.2? We do not know. Are there initial data for which the Sobolev norms of the solutions grow in time? We do not know. This "norm inflation" chaotic phenomenon would be the analogue of the Arnold diffusion problem for finite dimensional Hamiltonian systems. For the cubic NLS equation (2.22) on $\mathbb{T}^2$ some unstable orbits have been constructed in Colliander-Keel-Staffilani-Takaoka-Tao [50], Guardia-Kaloshin [81], and in Guardia-Haus-Procesi [86] for the Eq. (2.23).

Pure gravity water waves. In the case of pure gravity water waves, i.e., $\kappa=0$, the linear frequencies of oscillation are (see (2.40))

$$\omega_j := \omega_j(h) := \sqrt{gj\tanh(hj)}, \quad j \geq 1, \tag{2.44}$$

and three major further difficulties in proving the existence of time quasi-periodic solutions are:

(i) The nonlinear water waves system (2.33) (with $\kappa=0$) is a singular perturbation of (2.39) (with $\kappa=0$) in the sense that the linearized operator assumes the form

$$\omega\cdot\partial_\varphi+\mathrm{i}|D_x|^{\frac{1}{2}}\tanh^{\frac{1}{2}}(h|D_x|)+V(\varphi,x)\partial_x$$

and the term $V\partial_x$ is now a *singular* perturbation of the linear dispersion relation operator $\mathrm{i}|D_x|^{\frac{1}{2}}\tanh^{\frac{1}{2}}(h|D_x|)$ (on the contrary, for the gravity-capillary case the transport term $V\partial_x$ is a lower order perturbation of $|D_x|^{\frac{3}{2}}$).

(ii) The dispersion relation (2.44) is sublinear, i.e., $\omega_j\sim\sqrt{j}$ for $j\to\infty$, and therefore it is only possible to impose Melnikov non-resonance conditions which lose space derivatives.

(iii) The linear frequencies $\omega_j(h)$ in (2.44) vary with h of just exponentially small quantities.

The main result in Baldi-Berti-Haus-Montalto [6] proves the existence of pure gravity standing water waves solutions (a statement parallel to Theorem 2.1).

The difficulty (i) is solved proving a straightening theorem for a quasi-periodic transport operator: there is a quasi-periodic change of variables of the form $x\mapsto x+\beta(\omega t,x)$ which conjugates

$$\omega\cdot\partial_\varphi+V(\varphi,x)\partial_x$$

to the constant coefficient vector field $\omega\cdot\partial_\varphi$, for $V(\varphi,x)$ small. This perturbative rectification result is a classical small divisor problem, solved for perturbations of a Diophantine vector field at the beginning of KAM theory, see e.g., [131]. Notice that, despite the fact that $\omega\in\mathbb{R}^\nu$ is Diophantine, the constant vector field $\omega\cdot\partial_\varphi$ is resonant on the higher dimensional torus $\mathbb{T}^\nu_\varphi\times\mathbb{T}_x$. We exploit in a crucial way the symmetry induced by the reversible structure of the water waves equations, i.e., $V(\varphi,x)$ is odd in φ. This problem amounts to prove that all the solutions of the quasi periodically time-dependent scalar characteristic equation $\dot{x}=V(\omega t,x)$ are quasi-periodic in time with frequency ω.

The difficulty (ii) is overcome performing a regularizing procedure which conjugates the linearized operator, obtained along the Nash-Moser iteration, to a diagonal, constant coefficient linear one, up to a sufficiently smoothing operator. In this way the subsequent KAM reducibility scheme converges also in presence of very weak Melnikov non-resonance conditions which lose space derivatives. This regularization strategy is in principle applicable to a broad class of PDEs where the second order Melnikov non-resonance conditions lose space derivatives (this is always the case if the dispersion relation has a sublinear growth).

The difficulty (iii) is based on an improvement of the degenerate KAM theory for PDEs in Bambusi-Berti-Magistrelli [12], which allows to prove that the Melnikov non-resonance are fulfilled for most values of h.

Remark 2.2. We can introduce the space wavelength $2\pi\varsigma$ as an internal free parameter in the water waves equations (2.32), (2.33). Rescaling properly time, space and amplitude of the solution $(\eta(t,x),\psi(t,x))$ we obtain system (2.33) where the gravity constant g has been replaced by 1 and the depth parameter h depends linearly on the parameter ς, see [6]. In this way [6] proves existence results for a fixed equation, i.e., a fixed depth h, for most values of ς.

2.4 KAM without second order Melnikov

In order to overcome the difficulty posed by the presence of multiple normal frequencies Craig and Wayne [53, 57], proposed a novel approach, later largely extended by Bourgain [38, 41, 42], in which the linearized equations obtained at each step of the Newton iteration are not completely diagonalized. The main difficulty is that the linear operators (see (2.45)) have variable (quasi-periodic in time) coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues. Relying on a "resolvent" type analysis inspired to the work of Frölich-Spencer [88] in the context of Anderson localization, Craig-Wayne [57] solved this problem for periodic solutions in $d=1$, and Bourgain in [35] also for quasi-periodic solutions.

This approach is particularly inspiring for PDEs in higher space dimension.

Actually the first existence result of periodic solutions for the semilinear wave equation

$$y_{tt}-\Delta y+my=y^3+\text{h.o.t.},\quad x\in\mathbb{T}^d,$$

has been proved by Bourgain in [36] extending the Craig-Wayne approach [57]. Further existence results of periodic solutions have been proved in Berti-Bolle [21] for merely differentiable nonlinearities, Berti-Bolle-Procesi [26] for Zoll manifolds, Gentile-Procesi [79] using Lindstedt series techniques, and Delort [58] for NLS using paradifferential calculus.

The main goal is to prove that the linearized operators

$$(\omega\cdot\partial_\varphi)^2-\Delta_{xx}+m+\varepsilon b(x,\varphi) \tag{2.45}$$

obtained at each step of the Newton iteration are invertible, for most values of some parameter, and that their inverses satisfy tame estimates in high norms. It is necessary to require lower bounds for the eigenvalues of the self-adjoint non-diagonal matrices

$$\Pi_N\big((\omega\cdot\partial_\varphi)^2-\Delta_{xx}+m+\varepsilon b(x,\varphi)\big)\Pi_N, \tag{2.46}$$

where Π_N denotes the projection on the finite dimensional subspaces

$$E_N:=\Big\{h=\sum_{|(\ell,j)|\le N}h_{\ell j}e^{\mathrm{i}(\ell\cdot\varphi+jx)},\ \ell\in\mathbb{Z}^\nu,\ j\in\mathbb{Z}^d\Big\}$$

for large N. These "first order Melnikov" non-resonance conditions are essentially the minimal assumptions for proving the persistence of the KAM tori, and can be verified, also in presence of degenerate eigenvalues, by the self-adjointness of the matrix (2.46). This provides estimates of the inverse of the matrix (2.46) in L^2 norm. In order to prove estimates of the inverse in high norms, an essential ingredient is that the "singular" sites

$$(\ell,j)\in\mathbb{Z}^\nu\times\mathbb{Z}^d \quad \text{such that} \quad |-(\omega\cdot\ell)^2+|j|^2|\ll 1$$

are separated into clusters (otherwise the inverse operators would be so unbounded to prevent the convergence of the Nash-Moser scheme).

Bourgain [38] (in $d=2$) and in [41] (in any $d\geq 2$) succeeded in proving the existence of quasi-periodic solutions for analytic Hamiltonian equations like

$$\frac{1}{\mathrm{i}}u_t=Bu+\varepsilon\partial_{\bar{u}}H(u,\bar{u}), \quad y_{tt}+By+\varepsilon F'(y)=0, \quad x\in\mathbb{T}^d, \tag{2.47}$$

where $B:=\mathrm{Op}(\mu_j)_{j\in\mathbb{Z}^d}$ is the Fourier multiplier operator with

$$\mu_j:=\omega_j, \quad j\in\mathbb{S}, \quad \mu_j:=|j|^2, \quad j\in\mathbb{Z}^d\backslash\mathbb{S},$$

(thus $B=-\Delta$ restricted to the normal sites). The finitely many $\omega_j\in\mathbb{R}$, $j\in\mathbb{S}$ (= tangential sites), play the role of external parameters used for the measure estimates. Notice that the singular sites

$$|-(\omega\cdot\ell)^2+|j|^2|\ll 1 \ \ \text{(for NLW)}, \quad |\omega\cdot\ell+|j|^2|\ll 1 \ \ \text{(for NLS)}, \tag{2.48}$$

are integer vectors close to a "cone" for NLW, and a "paraboloid" for NLS. The NLW case is more difficult because the cone has flat directions which may contain lines of integers. That is why ω is required to satisfy the quadratic diophantine condition

$$\Big|\sum_{1\leq i\leq j\leq\nu}\omega_i\omega_j p_{ij}\Big|\geq\frac{\gamma_0}{|p|^{\tau_0}}, \quad \forall p\in\mathbb{Z}^{\frac{\nu(\nu+1)}{2}}\setminus\{0\},$$

which is an irrationality condition on the slopes of the "singular cone" in (2.48), see [133] to avoid this condition. For NLS this condition is not required. We also notice that for periodic solutions ($\nu=1$) the clusters of singular sites are more and more separated at infinity, unlike the case of quasi-periodic solutions ($\nu\geq 2$).

Bourgain's technique is a "multiscale" inductive analysis based on the repeated use of the "resolvent identity" which uses tools of sub-harmonic analysis previous developed for quasi-periodic Anderson localization theory to perform delicate measure estimates, see Bourgain-Goldstein-Schlag [42, 43].

Existence of quasi-periodic solutions for the completely resonant NLS equation (2.23) has been proved by Wang [126] using a Lyapunov-Schmidt decomposition and a normal form analysis to prove an effective nonlinear dependence of the perturbed frequencies with respect to the amplitudes. We also remark that existence of quasi-periodic solutions under a random perturbation has been proved in [44, 45]. The stochastic case is easier than the deterministic one because it is simpler to verify the non-resonance conditions with a random variable.

In the last years Berti-Bolle [22, 23] proved the existence of quasi-periodic solutions for quasi-periodically forced Hamiltonian semilinear wave equations

$$y_{tt}-\Delta y+V(x)y=\varepsilon f(\omega t,x,y), \quad x\in\mathbb{T}^d, \tag{2.49}$$

and Schrödinger equations

$$\mathrm{i}u_t-\Delta u+V(x)u=\varepsilon\partial_{\bar u}H(\omega t,x,u), \quad x\in\mathbb{T}^d, \tag{2.50}$$

where $H(\varphi,x,u)\in\mathbb{R}$, $\forall u\in\mathbb{C}$, and the multiplicative potential $V(x)$, as well as the nonlinearity f and the Hamiltonian H, are C^q for some $q\in\mathbb{N}$ large enough. The diophantine frequency vector $\omega\in\mathbb{R}^\nu$ is used as an external parameter. Actually it is possible to prescribe its direction

$$\omega=\lambda\bar\omega, \quad \lambda\in\Lambda:=[1/2,3/2],$$

and it is sufficient to use just the 1-dimensional parameter λ, i.e., a time-rescaling, to verify all the required non-resonance conditions. We also remark that the approach developed in [22, 23] never exploits properties of "localizations" of the eigenfunctions of $-\Delta+V(x)$ with respect to the exponentials, that actually might not be true. The main reason is that the Nash-Moser proof in [22, 23] makes use of the weakest possible tame estimates, required for the inverse linearized operators, in order to get the convergence of the iterative scheme in a scale of Sobolev functions. Such conditions are optimal, as a famous counterexample of Lojiaciewitz-Zehnder in [97] proves.

In [25], we look for the existence of quasi-periodic solutions for semilinear autonomous wave equations with a multiplicative potential

$$y_{tt}-\Delta y+V(x)y=a(x)y^3+\text{h.o.t.}, \quad x\in\mathbb{T}^d, \quad d\geq 2. \tag{2.51}$$

We use the symplectic construction developed in [24] which reduces the problem of constructing an approximate inverse of the linearized operator in action-angle and normal variables (where all the components are coupled) to the search of an approximate inverse of a quasi-periodically forced linear operator restricted to the normal directions. Then we perform the multiscale analysis in the exponential basis in the spirit of [22, 23]. On the other hand the main part of the quasi-periodic solutions of (2.51) is Fourier supported on finitely many eigenfunctions of $\psi_j(x)$ of $-\Delta+V(x)$, see (1.14), and so this basis has to be used to describe the bifurcation of the quasi-periodic solutions of (2.51). For generic choices of $V(x)$ and $a(x)$ we prove that it is possible to verify all the Melnikov non-resonance conditions required by the iterative scheme.

We also mention the recent work of Wang [127] for the autonomous, parameter independent, Klein-Gordon equation

$$y_{tt}-\Delta y+y+y^{p+1}=0, \quad p\in\mathbb{N}, \quad x\in\mathbb{T}^d.$$

A key step is a Birkhoff normal form type analysis of the frequency-amplitude modulation.

Remark 2.3. These results prove the existence of a quasi-periodic solution, not of a KAM constant coefficient Hamiltonian normal form around it. However, it is proved in [24] that, in a neighborhood of any quasi-periodic solution of a Hamiltonian PDE, there exists a set of symplectic coordinates in which the Hamiltonian is in variable coefficients KAM normal form.

A very interesting problem is to understand if these KAM results hold for more general compact manifolds, not only the torus. At present, the most general class of manifolds is considered in Berti-Corsi-Procesi [27] for forced NLW and NLS equations when

$$x \in \mathtt{M} = \text{compact Lie Group or Homogeneous Manifold},$$

namely there exists a compact Lie group which acts on $\mathtt{M}$ transitively and differentiably. Existence of time periodic solutions had been previously proved in [31]. Examples of compact connected Lie groups are, in addition to the standard torus $\mathbb{T}^d$, the special orthogonal group $SO(d)$, the special unitary group $SU(d)$. Examples of (compact) manifolds, homogeneous with respect to a compact Lie group, are the spheres $\mathbb{S}^d$, the real and complex Grassmannians, and the moving frames, namely, the manifold of the k-ples of orthonormal vectors in $\mathbb{R}^d$ with the natural action of the orthogonal group $O(d)$. The proofs in [27, 31] use only weak properties of "localization" of the eigenfunctions which follow by abstract properties of Lie groups.

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