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A Note on Weak Type $(1,1)$ Estimate for the Higher Order Commutators of Christ-Journé Type

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Abstract. In this paper, a weak type $(1,1)$ estimate is established for the higher order commutator introduced by Christ and Journé which is defined by

$$T[a_1,\cdots,a_l]f(x)=\text{p.v.}\int_{\mathbb{R}^d}K(x-y)\Big(\prod_{i=1}^{l}m_{x,y}a_i\Big)\cdot f(y)dy,$$

where K is the standard Calderón-Zygmund convolution kernel on $\mathbb{R}^d (d\geq 2)$ and $m_{x,y}a_i=\int_0^1 a_i(sx+(1-s)y)ds$.

Key Words: Weak type $(1,1)$, higher order, commutator.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

Suppose that K is the standard Calderón-Zygmund convolution kernel on $\mathbb{R}^d\setminus\{0\}$, $(d\geq 2)$, which means that K satisfies the following conditions:

$$|K(x)|\leq C|x|^{-d},\quad \int_{R<|x|<2R}K(x)dx=0\quad \text{holds for all } R>0, \tag{1.1a}$$

$$|K(x-y)-K(x)|\leq C|y|^{\delta}|x|^{-d-\delta}\quad \text{for some } 0<\delta\leq 1 \text{ if } |x|>2|y|. \tag{1.1b}$$

In 1987, Christ and Journé [5] introduced a higher dimensional commutator associated with K and $a_i\in L^{\infty}(\mathbb{R}^d)$ $(i=1,\cdots,l)$ by

$$T[a_1,\cdots,a_l]f(x)=\text{p.v.}\int_{\mathbb{R}^d}K(x-y)\Big(\prod_{i=1}^{l}m_{x,y}a_i\Big)\cdot f(y)dy,\quad f\in\mathcal{S}(\mathbb{R}^d),$$

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where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class and

$$m_{x,y}a_i = \int_0^1 a_i((1-t)x+ty)dt = \int_0^1 a_i(tx+(1-t)y)dt.$$

Note that $T[a_1,\cdots,a_l]f(x)$ can be seen as a higher dimensional generalization of the following commutator

$$\text{p.v.}\int_{\mathbb{R}}\prod_{i=1}^{l}\Big(\frac{A_i(x)-A_i(y)}{x-y}\Big)\frac{f(y)}{x-y}dy,$$

which is the famous Calderón commutator discussed in [3] and is related to the study of the Cauchy integral, boundary value problem of elliptic equation on non-smooth domain (see e.g., [4, 10, 15]).

Observe that the kernel $K(x-y)$ is smooth but $m_{x,y}a_i$ has no smoothness about variable x and y if $a_i \in L^\infty(\mathbb{R}^d)$. Therefore the standard Calderón-Zygmund theory cannot be applied directly. Christ and Journé [5] proved that $T[a_1,\cdots,a_l]$ is bounded on $L^p(\mathbb{R}^d)$ $(1<p<\infty)$ when $a_i \in L^\infty(\mathbb{R}^d)$ $(i=1,\cdots,l)$. In 1995, Hofmann [14] gave the weighted $L^p(\mathbb{R}^d)$ $(1<p<\infty)$ boundedness of $T[a_1,\cdots,a_l]$, when the kernel $K(x)=\Omega(x/|x|)|x|^{-d}$. Recently, there are renew interests on this singular integral of Christ-Journé type since it has some direct applications in the mixing flows problem (see e.g., [2, 13]). In 2015, A. Seeger, C. Smart and B. Street [19] further studied the commutator of Christ-Journé type and established some multilinear estimates. Later, the second author of the present paper established all multilinear estimates of the higher Calderón commutator (see [16]). For the endpoint case $p=1$, the weak type $(1,1)$ estimate seems to be difficulty and the previous result is only known for the first order commutator. In 2012, Grafakos and Honzík [12] proved that the commutator $T[a]$ is of weak type $(1,1)$ for $d=2$. Later, Seeger [18] showed that $T[a]$ is also of weak type $(1,1)$ for all $d\geq 2$. In [6], the authors established weighted L^p boundedness of $T[a]$ for A_p weight with $d\geq 2$ and weighted weak type $(1,1)$ boundedness for power weight $|x|^\alpha(-2<\alpha<0)$ with $d=2$ (later we extended this result to general A_1 weight for all $d\geq 2$ in [8]). However, the weak type $(1,1)$ estimate for the higher order commutator seems to be unexplored and may be very difficult since the kernel involves with more than two rough factors $\prod_{i=1}^{l} m_{x,y}a_i$ under the condition that all $a_i \in L^\infty(\mathbb{R}^d)(i=1,\cdots,l)$. In this paper, we try to give a weak type $(1,1)$ estimate for $T[a_1,\cdots,a_l]$ with some restricted condition of a_i. Our main result is as follows.

Theorem 1.1. *Suppose K satisfies* (1.1a) *and* (1.1b) *for $d\geq 2$. Let $a_1 \in L^\infty(\mathbb{R}^d)$. Assume $a_i, \widehat{a_i} \in L^1(\mathbb{R}^d)$, $i=2,\cdots,l$. Then there exists a constant $C>0$ such that*

$$m(\{x\in\mathbb{R}^d : |T[a_1,\cdots,a_l]f(x)|>\lambda\}) \leq C\lambda^{-1}\|a_1\|_\infty\Big(\prod_{i=2}^{l}\|\widehat{a_i}\|_1\Big)\|f\|_1$$

for all $\lambda>0$ and $f\in L^1(\mathbb{R}^d)$.

Remark 1.1. By using Fourier inversive formula, it is easy to see that the condition a, $\hat{a}\in L^1(\mathbb{R}^d)$ implies $a\in L^\infty(\mathbb{R}^d)$. To remove this kind of restricted condition, some new ideas may be needed.

When the dimension $d=2$, Grafakos and Honzík [12] used the TT^* method to show that the first order commutator $T[a]$ is of weak type $(1,1)$. In [18], Seeger used the microlocal decomposition of the kernel and the Littlewood-Paley decomposition of a. However it seems to be difficult to use these ideas from [12] and [18] to deal with the higher order commutator which involves more than two rough factors. In the present paper, we add some restricted condition $\hat{a}_i\in L^1(\mathbb{R}^d)$ for $i=2,\cdots,l$. So we can make a modified Calderón-Zygmund decomposition of a function with some parameters come from $a_2,\cdots,a_l$, but those bounds in the Calderón-Zygmund decomposition are independent of these parameters (see Lemma 2.1). Applying this kind of Calderón-Zygmund decomposition, the kernel essentially has only one rough kernel under the restricted condition $\hat{a}_i\in L^1(\mathbb{R}^d)$ for $i=2,\cdots,l$. Then using some idea from [7, 9, 17, 18], we may get the weak type $(1,1)$ bound for the higher order commutator.

This paper is organized as follows. In Section 2, we complete the proof of Theorem 1.1 based on some lemmas, their proofs will be given in Section 3 and Section 4. Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. $A\lesssim B$ means $A\leq CB$ for some constant C. $A\approx B$ means that $A\lesssim B$ and $B\lesssim A$. For a set $E\subset\mathbb{R}^d$, we denote Lebesgue measure of E by $|E|$ or $m(E)$. Denote by $\mathcal{F}f$ and $\hat{f}$ the Fourier transform of f, which is defined by

$$\mathcal{F}f(\xi)=\int_{\mathbb{R}^d}e^{-i\langle x,\xi\rangle}f(x)dx.$$

$\mathbb{Z}_+$ denote the set of all nonnegative integers and $\mathbb{Z}_+^d=\mathbb{Z}_+\times\cdots\times\mathbb{Z}_+$. $[x]$ denotes the integer part of x.

2 Proof of Theorem 1.1: setup

In this section we give the proof of Theorem 1.1 based on some lemmas, their proofs will be given in Section 3 and Section 4, respectively.

Using the inversive Fourier formula under the condition that $a_i,\widehat{a_i}\in L^1(\mathbb{R}^d)$, $i=2,\cdots,l$, we write each term

$$m_{x,y}a_i=\frac{1}{(2\pi)^d}\int_0^1\int_{\mathbb{R}^d}\widehat{a_i}(\eta_i)e^{is_i\langle\eta_i,x\rangle}e^{i(1-s_i)\langle y,\eta_i\rangle}d\eta_i ds_i.$$

Therefore by Fubini's theorem, we have

$$\begin{aligned} T[a_1,\cdots,a_l]f(x) &= \frac{1}{(2\pi)^{(l-1)d}}\text{p.v.}\int_{\mathbb{R}^d} K(x-y)m_{x,y}a_1\cdot f(y) \\ &\quad\times\Big(\iint_{[0,1]^{l-1}\times(\mathbb{R}^d)^{l-1}}\prod_{i=2}^{l}\widehat{a_i}(\eta_i)e^{is_i\langle x,\eta_i\rangle}e^{i(1-s_i)\langle y,\eta_i\rangle}d\vec{s}d\vec{\eta}\Big)dy \\ &= \iint_{E^l} u^{\vec{\eta},\vec{s}}(x)(T[a_1]W^{\vec{\eta},\vec{s}}f)(x)d\vec{\eta}d\vec{s}, \end{aligned} \tag{2.1}$$

where

$$E^l=[0,1]^{l-1}\times(\mathbb{R}^d)^{l-1}, \qquad u^{\vec{\eta},\vec{s}}(x)=(2\pi)^{-(l-1)d}\prod_{i=2}^{l}\widehat{a_i}(\eta_i)e^{is_i\langle x,\eta_i\rangle},$$

$$d\vec{\eta}=d\eta_2\cdots d\eta_l, \qquad W^{\vec{\eta},\vec{s}}(y)=\prod_{i=2}^{l}e^{i(1-s_i)\langle y,\eta_i\rangle} \quad\text{and}\quad d\vec{s}=ds_2\cdots ds_n.$$

In the following, we try to make a Calderón-Zygmund decomposition of $W^{\vec{\eta},\vec{s}}f$ with the underlying cubes independent of $\vec{\eta},\vec{s}$.

Lemma 2.1. *Let $f\in L^1(\mathbb{R}^d)$ and $\lambda>0$. Set $\Omega_\lambda=\{x\in\mathbb{R}^d: M(f)(x)>\lambda\}$ where M is the Hardy-Littlewood maximal operator. Then we have the following conclusions:*

(i) $\Omega_\lambda=\bigcup Q$, Q's are disjoint dyadic cubes. Let $\mathcal{Q}$ be the collection of all these cubes.

(ii) $m(\Omega_\lambda)\lesssim\frac{1}{\lambda}\|f\|_1$.

(iii) $fW^{\vec{\eta},\vec{s}}=g^{\vec{\eta},\vec{s}}+b^{\vec{\eta},\vec{s}}$.

(iv) $b^{\vec{\eta},\vec{s}}=\sum_Q b_Q^{\vec{\eta},\vec{s}}$, $\text{supp}b_Q^{\vec{\eta},\vec{s}}\subset Q$, $\int b_Q^{\vec{\eta},\vec{s}}=0$, $\|b_Q^{\vec{\eta},\vec{s}}\|_1\lesssim\lambda|Q|$, $\|b^{\vec{\eta},\vec{s}}\|_1\lesssim\|f\|_1$.

(v) $\|g^{\vec{\eta},\vec{s}}\|_2^2\lesssim\lambda\|f\|_1$.

All the explicit constants that appear in (i)-(v) above are independent of $\vec{\eta},\vec{s}$.

Proof. We first make a Whitney decomposition of the set Ω_λ. Then there exists a family of dyadic closed cubes $\{Q_j\}_j$ (e.g., see [11]) such that

(a) $\Omega_\lambda=\bigcup Q_j$ and Q_j's have disjoint interior.

(b) $\sqrt{d}\cdot l(Q_j)\le dist(Q_j,\Omega_\lambda^c)\le 4\sqrt{d}\cdot l(Q_j)$, where $l(Q_j)$ denotes the side's length of Q_j.

By the weak type $(1,1)$ of M, we have

$$m(\Omega_\lambda)\lesssim\frac{1}{\lambda}\|f\|_1. \tag{2.2}$$

We write $fW^{\vec{\eta},\vec{s}}=g^{\vec{\eta},\vec{s}}+b^{\vec{\eta},\vec{s}}$, where

$$g^{\vec{\eta},\vec{s}}=fW^{\vec{\eta},\vec{s}}\chi_{\Omega_\lambda^c}+\sum_Q\Big(\frac{1}{|Q|}\int_Q f(x)W^{\vec{\eta},\vec{s}}(x)dx\Big)\chi_Q,$$

$$b^{\vec{\eta},\vec{s}}=\sum_Q\Big\{fW^{\vec{\eta},\vec{s}}-\frac{1}{|Q|}\int_Q f(x)W^{\vec{\eta},\vec{s}}(x)dx\Big\}\chi_Q=:\sum_Q b_Q^{\vec{\eta},\vec{s}}.$$

So, $b_Q^{\vec{\eta},\vec{s}}$ is supported in Q and $\int b_Q^{\vec{\eta},\vec{s}}=0$. Let tQ denote the cube with t times the side length of Q and the same center. We now claim that

$$\frac{1}{|Q|}\int_Q |f(x)|dx \lesssim \lambda. \tag{2.3}$$

In fact, by the Whitney decomposition's property (b) we have $9\sqrt{d}Q\cap\Omega_\lambda^c\neq\emptyset$. Thus by the definition of Ω_λ^c, there exists $x_0\in 9\sqrt{d}Q$ such that $Mf(x_0)\leq\lambda$. Using the property of the maximal function, we have $\frac{1}{|9\sqrt{d}Q|}\int_{9\sqrt{d}Q}|f(x)|dx\lesssim\lambda$. Hence we conclude that

$$\frac{1}{|Q|}\int_Q |f(x)|dx \leq \frac{1}{|Q|}\int_{9\sqrt{d}Q}|f(x)|dx\lesssim\lambda.$$

For $b_Q^{\vec{\eta},\vec{s}}$ and $b^{\vec{\eta},\vec{s}}$, by (2.2) and (2.3) we have

$$\|b_Q^{\vec{\eta},\vec{s}}\|_1\leq 2\int_Q|f(x)|dx\lesssim\lambda|Q|,\quad \|b^{\vec{\eta},\vec{s}}\|_1\lesssim\|f\|_1+\lambda m(\Omega_\lambda)\lesssim\|f\|_1.$$

Note that $|f(x)|\leq\lambda$ almost everywhere in Ω_λ^c. Using this fact, (2.2) and (2.3), we have

$$\|g^{\vec{\eta},\vec{s}}\|_2^2\lesssim\lambda\|f\|_1+\lambda^2 m(\Omega_\lambda)\lesssim\lambda\|f\|_1.$$

Thus, we complete the proof. □

Now we set up the proof of Theorem 1.1 with a series of lemmas. We only focus on dimension $d\geq 2$. By using scaling arguments, we may assume $\|a_1\|_\infty=\|\widehat{a_i}\|_1=1$, $i=2,\cdots,l$.

By (2.1) and the property (iii) in Lemma 2.1, we obtain

$$\begin{aligned}\big|\{x:|T[a_1,\cdots,a_l]f(x)|>\lambda\}\big|\leq& m\Big(\Big\{x:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]g^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big)\\&+m\Big(\Big\{x:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]b^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big).\end{aligned}$$

Hence, using Chebyshev's inequality, the fact $T[a_1]$ is bounded on $L^2(\mathbb{R}^d)$ with bound $C\|a_1\|_\infty$ (see [5]) and the property (iv) in Lemma 2.1, we get

$$\begin{aligned}&\Big|\Big\{x\in\mathbb{R}^d:\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]g^{\vec{\eta},\vec{s}}(x)d\vec{\eta}d\vec{s}|>\lambda/2\Big\}\Big|\\\lesssim&\lambda^{-2}\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]g^{\vec{\eta},\vec{s}}(x)d\vec{\eta}d\vec{s}\Big\|_2^2\\\lesssim&\lambda^{-2}\Big(\iint_{E^l}\Big(\prod_{i=2}^l|\widehat{a_i(\eta_i)}|\Big)\|T[a_1]g^{\vec{\eta},\vec{s}}(x)\|_2d\vec{\eta}d\vec{s}\Big)^2\\\lesssim&\lambda^{-2}\Big(\iint_{E^l}\Big(\prod_{i=2}^l|\widehat{a_i(\eta_i)}|\Big)\|a_1\|_\infty\|g^{\vec{\eta},\vec{s}}\|_2d\vec{\eta}d\vec{s}\Big)^2\\\lesssim&\lambda^{-1}\|f\|_1.\end{aligned}$$

Define $E^*=\bigcup_{Q\in\mathcal{Q}}2^{200}Q$. Then we have

$$\begin{aligned}&m\Big(\Big\{x:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]b^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big)\\ \le& m(E^*)+m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]b^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big).\end{aligned}$$

By the property (ii) in Lemma 2.1, the set E^* satisfies

$$m(E^*)\lesssim m(\Omega_\lambda)\lesssim\lambda^{-1}\|f\|_1.$$

Thus, to finish the proof of Theorem 1.1, it suffices to show

$$m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]b^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big)\lesssim\frac{\|f\|_1}{\lambda}.\tag{2.4}$$

Denote $\mathfrak{Q}_k=\{Q\in\mathcal{Q}:l(Q)=2^k\}$ and let $\mathfrak{B}_k^{\vec{\eta},\vec{s}}=\sum_{Q\in\mathfrak{Q}_k}b_Q^{\vec{\eta},\vec{s}}$. Then $b^{\vec{\eta},\vec{s}}$ can be rewritten as $b^{\vec{\eta},\vec{s}}=\sum_{j\in\mathbb{Z}}\mathfrak{B}_j^{\vec{\eta},\vec{s}}$. Let ψ be a radial C^∞ function such that $\psi(\xi)=1$ for $|\xi|\le1$, $\psi(\xi)=0$ for $|\xi|\ge2$ and $0\le\psi(\xi)\le1$ for all $\xi\in\mathbb{R}^d$. Define $\phi(x)=\psi(x)-\psi(2x)$. Then supp $\phi\subset\{x:\frac12\le|x|\le2\}$ and $\sum_j\phi_j(x)=1$ for all $x\in\mathbb{R}^d\backslash\{0\}$, where $\phi_j(x)=\phi(2^{-j}x)$. Now we define the operator $T_j[a_1]$ as

$$T_j[a_1]f(x)=\int_{\mathbb{R}^d}\phi_j(x-y)K(x-y)m_{x,y}a_1\cdot f(y)dy.\tag{2.5}$$

Then $T[a_1]=\sum_j T_j[a_1]$. For simplicity, we set $K_j(x)=\phi_j(x)K(x)$. We write

$$T[a_1]b^{\vec{\eta},\vec{s}}(x)=\sum_{n\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}T_j[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}.$$

Note that $T_j[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)=0$ for $x\in(E^*)^c$ and $n<100$. Therefore we only consider $n\ge100$. Here we should point out that the number 100 is not very important. In fact, it is sufficient to consider large n once we choose the set E^* as large as we want. Write

$$\begin{aligned}&m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)T[a_1]b^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big)\\ =&m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge100}\sum_{j\in\mathbb{Z}}T_j[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big).\end{aligned}$$

Hence, to finish the proof of of Theorem 1.1, it suffices to verify the following estimate:

$$m\Big(\Big\{x\in\mathbb{R}^d:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge100}\sum_{j\in\mathbb{Z}}T_j[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)d\vec{\eta}d\vec{s}\Big|>\lambda/2\Big\}\Big)\lesssim\frac{\|f\|_1}{\lambda}.\tag{2.6}$$

2.1 Some key estimates

Some important estimates play a key role in the proof of (2.6). We present them by some lemmas, which will be proved in Section 3 and Section 4. The first estimate tells us that the operator $T_j[a_1]$ can be approximated by an operator $T_j^n[a_1]$ in measure, which is defined below.

Let $l_\tau(n)=\tau\delta^{-1}n+2$, where $\tau>\frac{1}{n}>0$ and $0<\tau\delta^{-1}<1$ will be chosen later. As we mention before, we only need to consider sufficient larger n, so the constant τ could be chosen as small as want and it will be chosen at the end of this paper. Let η be a nonnegative, radial C^∞ function which is supported in $\{|x|\le 1\}$ and satisfies $\int_{\mathbb{R}^d}\eta(x)dx=1$. Set $\eta_i(x)=2^{-id}\eta(2^{-i}x)$. Define the operator P_t by $P_tf(x)=\eta_t*f(x)$. Set

$$K_j^n(x)=P_{j-l_\tau(n)}K_j(x).$$

Since $K_j(x)$ is supported in $\{2^{j-1}\le|x|\le 2^{j+1}\}$ and $\eta_{j-l_\tau(n)}(x)$ is supported in $\{|x|\le 2^{j-l_\tau(n)}\}$, we see that $K_j^n(x)$ is supported in $\{2^{j-2}\le|x|\le 2^{j+2}\}$. Therefore

$$|K_j^n(x)|\lesssim 2^{-jd}\chi_{\{2^{j-2}\le|x|\le 2^{j+2}\}} \tag{2.7}$$

and similarly for multi-indices α,

$$|\partial^\alpha K_j^n(x)|\lesssim 2^{-jd+(l_\tau(n)-j)|\alpha|}\chi_{\{2^{j-2}\le|x|\le 2^{j+2}\}}. \tag{2.8}$$

Let ρ_n be a smooth, nonnegative function such that $\rho_n(s)=1$ on $[2^{-\tau n},1-2^{-\tau n}]$, supp $\rho_n\subset(2^{-\tau n-1},1-2^{-\tau n-1})$, and the derivatives of ρ_n satisfy the natural estimates

$$\left|\frac{d^k}{ds^k}\rho_n(s)\right|\lesssim 2^{k\tau n}\quad\text{for all } k\in\mathbb{Z}_+.$$

Let

$$m_{x,y}^n a_1=\int_0^1\rho_n(s)a_1(sx+(1-s)y)ds.$$

Define the operator $T_j^n[a_1]$ by

$$T_j^n[a_1]f(x)=\int_{\mathbb{R}^d}K_j^n(x-y)m_{x,y}^n a_1\cdot f(y)dy.$$

Lemma 2.2. *With those definitions above, we have*

$$m\Big(\Big\{x\in\mathbb{R}^d:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge100}\sum_{j\in\mathbb{Z}}\Big(T_j[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)-T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\Big)d\vec{\eta}d\vec{s}\Big|>\lambda/4\Big\}\Big)\lesssim\frac{\|f\|_1}{\lambda}.$$

By Lemma 2.2, the proof of (2.6) now is reduced to verify the following estimate:

$$m\Big(\Big\{x\in\mathbb{R}^d:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge100}\sum_{j\in\mathbb{Z}}T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/4\Big\}\Big)\lesssim\frac{\|f\|_1}{\lambda}.$$

Below we separate $T_j^n[a_1]$ as $P_{j-n\kappa}T_j^n[a_1]+(I-P_{j-n\kappa})T_j[a_1]$, where κ satisfies $0<\kappa<1$ which will be chosen later. For $P_{j-n\kappa}T_j^n[a_1]$, we have the following lemma.

Lemma 2.3. *With those definitions above, for $n\geq 100$, we have*

$$\|P_{j-n\kappa}T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\|_1\lesssim n\big(2^{-(1-\kappa)n}+2^{-(1-\tau\delta^{-1})n}\big)\|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\|_1.$$

Applying Chebyshev's inequality, Minkowski's inequality and Lemma 2.3 and the property (iv) in Lemma 2.1, one may get

$$\begin{aligned}
&m\Big(\Big\{x\in\mathbb{R}^d:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\geq 100}\sum_{j\in\mathbb{Z}}P_{j-n\kappa}T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/4\Big\}\Big)\\
\lesssim&\lambda^{-1}\iint_{E^l}\sum_{n\geq 100}\sum_j|\prod_{i=2}^l\hat{a}_i(\eta_i)|\|P_{j-n\kappa}T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\|_1d\vec{\eta}d\vec{s}\\
\lesssim&\lambda^{-1}\sum_{n\geq 100}n(2^{(-1+\kappa)n}+2^{(-1+\tau\delta^{-1})n})\|f\|_1\lesssim\lambda^{-1}\|f\|_1.
\end{aligned}$$

Now the problem is reduced to prove the estimate below

$$m\Big(\Big\{x\in\mathbb{R}^d:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\geq 100}\sum_{j\in\mathbb{Z}}(I-P_{j-n\kappa})T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/4\Big\}\Big)\lesssim\frac{\|f\|_1}{\lambda}.\quad(2.9)$$

In the following, we need to make a microlocal decomposition of the kernel. To do this, we need to give a partition of unity on the unit surface $\mathbb{S}^{d-1}$. Choose $n\geq 100$. Let $\Theta_n=\{e_v^n\}_v$ be a collection of unit vectors on $\mathbb{S}^{d-1}$ which satisfies the following two conditions:

(a) $|e_v^n-e_{v'}^n|\geq 2^{-n\gamma-4}$, if $v\neq v'$;

(b) If $\theta\in\mathbb{S}^{d-1}$, there exists an e_v^n such that $|e_v^n-\theta|\leq 2^{-n\gamma-4}$.

The constant γ in (a) and (b) satisfying $0<\tau\delta^{-1}<\gamma<\kappa<1$ which will be chosen later. In fact, we may simply take a maximal collection $\{e_v^n\}_v$ for which (a) holds. Notice that there are $C2^{n\gamma(d-1)}$ elements in the collection $\{e_v^n\}_v$. For every $\theta\in\mathbb{S}^{d-1}$, there only exists finite e_v^n such that $|e_v^n-\theta|\leq 2^{-n\gamma-4}$. Now we can construct an associated partition of unity on the unit surface $\mathbb{S}^{d-1}$. Let ζ be a smooth, nonnegative, radial function with $\zeta(u)=1$ for $|u|\leq\frac{1}{2}$ and $\zeta(u)=0$ for $|u|>1$. Define

$$\Psi_v^n(\xi)=\zeta\Big(2^{n\gamma}\Big(\frac{\xi}{|\xi|}-e_v^n\Big)\Big),\quad\Gamma_v^n(\xi)=\Psi_v^n(\xi)\Big(\sum_{e_v^n\in\Theta_n}\Psi_v^n(\xi)\Big)^{-1}.$$

Then it is easy to see that Γ_v^n is homogeneous of degree 0 with $\sum_v\Gamma_v^n(\xi)=1$ holds for all $\xi\neq 0$ and all n. In addition, we have the following estimate for multi-indices α and $\xi\neq 0$,

$$|\partial_\xi^\alpha\Gamma_v^n(\xi)|\lesssim 2^{n\gamma|\alpha|}|\xi|^{-|\alpha|}.\quad(2.10)$$

Now we define operator $T_j^{n,v}[a_1]$ by

$$T_j^{n,v}[a_1]f(x)=\int_{\mathbb{R}^d}K_j^{n,v}(x-y)m_{x,y}^n a_1\cdot f(y)dy,\quad(2.11)$$

where $K_j^{n,v}(x)=K_j^n(x)\Gamma_v^n(x)$. Therefore, we have

$$T_j^n[a_1]=\sum_v T_j^{n,v}[a_1].$$

In the sequel, we need to sperate the phase of the frequent space into different direction. Hence we define a multiplier operator by

$$\widehat{G_{n,v}h}(\xi)=\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)\hat{h}(\xi),$$

where h is a Schwartz function and Φ is a smooth, nonnegative, radial function such that $0\le\Phi(x)\le 1$ and $\Phi(x)=1$ on $|x|\le 2$, $\Phi(x)=0$ on $|x|>4$. Now let $G_{n,v}+(I-G_{n,v})$ act on $T_j^{n,v}[a_1]$. Then we can split $T_j^{n,v}[a_1]$ into two parts:

$$T_j^{n,v}[a_1]=G_{n,v}T_j^{n,v}[a_1]+(I-G_{n,v})T_j^{n,v}[a_1].$$

The following lemmas give the L^2 estimate involving the term $G_{n,v}$, which will be proved in next section.

Lemma 2.4. *With those definitions above, for $n\ge 100$,*

$$\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{j\in\mathbb{Z}}\sum_v G_{n,v}(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)d\vec{\eta}d\vec{s}\Big\|_2^2\lesssim 2^{-n\gamma}\lambda\|f\|_1.$$

The estimates of the terms involving $(I-G_{n,v})(I-P_{j-n\kappa})T_j^{n,v}[a_1]$ are more complicated. In Section 4, we shall prove the following lemma.

Lemma 2.5. *We have*

$$\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge 100}\sum_{j\in\mathbb{Z}}\sum_v(I-G_{n,v})(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big\|_1\lesssim\|f\|_1.$$

2.2 Proof of Theorem 1.1

We now complete the proof of (2.9). By Chebyshev's inequality, we have

$$\begin{aligned}
&m\Big(\Big\{x\in(E^*)^c:\Big|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge 100}\sum_{j\in\mathbb{Z}}(I-P_{j-n\kappa})T_j^n[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big|>\lambda/4\Big\}\Big)\\
\lesssim&\lambda^{-2}\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge 100}\sum_{j\in\mathbb{Z}}\sum_v G_{n,v}(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big\|_2^2\\
&+\lambda^{-1}\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{n\ge 100}\sum_{j\in\mathbb{Z}}\sum_v(I-G_{n,v})(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big\|_1.
\end{aligned}$$

Using Lemma 2.5, we can get the desired estimate of the second term above. By Minkowski's inequality and Lemma 2.4, the first term above is bounded by

$$\begin{aligned}&\lambda^{-2}\Big(\sum_{n\geq 100}\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{j\in\mathbb{Z}}\sum_{v}G_{n,v}(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big\|_2\Big)^2\\ \lesssim&\lambda^{-2}\Big(\sum_{n\geq 100}(2^{-n\gamma}\lambda\|f\|_1)^{\frac{1}{2}}\Big)^2\lesssim\lambda^{-1}\|f\|_1.\end{aligned}$$

We hence complete the proof of Theorem 1.1 once Lemmas 2.2-2.5 hold.

3 Proofs of Lemmas 2.2-2.4

3.1 Proof of Lemma 2.2

By the definitions of $T_j[a_1]$ and $T_j^n[a_1]$, we have

$$\begin{aligned}\|T_j[a_1]f-T_j^n[a_1]f\|_1=&\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\Big(K_j(x-y)m_{x,y}a_1-K_j^n(x-y)m_{x,y}^na_1\Big)f(y)dy\Big|dx\\ \leq& I+II,\end{aligned}$$

where

$$\begin{aligned}I=&\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}(K_j(x-y)-K_j^n(x-y))m_{x,y}a_1\cdot f(y)dy\Big|dx,\\ II=&\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}K_j^n(x-y)\Big(m_{x,y}a_1-m_{x,y}^na_1\Big)f(y)dy\Big|dx.\end{aligned}$$

Consider I firstly. By the definition of $K_j^n(x)$, we have

$$K_j(x-y)-K_j^n(x-y)=\int\eta_{j-l_\tau(n)}(z)(K_j(x-y)-K_j(x-y-z))dz.$$

Notice that

$$\begin{aligned}&|K_j(x-y)-K_j(x-y-z)|\\ \leq&|\phi_j(x-y)(K(x-y)-K(x-y-z))|+|\phi_j(x-y)-\phi_j(x-y-z)||K(x-y-z)|\\ :=&A+B.\end{aligned}$$

Consider the first term A. Note that $|z|\leq 2^{j-l_\tau(n)}$ and $2^{j-1}\leq|x-y|\leq 2^{j+1}$, then we have $2|z|<|x-y|$. By the regularity condition (1.1b), we have

$$A\lesssim\frac{|z|^\delta}{|x-y|^{d+\delta}}\chi_{\{2^{j-1}\leq|x-y|\leq 2^{j+1}\}}\lesssim 2^{-\tau n}2^{-jd}\chi_{\{2^{j-1}\leq|x-y|\leq 2^{j+1}\}}.$$

For the second therm B, by the fact $|z|\le 2^{j-l_\tau(n)}$ and the support of ϕ_j, we have $|x-y|\approx |x-z-y|$ and $2^{j-2}\le|x-y|\le 2^{j+2}$. By (1.1a), we get

$$B\lesssim \frac{2^{-j}|z|}{|x-z-y|^d}\chi_{\{2^{j-2}\le|x-y|\le 2^{j+2}\}}\lesssim 2^{-\tau n}2^{-jd}\chi_{\{2^{j-2}\le|x-y|\le 2^{j+2}\}}.$$

As for II, we have

$$\left|m_{x,y}a_1-m^n_{x,y}a_1\right|=\left|\int_0^1(1-\rho_n(s))a_1(sx+(1-s)y)ds\right|\lesssim 2^{-\tau n}\|a_1\|_\infty.$$

Combining the above three estimates and applying Minkowski's inequality, we have

$$\begin{aligned}
&\|T_j[a_1]f-T^n_j[a_1]f\|_1\\
\lesssim&2^{-\tau n}\|a_1\|_\infty\int_{\mathbb{R}^d}\int_{2^{j-2}\le|x-y|\le 2^{j+2}}2^{-jd}\int_{\mathbb{R}^d}\eta_{j-l_\tau(n)}(z)dz|f(y)|dydx\\
\lesssim&2^{-\tau n}\|a_1\|_\infty 2^{-jd}\int_{\mathbb{R}^d}\int_{2^{j-2}\le|x-y|\le 2^{j+2}}|f(y)|dydx\\
\lesssim&2^{-\tau n}\|a_1\|_\infty\|f\|_1.
\end{aligned}\tag{3.1}$$

By Chebyshev's inequality, Minkowski's inequality, the estimates above and the property (iv) in Lemma 2.1, we get the bound

$$\begin{aligned}
&m\Big(\Big\{x\in\mathbb{R}^d:\Big|\iint_{E^l}u^{\vec\eta,\vec s}(x)\sum_{n\ge100}\sum_{j\in\mathbb{Z}}\Big(T_j[a_1]\mathfrak{B}^{\vec\eta,\vec s}_{j-n}(x)-T^n_j[a_1]\mathfrak{B}^{\vec\eta,\vec s}_{j-n}(x)\Big)d\vec\eta d\vec s\Big|>\lambda/2\Big\}\Big)\\
\lesssim&\lambda^{-1}\sum_{n\ge100}\iint_{E^l}\Big|\prod_{i=2}^l\hat a_i(\eta_i)\Big|\sum_j\Big\|T_j[a_1]\mathfrak{B}^{\vec\eta,\vec s}_{j-n}-T^n_j[a_1]\mathfrak{B}^{\vec\eta,\vec s}_{j-n}\Big\|_1d\vec\eta d\vec s\\
\lesssim&\lambda^{-1}\|a_1\|_\infty\Big(\prod_{i=2}^l\|\hat a_i\|_1\Big)\sum_{n\ge100}2^{-\tau n}\|f\|_1\lesssim\lambda^{-1}\|f\|_1,
\end{aligned}$$

which is the required estimate. □

3.2 Proof of Lemma 2.3

Since $\mathfrak{B}^{\vec\eta,\vec s}_{j-n}=\sum_{l(Q)=2^{j-n}}b^{\vec\eta,\vec s}_Q$, we only need to consider a fixed $b^{\vec\eta,\vec s}_Q$ with $l(Q)=2^{j-n}$. By applying Fubini's theorem, one may write

$$\begin{aligned}
&P_{j-n\kappa}T^n_j[a_1]b^{\vec\eta,\vec s}_Q(x)\\
=&\int_0^1\rho_n(s)\int_{\mathbb{R}^d}b^{\vec\eta,\vec s}_Q(y)\int_{\mathbb{R}^d}\eta_{j-n\kappa}(x-w)K^n_j(w-y)a_1(sw+(1-s)y)dwdyds.
\end{aligned}$$

Making a change of variables $z=w+\frac{1-s}{s}y$, one may get

$$\begin{aligned}&P_{j-n\kappa}T_j^n[a_1]b_Q^{\vec{\eta},\vec{s}}(x)\\&=\int_0^1\rho_n(s)\int_{\mathbb{R}^d}a_1(sz)\int_{\mathbb{R}^d}\eta_{j-n\kappa}\Big(x-z+\frac{1-s}{s}y\Big)K_j^n\Big(z-\frac{y}{s}\Big)b_Q^{\vec{\eta},\vec{s}}(y)dydzds.\end{aligned}$$

By using the cancellation of $b_Q^{\vec{\eta},\vec{s}}$(see the property of (iv) in Lemma 2.1), one may write

$$\begin{aligned}&\big|P_{j-n\kappa}T_j^n[a_1]b_Q^{\vec{\eta},\vec{s}}(x)\big|\\&=\Big|\int_0^1\rho_n(s)\int_{\mathbb{R}^d}a_1(sz)\int_{\mathbb{R}^d}b_Q^{\vec{\eta},\vec{s}}(y)\\&\quad\times\Big(\eta_{j-n\kappa}\Big(x-z+\frac{1-s}{s}y\Big)K_j^n\Big(z-\frac{y}{s}\Big)-\eta_{j-n\kappa}\Big(x-z+\frac{1-s}{s}y_0\Big)K_j^n\Big(z-\frac{y_0}{s}\Big)\Big)dydzds\Big|\\&\le I+II,\end{aligned}$$

where y_0 is the center of Q and

$$\begin{aligned}I=&\Big|\int_0^1\rho_n(s)\int_{\mathbb{R}^d}|a_1(sz)|\int_{\mathbb{R}^d}|b_Q^{\vec{\eta},\vec{s}}(y)|\\&\times\Big|\eta_{j-n\kappa}\Big(x-z+\frac{1-s}{s}y\Big)-\eta_{j-n\kappa}\Big(x-z+\frac{1-s}{s}y_0\Big)\Big|\Big|K_j^n\Big(z-\frac{y}{s}\Big)\Big|dydzds\end{aligned}$$

and

$$\begin{aligned}II=&\Big|\int_0^1\rho_n(s)\int_{\mathbb{R}^d}|a_1(sz)|\int_{\mathbb{R}^d}|b_Q^{\vec{\eta},\vec{s}}(y)|\\&\times\Big|\eta_{j-n\kappa}\Big(x-z+\frac{1-s}{s}y_0\Big)\Big|\Big|K_j^n\Big(z-\frac{y}{s}\Big)-K_j^n\Big(z-\frac{y_0}{s}\Big)\Big|dydzds.\end{aligned}$$

Since $l(Q)=2^{j-n}$, we have $|y-y_0|\lesssim 2^{j-n}$. Using the mean value formula, one may have

$$\begin{aligned}I\lesssim&\|a_1\|_\infty\int_0^1\rho_n(s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|b_Q^{\vec{\eta},\vec{s}}(y)\Big|\frac{1-s}{s}2^{-j+\kappa n}|y-y_0|\Big|K_j^n\Big(z-\frac{y}{s}\Big)\Big|dydzds\\\lesssim&2^{(-1+\kappa)n}\|a_1\|_\infty\int_0^1\rho_n(s)s^{-1}ds\|K_j^n\|_1\|b_Q^{\vec{\eta},\vec{s}}\|_1\lesssim n2^{(-1+\kappa)n}\|b_Q^{\vec{\eta},\vec{s}}\|_1.\end{aligned}$$

Similarly, by the mean value formula and (2.8), one may get

$$\begin{aligned}II\lesssim&\|a_1\|_\infty\int_0^1\rho_n(s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|b_Q^{\vec{\eta},\vec{s}}(y)|s^{-1}|y-y_0|\int_0^1\Big|\nabla K_j^n\Big(z-\frac{ty+(1-t)y_0}{s}\Big)\Big|dtdydzds\\\lesssim&2^{j-n}\|a_1\|_\infty\int_0^1\rho_n(s)s^{-1}ds\|\nabla K_j^n\|_1\|b_Q^{\vec{\eta},\vec{s}}\|_1\lesssim n2^{(-1+\tau\delta^{-1})n}\|b_Q^{\vec{\eta},\vec{s}}\|_1.\end{aligned}$$

Combining the estimates of I and II and summing over all Q with $l(Q)=2^{j-n}$, we finish the proof of Lemma 2.3. □

3.3 Proof of Lemma 2.4

The proof of this lemma is quite similar to that of Proposition 2.4 in [18]. As usually, we adopt the TT^* method in the L^2 estimate. We also use some orthogonality argument based on the following observation of the support of $\mathcal{F}(G_{n,v}(I-P_{j-n\kappa})T_j^{n,v}[a_1])$: For a fixed $n\geq 100$, we have

$$\sup_{\xi\neq 0}\sum_v |\Phi^2(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)|\lesssim 2^{n\gamma(d-2)}. \tag{3.2}$$

In fact, by the homogeneity of $\Phi^2(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)$, it suffices to take the supremum over the surface $\mathbb{S}^{d-1}$. For $|\xi|=1$ and $\xi\in\operatorname{supp}\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)$, denote by $\xi^\perp$ the hyperplane perpendicular to ξ. Then

$$\operatorname{dist}(e_v^n,\xi^\perp)\lesssim 2^{-n\gamma}. \tag{3.3}$$

Since the mutual distance of e_v^n's is bounded by $2^{-n\gamma-4}$, there are at most $2^{n\gamma(d-2)}$ vectors satisfy (3.3). We hence get (3.2).

By applying Plancherel's theorem and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\Big\|\iint_{E^l} u^{\vec{\eta},\vec{s}}(x)\sum_{j\in\mathbb{Z}}\sum_v G_{n,v}(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big\|_2^2\\
=&\Big(\iint_{E^l}\Big|\prod_{i=2}^l\hat{a}_i(\eta_i)\Big|\Big\|\sum_v\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)\mathcal{F}\Big(\sum_j(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\Big)(\xi)\Big\|_2 d\vec{\eta}d\vec{s}\Big)^2\\
\lesssim&2^{n\gamma(d-2)}\Big(\iint_{E^l}\Big|\prod_{i=2}^l\hat{a}_i(\eta_i)\Big|\Big\|\sum_v\Big|\mathcal{F}\Big(\sum_j(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\Big)\Big|^2\Big\|_1^{\frac12}d\vec{\eta}d\vec{s}\Big)^2\\
\lesssim&2^{n\gamma(d-2)}\Big(\iint_{E^l}\Big|\prod_{i=2}^l\hat{a}_i(\eta_i)\Big|\Big(\sum_v\Big\|\sum_j(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\Big\|_2^2\Big)^{\frac12}d\vec{\eta}d\vec{s}\Big)^2.
\end{aligned}\tag{3.4}$$

Once it is showed that for a fixed e_v^n, $\vec{\eta}$, $\vec{s}$,

$$\Big\|\sum_j(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\Big\|_2^2\lesssim 2^{-2n\gamma(d-1)}\lambda\|f\|_1, \tag{3.5}$$

then by $\operatorname{card}(\Theta_n)\lesssim 2^{n\gamma(d-1)}$, and apply (3.4) and (3.5) we get

$$\Big\|\iint_{E^l}u^{\vec{\eta},\vec{s}}(x)\sum_{j\in\mathbb{Z}}\sum_v G_{n,v}(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)\,d\vec{\eta}d\vec{s}\Big\|_2^2\lesssim 2^{-n\gamma}\lambda\|f\|_1,$$

which is just the desired bound of Lemma 2.4. Thus, to finish the proof of Lemma 2.4, it is enough to prove (3.5). By applying (2.7), the support of Γ_v^n and $0<\gamma<\kappa<1$, we have

$$|(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)|\lesssim H_j^{n,v}*|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}|(x),$$

where $H_j^{n,v}(x):=2^{-jd}\chi_{E_j^{n,v}}(x)$ and $\chi_{E_j^{n,v}}(x)$ is a characteristic function of the set

$$E_j^{n,v}:=\{x\in\mathbb{R}^d:|\langle x,e_v^n\rangle|\le 2^{j+2},\ |x-\langle x,e_v^n\rangle e_v^n|\le 2^{j+2-n\gamma}\}.$$

For a fixed e_v^n, we write

$$\begin{aligned}\Big\|\sum_j T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\Big\|_2^2\lesssim&\sum_j\int_{\mathbb{R}^d}H_j^{n,v}*H_j^{n,v}*|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}|(x)\cdot|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)|dx\\&+\sum_j\sum_{i=-\infty}^{j-1}\int_{\mathbb{R}^d}H_j^{n,v}*H_i^{n,v}*|\mathfrak{B}_{i-n}^{\vec{\eta},\vec{s}}|(x)\cdot|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)|dx.\end{aligned}\tag{3.6}$$

Observe that $\|H_i^{n,v}\|_1\lesssim 2^{-id}m(E_i^{n,v})\lesssim 2^{-n\gamma(d-1)}$, therefore for any $i\le j$,

$$H_j^{n,v}*H_i^{n,v}(x)\le 2^{-n\gamma(d-1)}2^{-jd}\chi_{\widetilde{E}_j^{n,v}},$$

where $\widetilde{E}_j^{n,v}=E_j^{n,v}+E_j^{n,v}$. Hence for a fixed $\vec{\eta}$, $\vec{s}$, j, n, e_v^n and x, we have

$$\begin{aligned}&H_j^{n,v}*H_j^{n,v}*|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}|(x)+\sum_{i=-\infty}^{j-1}H_j^{n,v}*H_i^{n,v}*|\mathfrak{B}_{i-n}^{\vec{\eta},\vec{s}}|(x)\\\lesssim&2^{-n\gamma(d-1)}2^{-jd}\sum_{i\le j}\int_{x+\widetilde{E}_j^{n,v}}|\mathfrak{B}_{i-n}^{\vec{\eta},\vec{s}}(y)|dy\\\lesssim&2^{-n\gamma(d-1)}2^{-jd}\sum_{i\le j}\sum_{\substack{Q\in\mathfrak{Q}_{i-n}\\Q\cap\{x+\widetilde{E}_j^{n,v}\}\neq\emptyset}}\int_{\mathbb{R}^d}|b_Q^{\vec{\eta},\vec{s}}(y)|dy\\\lesssim&2^{-n\gamma(d-1)}2^{-jd}\sum_{i\le j}\sum_{\substack{Q\in\mathfrak{Q}_{i-n}\\Q\cap\{x+\widetilde{E}_j^{n,v}\}\neq\emptyset}}\lambda|Q|\\\lesssim&\lambda 2^{-2n\gamma(d-1)},\end{aligned}\tag{3.7}$$

where in third inequality above, we use $\int|b_Q^{\vec{\eta},\vec{s}}(y)|dy\lesssim\lambda|Q|$ (see the property (iv) in Lemma 2.1) and in the fourth inequality we use fact that the cubes in $\mathfrak{Q}$ are disjoint (see the property (i) in Lemma 2.1). By (3.6), (3.7) and the property (iv) in Lemma 2.1, we obtain

$$\Big\|\sum_j(I-P_{j-n\kappa})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\Big\|_2^2\lesssim\lambda 2^{-2n\gamma(d-1)}\sum_j\|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\|_1\lesssim\lambda 2^{-2n\gamma(d-1)}\|f\|_1.$$

Hence, we complete the proof of Lemma 2.4. □

4 Proof of Lemma 2.5

To prove Lemma 2.5, we have to deal with some oscillatory integrals which come from $(I-G_{n,v})T_j^{n,v}[a_1]$.

Before stating the proof of Lemma 2.5, let us give some notations. We first introduce the Littlewood-Paley decomposition. Let ψ be a radial C^∞ function such that $\psi(\xi)=1$ for $|\xi|\le 1$, $\psi(\xi)=0$ for $|\xi|\ge 2$ and $0\le\psi(\xi)\le 1$ for all $\xi\in\mathbb{R}^d$. Define $\beta_k(\xi)=\psi(2^k\xi)-\psi(2^{k+1}\xi)$, then β_k is supported in $\{\xi: 2^{-k-1}\le|\xi|\le 2^{-k+1}\}$ and $\sum_k\beta_k(\xi)=1$ for $\xi\in\mathbb{R}^d\setminus\{0\}$. Choose $\tilde\beta$ be a radial C^∞ function such that $\tilde\beta(\xi)=1$ for $\frac{1}{2}\le|\xi|\le 2$, $\tilde\beta$ is supported in $\{\xi:\frac{1}{4}\le|\xi|\le 4\}$ and $0\le\tilde\beta(\xi)\le 1$ for all $\xi\in\mathbb{R}^d$. Set $\tilde\beta_k(\xi)=\tilde\beta(2^k\xi)$, then it is easy to see $\beta_k=\tilde\beta_k\beta_k$. Define the convolution operators Λ_k and $\tilde\Lambda_k$ with Fourier multipliers β_k and $\tilde\beta_k$, respectively. That is,

$$\widehat{\Lambda_k f}(\xi)=\beta_k(\xi)\hat f(\xi),\quad \widehat{\tilde\Lambda_k f}(\xi)=\tilde\beta_k(\xi)\hat f(\xi).$$

Then by the construction of β_k and $\tilde\beta_k$, we have

$$\Lambda_k=\tilde\Lambda_k\Lambda_k,\quad I=\sum_{k\in\mathbb{Z}}\Lambda_k.$$

where I is the identity. Write

$$(I-G_{n,v})T_j^{n,v}[a_1]=\sum_k(I-G_{n,v})\Lambda_kT_j^{n,v}[a_1].$$

By using Minkowski's inequality,

$$\begin{aligned}&\Big\|\iint_{E^l}u^{\vec\eta,\vec s}(x)\sum_{n\ge100}\sum_v\sum_j(I-P_{j-n\kappa})(I-G_{n,v})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec\eta,\vec s}(x)d\vec\eta d\vec s\Big\|_1\\ \le&\sum_{n\ge100}\sum_v\sum_j\sum_k\sum_{l(Q)=2^{j-n}}\iint_{E^l}\Big|\prod_{i=2}^l\hat a_i(\eta_i)\Big|\cdot\|(I-P_{j-n\kappa})(I-G_{n,v})\Lambda_kT_j^{n,v}[a_1]b_Q^{\vec\eta,\vec s}\|_1d\vec\eta d\vec s.\end{aligned}\tag{4.1}$$

Lemma 4.1. *For a fixed $b_Q^{\vec\eta,\vec s}$ with $l(Q)=2^{j-n}$, there exists $N>0$, such that*

$$\|(I-P_{j-n\kappa})(I-G_{n,v})\Lambda_kT_j^{n,v}[a_1]b_Q^{\vec\eta,\vec s}\|_1\lesssim 2^{\delta^{-1}\tau n-n\gamma(d-1)+(-j+k)+n\gamma(1+2N)}\|b_Q^{\vec\eta,\vec s}\|_1.\tag{4.2}$$

Proof. First write

$$\begin{aligned}&\|(I-P_{j-n\kappa})(I-G_{n,v})\Lambda_kT_j^{n,v}[a_1]b_Q^{\vec\eta,\vec s}\|_1\\ =&\|(I-P_{j-n\kappa})\tilde\Lambda_k(I-G_{n,v})\Lambda_kT_j^{n,v}[a_1]b_Q^{\vec\eta,\vec s}\|_1\\ \le&\|(I-P_{j-n\kappa})\tilde\Lambda_k\|_{L^1\to L^1}\|(I-G_{n,v})\Lambda_kT_j^{n,v}[a_1]b_Q^{\vec\eta,\vec s}\|_1.\end{aligned}$$

It is easy to see that $\|(I-P_{j-n\kappa})\tilde{\Lambda}_k\|_{L^1\to L^1}\lesssim 1$ uniformly with $j,k,n\kappa$. Denote $h_{k,n,v}(\xi)=(1-\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle))\beta_k(\xi)$. Applying Fubini's theorem, we may write

$$(I-G_{n,v})\Lambda_k T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}(x)=:\int_{\mathbb{R}^d}D_k(x,y)b_Q^{\vec{\eta},\vec{s}}(y)dy, \tag{4.3}$$

where

$$D_k(x,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}h_{k,n,v}(\xi)\int_{\mathbb{R}^d}e^{-i\xi\cdot\omega}K_j^{n,v}(\omega-y)m_{\omega,y}^n a_1 d\omega d\xi.$$

Next we make a change of variables to polar coordinate $\omega-y=r\theta$. By Fubini's theorem, $D_k(x,y)$ can be written as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Gamma_v^n(\theta)\left\{\int_{\mathbb{R}^d}\int_0^\infty e^{i\langle x-y-r\theta,\xi\rangle}h_{k,n,v}(\xi)K_j^n(r\theta)r^{d-1}(m_{y+r\theta,y}^n a_1)drd\xi\right\}d\sigma(\theta). \tag{4.4}$$

By the support of $K_j^n(x)$ in (2.7), we have $2^{j-2}\le r\le 2^{j+2}$. Since $\theta\in\operatorname{supp}\Gamma_v^n$, then $|\theta-e_v^n|\le 2^{-n\gamma}$. Using the support of Φ, we see $|\langle e_v^n,\xi/|\xi|\rangle|\ge 2^{1-nr}$. Thus,

$$|\langle\theta,\xi/|\xi|\rangle|\ge|\langle e_v^n,\xi/|\xi|\rangle|-|\langle e_v^n-\theta,\xi/|\xi|\rangle|\ge 2^{-n\gamma}. \tag{4.5}$$

Integrating by parts with r, $D_k(x,y)$ can be rewritten as

$$\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Gamma_v^n(\theta)\left\{\int_{\mathbb{R}^d}\int_0^\infty e^{i\langle x-y-r\theta,\xi\rangle}\frac{h_{k,n,v}(\xi)}{i\langle\theta,\xi\rangle}\partial_r\left[K_j^n(r\theta)r^{d-1}m_{y+r\theta,y}^n a_1\right]drd\xi\right\}d\sigma(\theta).$$

After integrating by parts with r, integrating by parts with ξ, the integral $D_k(x,y)$ can be rewritten as

$$\begin{aligned}&\frac{1}{(2\pi)^d}\int_{\mathbb{S}^{d-1}}\Gamma_v^n(\theta)\int_{\mathbb{R}^d}e^{i\langle x-y-r\theta,\xi\rangle}\int_0^\infty\partial_r\left(K_j^n(r\theta)r^{d-1}m_{y+r\theta,y}^n a_1\right)\\&\times\frac{(I-2^{-2k}\Delta_\xi)^N}{(1+2^{-2k}|x-y-r\theta|^2)^N}\left(h_{k,n,v}(\xi)(i\langle\theta,\xi\rangle)^{-1}\right)drd\xi d\sigma(\theta).\end{aligned} \tag{4.6}$$

In the following, we give an explicit estimate of the term in (4.6). By the definition of $K_j^n(x)$, we have

$$\begin{aligned}|\partial_x^\alpha K_j^n(x)|&=2^{-(j-l_\tau(n))|\alpha|}\left|\int(\partial_x^\alpha\eta)_{j-l_\tau(n)}(x-z)K_j(z)dz\right|\\&\le 2^{-(j-l_\tau(n))|\alpha|}\|K_j\|_\infty\|\partial_x^\alpha\eta\|_1\lesssim 2^{-(j-l_\tau(n))|\alpha|-jd},\end{aligned} \tag{4.7}$$

where the third inequality follows from (2.7). Observe

$$|\partial_r(m_{y+r\theta,y}^n a_1)|=\left|\partial_r\left(\frac{1}{r}\int_0^r\rho_n\left(\frac{s}{r}\right)a_1(y+s\theta)ds\right)\right|\lesssim\frac{1}{r}2^{l_\tau(n)}\|a_1\|_\infty. \tag{4.8}$$

By using product rule, (4.7), (4.8) and $2^{j-2}\le r\le 2^{j+2}$,

$$\left|\partial_r\left(K_j^n(r\theta)r^{d-1}(m_{y+r\theta,y}^n a_1)\right)\right|\lesssim 2^{l_\tau(n)-2j}\|a_1\|_\infty. \tag{4.9}$$

By (4.5), we have

$$|(-i\langle\theta,\xi\rangle)^{-1}\cdot h_{k,n,v}(\xi)|\lesssim|\langle\theta,\xi\rangle|^{-1}\lesssim 2^{n\gamma+k}.$$

Now using product rule,

$$|\partial_{\xi_i}h_{k,n,v}(\xi)|=\left|-\partial_{\xi_i}[\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle)]\cdot\beta_k(\xi)+\partial_{\xi_i}\beta_k(\xi)\cdot(1-\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle))\right|\lesssim 2^{n\gamma+k}.$$

Therefore by induction, we have $|\partial_\xi^\alpha h_{k,n,v}(\xi)|\lesssim 2^{(n\gamma+k)|\alpha|}$ for any multi-indices $\alpha\in\mathbb{Z}_+^n$. By using product rule again and (4.5), we have

$$\begin{aligned}\left|\partial_{\xi_i}^2(\langle\theta,\xi\rangle)^{-1}h_{k,n,v}(\xi))\right|&=\left|\langle\theta,\xi\rangle^{-3}\cdot 2\theta_i^2\cdot h_{k,n,v}-2\langle\theta,\xi\rangle^{-2}\cdot\theta_i\partial_{\xi_i}h_{k,n,v}(\xi)+\langle\theta,\xi\rangle^{-1}\partial_{\xi_i}^2h_{k,n,v}(\xi)\right|\\&\lesssim 2^{3(n\gamma+k)}.\end{aligned}$$

This implies the follow inequality

$$2^{-2k}\left|\Delta_\xi[(\langle\theta,\xi\rangle)^{-1}h_{k,n,v}(\xi)]\right|\lesssim 2^{(n\gamma+k)+2n\gamma}.$$

Proceeding by induction, we have

$$\left|(I-2^{-2k}\Delta_\xi)^N[\langle\theta,\xi\rangle^{-1}h_{k,n,v}(\xi)]\right|\lesssim 2^{(n\gamma+k)+2n\gamma N}. \tag{4.10}$$

Now we choose $N=[\frac{d}{2}]+1$. By (4.3) and Minkowski's inequality,

$$\left\|(I-G_{n,v})\Lambda_k T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\right\|_1\le\int\|D_k(\cdot,y)\|_1|b_Q^{\vec{\eta},\vec{s}}(y)|dy.$$

So we need to get the L^1 estimate of (4.4), by the support of $h_{k,n,v}$,

$$\int_{\mathrm{supp}(h_{k,n,v})}\int\left(1+2^{-2k}|x-y-r\theta|^2\right)^{-N}dxd\xi\le C.$$

Integrating with r, we get a bound 2^j. Then integrating with θ, we get a bound $2^{-n\gamma(d-1)}$. Combining (4.9), (4.10) and the above estimates,

$$\|D_k(\cdot,y)\|_1\lesssim 2^{\delta^{-1}\tau n-n\gamma(d-1)+(-j+k)+n\gamma(1+2N)}.$$

Hence we complete the proof of Lemma 4.1 with $N=[\frac{d}{2}]+1$. □

Lemma 4.2. *For a fixed $b_Q^{\vec{\eta},\vec{s}}$ with $l(Q)=2^{j-n}$, then*

$$\|(I-P_{j-n\kappa})(I-G_{n,v})\Lambda_k T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\|_1\lesssim 2^{-n\gamma(d-1)+j-n\kappa-k}\|b_Q^{\vec{\eta},\vec{s}}\|_1.$$

Proof. We write

$$\begin{aligned}&\|(I-P_{j-n\kappa})(I-G_{n,v})\Lambda_k T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\|_1\\=&\|(I-P_{j-n\kappa})\tilde{\Lambda}_k(I-G_{n,v})\Lambda_k T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\|_1\\\leq&\|(I-P_{j-n\kappa})\tilde{\Lambda}_k\|_{L^1\to L^1}\|(I-G_{n,v})\Lambda_k\|_{L^1\to L^1}\|T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\|_1.\end{aligned}$$

By using Minkowski's inequality, one can easily get

$$\|T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\|_1\lesssim 2^{-n\gamma(d-1)}\|b_Q^{\vec{\eta},\vec{s}}\|_1. \tag{4.11}$$

Now we claim that

$$\|(I-P_{j-n\kappa})\tilde{\Lambda}_k\|_{L^1\to L^1}\lesssim 2^{j-n\kappa-k}, \tag{4.12a}$$

$$\|(I-G_{n,v})\Lambda_k\|_{L^1\to L^1}\lesssim 1. \tag{4.12b}$$

Combining (4.11), (4.12a) and (4.12b), one get the asserted bound. So to finish the proof, it suffice to show (4.12a) and (4.12b). Write

$$\begin{aligned}(I-P_{j-n\kappa})\tilde{\Lambda}_k f(x)=&\widehat{\tilde{\beta}}_k * f(x)-\eta_{j-n\kappa}*\widehat{\tilde{\beta}}_k f(x)\\=&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(\widehat{\tilde{\beta}}_k(x-y)-\widehat{\tilde{\beta}}_k(x-y-z)\big)\eta_{j-n\kappa}(z)dz\cdot f(y)dy.\end{aligned}$$

By using the mean value formula, one may get

$$\widehat{\tilde{\beta}}_k(x-y)-\widehat{\tilde{\beta}}_k(x-y-z)=\int_0^1\langle z,\nabla\widehat{\tilde{\beta}}_k(x-y-sz)\rangle ds.$$

Utilize the Fubini's theorem, one may get

$$\|(I-P_{j-n\kappa})\tilde{\Lambda}_k f\|_1\lesssim 2^{j-n\kappa-k}\|\nabla\widehat{\tilde{\beta}}\|_1\|\eta\|_1\|f\|_1\lesssim 2^{j-n\kappa-k}\|f\|_1.$$

Thus we prove (4.12a). To prove (4.12b), it is enough to show that the function $h_{k,n,v}(\xi)=(1-\Phi(2^{n\gamma}\langle e_v^n,\xi/|\xi|\rangle))\beta_k(\xi)$ is a L^1 Fourier multiplier. Let $A_k^{n,v}$ be an invertible transform with $A_k^{n,v}e^{n,v}=2^{-n\gamma-k}e^{n,v}$ and $A_k^{n,v}y=2^{-k}y$ if $\langle y,e^{n,v}\rangle=0$. For all $\alpha\in\mathbb{Z}_+^d$, it is straightforward to check that

$$\|\partial^\alpha\big(h_{k,n,v}(A_k^{n,v}\cdot)\big)\|_2\lesssim 1$$

uniformly with k,n,v. By the Berenstein multiplier theorem (e.g., see Lemma 6.1.5 in [1]),

$$\|(I-G_{n,v})\Lambda_k\|_{L^1\to L^1}\lesssim\|h_{k,n,v}(A_k^{n,v}\cdot)\|_2^{\frac{1}{2}}\sum_{|\alpha|=n}\|\partial^\alpha\big(h_{k,n,v}(A_k^{n,v}\cdot)\big)\|_2^{\frac{1}{2}}\lesssim 1,$$

which completes the proof of (4.12b). □

Proof of Lemma 2.5. Let ε_0 satisfy $0<\varepsilon_0<1$ and will be chosen later. By (4.1), Lemma 4.1, Lemma 4.2 $\text{card}\Theta_n \lesssim 2^{n\gamma(d-1)}$, the property (iv) in Lemma 2.1 and the fact $[n\varepsilon_0]\le n\varepsilon_0 < [n\varepsilon_0]+1$, one obtain

$$\begin{aligned}
&\Big\| \iint_{E^l} u^{\vec{\eta},\vec{s}}(x) \sum_{n\ge 100}\sum_{v}\sum_{j}(I-P_{j-n\kappa})(I-G_{n,v})T_j^{n,v}[a_1]\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}(x)d\eta ds\Big\|_1\\
&\le \Big(\sum_{n\ge 100}\sum_{v}\sum_{j}\sum_{k<j-[n\varepsilon_0]}\sum_{l(Q)=2^{j-n}} + \sum_{n\ge 100}\sum_{v}\sum_{j}\sum_{k\ge j-[n\varepsilon_0]}\sum_{l(Q)=2^{j-n}}\Big)\\
&\quad \iint_{E^l}\Big|\prod_{i=2}^{l}\hat{a}_i(\eta_i)\Big|\cdot\|(I-P_{j-n\kappa})(I-G_{n,v})\Lambda_k T_j^{n,v}[a_1]b_Q^{\vec{\eta},\vec{s}}\|_1 d\eta ds\\
&\lesssim \sum_{n\ge 100}(2^{s_1 n}+2^{s_2 n})\iint_{E^l}\Big|\prod_{i=2}^{l}\hat{a}_i(\eta_i)\Big|\cdot\sum_{j}\|\mathfrak{B}_{j-n}^{\vec{\eta},\vec{s}}\|_1 d\eta ds\\
&\lesssim \sum_{n\ge 100}(2^{s_1 n}+2^{s_2 n})\|f\|_1,
\end{aligned}$$

where

$$s_1=-\varepsilon_0+\gamma+2\Big(\Big[\frac{d}{2}\Big]+1\Big)\gamma+\delta^{-1}\tau,\quad s_2=-\kappa+\varepsilon_0.$$

Now we choose $\tau>\frac{1}{n}$ and $0<\delta^{-1}\tau\ll\gamma\ll\varepsilon_0\ll\kappa\ll 1$ such that

$$\max\{s_1,s_2\}<0.$$

Therefore the above sum is convergent and we finish the proof of Lemma 2.5. □

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