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Weighted Norm Inequalities for Toeplitz Type Operator Related to Singular Integral Operator with Variable Kernel

Yuexiang He*

Department of Mathematics, Jiaozuo University, Jiaozuo 454003, Henan, China

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Abstract. Let $T^{k,1}$ be the singular integrals with variable Calderón-Zygmund kernels or $\pm I$ (the identity operator), let $T^{k,2}$ and $T^{k,4}$ be the linear operators, and let $T^{k,3}=\pm I$. Denote the Toeplitz type operator by

$$T^b=\sum_{k=1}^{t}(T^{k,1}M^bI_\alpha T^{k,2}+T^{k,3}I_\alpha M^bT^{k,4}),$$

where $M^bf=bf$, and I_α is the fractional integral operator. In this paper, we investigate the boundedness of the operator on weighted Lebesgue space when b belongs to weighted Lipschitz space.

Key Words: Toeplitz type operator, variable Calderón-Zygmund kernel, fractional integral, weighted Lipschitz space.

AMS Subject Classifications: 42B20, 42B25

1 Introduction and results

As the development of the singular integral operators, their commutators have been well studied (see [1-3]). In [1], the authors proved that the commutators $[b,T]$, which generated by Calderón-Zygmund singular integral operators and BMO functions, are bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Chanillo [4] obtained a similar result when Calderón-Zygmund singular integral operators are replaced by the fractional integral operators. Recently, some Toeplitz type operators related to the singular integral operators are introduced, and the boundedness for the operators generated by singular integral operators and BMO functions or Lipschitz functions are obtained (see [5-8]).

*Corresponding author. *Email address:* heyuexiang63@163.com (Y. X. He)

Let $K(x,\xi):\ \mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be a variable Calderón-Zygmund kernel, which depends on some parameter x and possesses 'good' properties with respect to the second variable ξ. The singular integral operator with variable Calderón-Zygmund kernel is defined by

$$Tf(x)=p.v.\int_{\mathbb{R}^n}K(x,x-y)f(y)dy. \tag{1.1}$$

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator associated to singular integrals with variable Calderón-Zygmund kernel and fractional integral operator I_α is defined by

$$T^b=\sum_{k=1}^{t}(T^{k,1}M^bI_\alpha T^{k,2}+T^{k,3}I_\alpha M^bT^{k,4}), \tag{1.2}$$

where $M^bf=bf$, and $T^{k,1}$ are the singular integral operators with variable Calderón-Zygmund kernel or $\pm I$ (the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3}=\pm I, k=1,\cdots,t$.

Note that the commutators $[b,I_\alpha](f)=bI_\alpha(f)-I_\alpha(bf)$ are the particular operators of the Toeplitz type operators T^b. The Toeplitz type operators T^b are the non-trivial generalizations of these commutators.

It is well known that the commutators of fractional integral have been widely studied by many authors. Paluszyński [9] showed that $b\in Lip_\beta$ (homogeneous Lipschitz space) if and only if $[b,I_\alpha]$ is bounded from L^p to L^q, where $0<\beta<1, 1<p<n/(\alpha+\beta)$ and $1/q=1/p-(\alpha+\beta)/n$. When b belongs to the weighted Lipschitz spaces $Lip_{\beta,\omega}$, Hu and Gu [10] proved that $[b,I_\alpha]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$ for $1/q=1/p-(\alpha+\beta)/n$ with $1<p<n/(\alpha+\beta)$. A similar result obtained when I_α is replaced by the generalized fractional integral operator [7].

Motivated by these papers, in this paper, we investigate the boundedness of the Toeplitz type operator as (1.2) on weighted Lebesgue space when b belongs to weighted Lipschitz space, and get the following result.

Theorem 1.1. *Suppose that T is a singular integral operator with variable Calderón-Zygmund kernel as (1.1), $\omega^{q/p}\in A_1$, and $b\in Lip_{\beta,\omega}(0<\beta<1)$. Let $0<\alpha+\beta<n, 1<p<n/(\alpha+\beta)$ and $1/q=1/p-(\alpha+\beta)/n$. If $T^1(f)=0$ for any $f\in L^p(\omega)(1<p<\infty)$, $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^p(\omega)$, $k=1,\cdots,t$, then there exists a constant $C>0$ such that,*

$$\|T^b(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})}\le C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^p(\omega)}.$$

2 Some preliminaries

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B=B_r(x_0)$ denote the ball with the center x_0 and radius r, and $\lambda B=B_{\lambda r}(x_0)$ for any $\lambda>0$. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E)=\int_E \omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $0<p<\infty$, denote by $L^p(\omega)$ the space of all function f satisfying

$$\|f\|_{L^p(\omega)}=\left(\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx\right)^{1/p}<\infty.$$

A weight ω is said to belong to the Muckenhoupt class A_p for $1<p<\infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1}\le C \tag{2.1}$$

for every ball B. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B \omega(y)dy\le C\cdot \operatorname*{ess\,inf}_{x\in B} w(x) \tag{2.2}$$

for every ball B.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [11]. We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [12]. Let p' be the dual of p such that $1/p+1/p'=1$. A weight function ω belongs to $A_{p,q}$ for $1<p<q<\infty$, if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|}\int_B \omega(y)^{-p'}dy\right)^{1/p'}\left(\frac{1}{|B|}\int_B \omega(y)^q dy\right)^{1/q}\le C. \tag{2.3}$$

From the definition of $A_{p,q}$, we can get that

$$\omega\in A_{p,q},\quad \text{iff } \omega^q\in A_{1+q/p'}. \tag{2.4}$$

Lemma 2.1 ([13]). *Suppose $\omega\in A_1$. Then*

(i) there exists a $\epsilon>0$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)^{1+\epsilon}dx\right)^{1/(1+\epsilon)}\le \frac{C}{|B|}\int_B \omega(x)dx; \tag{2.5}$$

(ii) there exist two constants C_1 and C_2, such that

$$C_1\omega(B)\le |B|\inf_{x\in B}\omega(x)\le C_2\omega(B). \tag{2.6}$$

Definition 2.1. For $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The maximal fractional function $M_\alpha(f)$ is defined by

$$M_\alpha(f)(x) = \sup_{x\in B} \frac{1}{|B|^{1-\alpha/n}} \int_{\mathbb{R}^n} |f(y)| dy.$$

Lemma 2.2 ([12]). *Let $0 < \alpha < n, 1/q = 1/p - \alpha/n$ and $\omega \in A_{p.q}$. Then*

$$\|I_\alpha(f)\|_{L^q(\omega^q)} \le C\|f\|_{L^p(\omega^p)} \quad and \quad \|M_\alpha(f)\|_{L^q(\omega^q)} \le C\|f\|_{L^p(\omega^p)}. \tag{2.7}$$

Let us recall the definition of weighted Lipschitz function space.

Definition 2.2. For $1 \le p < \infty, 0 < \beta < 1$, and $\omega \in A_\infty$. A locally integrable function b is said to be in the weighted Lipschitz function space if

$$\sup_B \frac{1}{\omega(B)^{\beta/n}} \left[\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x)^{1-p} dx \right]^{1/p} \le C < \infty, \tag{2.8}$$

where $b_B = |B|^{-1} \int_B b(y) dy$, and the supremum is taken over all ball $B \subset \mathbb{R}^n$.

The Banach space of such functions modulo constants is denoted by $Lip_{\beta,p}(\omega)$. The smallest bound C satisfying conditions above is then taken to be the norm of b denoted by $\|b\|_{Lip_{\beta,p}(\omega)}$. Put $Lip_{\beta,\omega} = Lip_{\beta,1}(\omega)$. Obviously, for the case $\omega = 1$, the $Lip_{\beta,p}(\omega)$ space is the classical Lip_β space. Let $\omega \in A_1$. García-Cuerva in [14] proved that the spaces $Lip_{\beta,p}(\omega)$ coincide, and the norms $\|b\|_{Lip_{\beta,p}(\omega)}$ are equivalent with respect to different values of p provided that $1 \le p < \infty$. Since we always discuss under the assumption $\omega \in A_1$ in the following, then we denote the norm of $Lip_{\beta,p}(\omega)$ by $\|\cdot\|_{Lip_{\beta,\omega}}$ for $1 \le p < \infty$.

Now we shall introduce the Hardy-Littlewood maximal operator and several variants.

For a given measurable function $f \in L^1_{loc}(\mathbb{R}^n)$, define the Hardy-Littlewood maximal operator Mf and the sharp maximal operator $M^\sharp f$ as

$$\begin{aligned} M(f)(x) &= \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)| dy, \\ M^\sharp(f)(x) &= \sup_{x\in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy \sim \sup_{x\in B} \inf_c \frac{1}{|B|} \int_B |f(y) - c| dy. \end{aligned}$$

For $0 < \beta < 1$, and $1 \le r < \infty$, we define the fractional weighted maximal operator $M_{\beta,\omega,r} f$ by

$$M_{\beta,\omega,r}(f)(x) = \sup_{x\in B} \left(\frac{1}{\omega(B)^{1-r\beta/n}} \int_B |f(y)|^r \omega(y) dy \right)^{1/r},$$

where the supremum is taken over is taken over all balls B.

The basic estimate is contained the following lemma of Fefferman and Stein; see [14].

Lemma 2.3. *Let* $0 < p < \infty$, *and* $\omega \in \cup_{1\le r<\infty} A_r$. *There exists a constant* C *such that*

$$\|Mf\|_{L^p(\omega)} \le C\|M^\sharp f\|_{L^p(\omega)}. \tag{2.9}$$

The following two lemmas play key roles in the proof of Theorem 1.1.

Lemma 2.4. *Let* $b \in Lip_{\beta,\omega}(0 < \beta < 1)$, $1/q = 1/r_0 - \beta/n$, $1/r_0 = 1/p - \alpha/n$, *and* $\mu = \omega^{r_0/p} \in A_1$. *Then there exist a sufficiently large number* s *and a constant* $C > 0$ *such that, for every* $f \in L^p(\omega)$ *with* $1 < r < p$, *we have*

$$\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'} \le C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(f)(x), \tag{2.10}$$

where $1/s + 1/s' = 1$.

Proof. Let $r_2 = r/s'$,$r_3 = \epsilon/(s'-1)$ and $1/r_1 + 1/r_2 + 1/r_3 = 1$, where ϵ is the constant in (2.5). Choosing a sufficiently large number s such that $1 < s' < r(1+\epsilon)/(r+\epsilon)$, then $r_1, r_2, r_3 > 1$. By Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'}\\
=&|B|^{-1/s'}\left(\int_B |b(x)-b_B|^{s'}\omega(x)^{1/r_1-s'}|f(x)|^{s'}\omega(x)^{\frac{1}{r_2}}\omega(x)^{s'-1/r_1-1/r_2}dx\right)^{1/s'}\\
\le&C|B|^{-1/s'}\left(\int_B |b(x)-b_B|^{r_1s'}\omega(x)^{1-r_1s'}dx\right)^{1/(r_1s')}\\
&\times\left(\int_B |f(x)|^{r_2s'}\omega(x)dx\right)^{1/(r_2s')}\left(\int_B \omega(x)^{1+r_3(s'-1)}dx\right)^{1/(r_3s')}.
\end{aligned}$$

Since $\mu = \omega^{r_0/p} \in A_1$, then

$$\left(\inf_{x\in B}\omega(x)\right)^{r_0/p-1}\omega(B) \le \mu(B) \le C\left(\inf_{x\in B}\omega(x)\right)^{r_0/p-1}\omega(B). \tag{2.11}$$

By $b \in Lip_{\beta,\omega}$, and (2.5), we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'}\\
\le& C\|b\|_{Lip_{\beta,\omega}}|B|^{-1/s'}\omega(B)^{\beta/n+1/(r_1s')}\\
&\times\left(\int_B |f(x)|^r\omega(x)dx\right)^{1/r}\left(\int_B \omega(x)^{1+\epsilon}dx\right)^{1/(r_3s')}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\frac{\omega(B)^{1+\beta/n}}{|B|}\left(\frac{1}{\omega(B)}\int_B |f(x)|^r\omega(x)dx\right)^{1/r}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\frac{\omega(B)^{1+\beta/n}}{|B|}\left(\frac{1}{\mu(B)}\int_B |f(x)|^r\mu(x)dx\right)^{1/r}.
\end{aligned}$$

But

$$\omega(B)^{\beta/n} = \left(\inf_{x\in B}\omega(x)\right)^{-\alpha\beta r_0/n^2}\left(\int_B \omega(x)\left(\inf_{x\in B}\omega(x)\right)^{\alpha r_0/n}dx\right)^{\beta/n}$$
$$\leq C\left(\inf_{x\in B}\omega(x)\right)^{-\alpha\beta r_0/n^2}\mu(B)^{\beta/n}. \tag{2.12}$$

By $1/q=1/r_0-\beta/n$, we have $1-\alpha\beta r_0/n^2>0$. Then

$$\left(\frac{1}{|B|}\int_B |b(x)-b_B|^{s'}|f(x)|^{s'}dx\right)^{1/s'}$$
$$\leq C\|b\|_{Lip_{\beta,\omega}}\left(\inf_{x\in B}\omega(x)\right)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(f)(x)$$
$$\leq C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(f)(x).$$

Thus, we complete the proof. □

Lemma 2.5 ([15]). *Let I_α be fractional integral operator, and let E be a measurable set in $\mathbb{R}^n$. Then for any $f\in L^1(\mathbb{R}^n)$, there exists a constant C such that*

$$\int_E |I_\alpha f(x)|\,dx \leq C\|f\|_{L^1}|E|^{\alpha/n}.$$

Finally, we recall the definition of variable Calderón-Zygmund kernel and its properties [16].

Definition 2.3. The function $K(x,\xi):\ \mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ is called a variable Calderón-Zygmund kernel if:

(i) for every fixed x, the function $K(x,\cdot)$ is a constant kernel satisfying

  (1) $K(x,\cdot)\in C^\infty(\mathbb{R}^n\setminus\{0\})$;
  (2) for any $\mu>0$, $K(x,\mu\xi)=\mu^{-n}K(x,\xi)$;
  (3) $\int_{\mathbb{S}^{n-1}}K(x,\xi)d\xi=0$ and $\int_{\mathbb{S}^{n-1}}|K(x,\xi)|d\xi<\infty$;

(ii) for every multiindex β, $\sup_{\xi\in\mathbb{S}^{n-1}}|D_\xi^\beta K(x,\xi)|\leq C(\beta)$ independent of x.

We need the spherical harmonics and their properties (see more detail in [17, 18]). Recall that any homogeneous polynomial $P:\mathbb{R}^n\to\mathbb{R}$ of degree m that satisfies $\Delta P=0$ is called an n-dimensional solid harmonic of degree m. Its restriction to the unit sphere $\mathbb{S}^{n-1}$ will be called an n-dimensional spherical harmonic of degree m Denote by $\mathbb{H}_m$ the space of all n-dimensional spherical harmonics of degree m. In general it results in a finite dimensional linear space with $g_m=dim\mathbb{H}_m$ such that $g_0=1, g_1=n$ and

$$g_m=C_{m+n-1}^{n-1}-C_{m+n-3}^{n-1}\leq C(n)m^{n-2},\qquad m\geq 2. \tag{2.13}$$

Furthermore, let $\{Y_{sm}\}_{s=1}^{g_m}$ be an orthonormal base of $\mathbb{H}_m$, then $\{Y_{sm}\}_{s=1}^{g_m}{}_{m=0}^{\infty}$ is a complete orthonormal system in $L^2(\mathbb{S}^{n-1})$ and

$$\sup_{x\in\mathbb{S}^{n-1}} |D_x^{\beta} Y_{sm}(x)| \le C(n) m^{|\beta|+(n-2)/2}, \qquad m=1,2,\cdots. \tag{2.14}$$

If, for instance, $\phi \in C^{\infty}(\mathbb{S}^{n-1})$, then $\sum_{s,m} a_{sm} Y_{sm}$ is the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}\}_{sm}$ then

$$a_{sm} = \int_{\mathbb{S}^{n-1}} \phi(y) Y_{sm}(y) d\sigma, \qquad |a_{sm}| \le C(n,l) m^{-2l} \sup_{|\beta|=2l} \sup_{y\in\mathbb{S}^{n-1}} |D_y^{\beta}\phi(y)|, \tag{2.15}$$

for any integer l. In particular, the expansion of ϕ into spherical harmonics converges uniformly to ϕ. The proof of the above results can be found in [19].

Let $x,y\in\mathbb{R}^n$, and $\overline{y}=y/|y|\in\mathbb{S}^{n-1}$. In view of the properties of the kernel K with respect to the second variable and the complete of $\{Y_{sm}(x)\}$ in $L^2(\mathbb{S}^{n-1})$, we get

$$\begin{aligned} K(x,x-y) &= |x-y|^{-n} K(x,\overline{x-y}) \\ &= |x-y|^{-n} \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x) Y_{sm}(\overline{x-y}). \end{aligned}$$

Replacing the kernel with its series expansion, (1.1) can be written as

$$\begin{aligned} T(f)(x) &= \lim_{\epsilon\to 0} T_{\epsilon}(f)(x) \\ &= \lim_{\epsilon\to 0} \int_{\rho(x-y)>\epsilon} \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x)|x-y|^{-n} Y_{sm}(\overline{x-y}) f(y) dy. \end{aligned}$$

From the properties of (2.13)-(2.15), the series expansion

$$\sum_{m=1}^{N}\sum_{s=1}^{g_m} |a_{sm}(x)|x-y|^{-n} Y_{sm}(\overline{x-y}) f(y)| \le C \frac{|f(y)|}{|x-y|^n} \sum_{m=1}^{\infty} m^{3(n-2)/2-2l},$$

where the integer l is preliminarily chosen greater than $(3n-4)/4$. Along with the $|x-y|^{-n} f(y)\in L^1(\mathbb{R}^n)$ for almost everywhere $x\in\mathbb{R}^n$, by the Fubini dominated convergence theorem, we have

$$\begin{aligned} T(f)(x) &= \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x) \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} H_{sm}(x-y) f(y) dy \\ &= \sum_{m=1}^{\infty}\sum_{s=1}^{g_m} a_{sm}(x) T_{sm} f(x), \end{aligned} \tag{2.16}$$

where

$$H_{sm}(x-y) = |x-y|^{-n} Y_{sm}(\overline{x-y}),$$

and H_{sm} satisfies pointwise Hörmander condition as following

$$|H_{sm}(x-y)-H_{sm}(x_0-y)|\leq Cm^{n/2}\frac{|x_0-x|}{|x-y|^{n+1}} \tag{2.17}$$

for each $x\in B$ and $y\notin 2B$ (see Lemma 3.2 in [19]). Then

$$\begin{aligned} T_{sm}f(x)=&\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}H_{sm}(x-y)f(y)dy\\ =&p.v.\int_{\mathbb{R}^n}H_{sm}(x-y)f(y)dy \end{aligned} \tag{2.18}$$

is a classical Calderón-Zygmund operator with a constant kernel.

3 Proof of the theorem

Let

$$T^b(f)(x)=\sum_{k=1}^{t}(T^{k,1}M^bI_\alpha T^{k,2}(f)(x)+T^{k,3}I_\alpha M^bT^{k,4}(f)(x)).$$

Without loss generality, we may assume $T^{k,1}$ $(k=1,\cdots,t)$ are singular integral operators with variable Calderón-Zygmund kernel. By (2.16)

$$\begin{aligned} T^b(f)(x)=&\sum_{k=1}^{t}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a_{sm}^{k,1}(x)T_{sm}^{k,1}M^bI_\alpha T^{k,2}(f)(x)\\ &+\sum_{k=1}^{t}T^{k,3}I_\alpha M^bT^{k,4}(f)(x)), \end{aligned} \tag{3.1}$$

where,

$$T_{sm}^{k,1}(f)(x)=\int_{\mathbb{R}^n}H_{sm}^{k,1}(x-y)f(y)dy \tag{3.2}$$

are classical Calderón-Zygmund operator with constant kernel as (2.18). For arbitrary $x\in\mathbb{R}^n$, set B for the ball centered at x_0 and of radius r, and $B\ni x$. Since $T^1(f)=0$ for any $f\in L^p(\omega)$, then by (3.1),

$$\begin{aligned} &T^b(f)(x)=T^{b-b_{2B}}(f)(x)\\ =&\sum_{k=1}^{t}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a_{sm}^{k,1}(x)T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(x)+\sum_{k=1}^{t}T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(x)). \end{aligned} \tag{3.3}$$

Let us first prove the following inequality:

$$\begin{aligned} &M^\sharp T_{sm}^{k,1}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(x)\\ \leq& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-r_0\alpha\beta/n^2}(M_{\beta,\mu,r}(I_\alpha T^{k,2}f)(x)+M_{\beta,\mu,1}(I_\alpha T^{k,2}(f))(x)), \end{aligned} \tag{3.4}$$

where $1<r<\infty, 1/r_0=1/p-\alpha/n$, and $\mu=\omega^{r_0/p}$. Write $T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(x)$ as

$$\begin{aligned}&T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(y)\\=&T^{k,1}_{sm}M^{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)+T^{k,1}_{sm}M^{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,2}(f)(y)\\=&\mathrm{U}_1(y)+\mathrm{U}_2(y).\end{aligned}$$

Taking $c=\mathrm{U}_2(x_0)$, then

$$\begin{aligned}&\frac{1}{|B|}\int_B|T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)(y)-c|dy\\\le&\frac{1}{|B|}\int_B|\mathrm{U}_1(y)|dy+\frac{1}{|B|}\int_B|\mathrm{U}_2(y)-\mathrm{U}_2(x_0)|dy\\=:&M_1+M_2.\end{aligned}$$

Choosing a sufficiently large number s and by Hölder's inequality, the boundedness of $T^{k,1}_{sm}$ in $L^{s'}(\mathbb{R}^n)$ and Lemma 2.4, we have

$$\begin{aligned}M_1=&\frac{1}{|B|}\int_B|T^{k,1}_{sm}M^{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|dy\\\le&\left(\frac{1}{|B|}\int_B|T^{k,1}_{sm}M^{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|^{s'}dy\right)^{1/s'}\\\le&C\left(\frac{1}{|B|}\int_{\mathbb{R}^n}|M^{(b-b_{2B})\chi_{2B}}I_\alpha T^{k,2}(f)(y)|^{s'}dy\right)^{1/s'}\\\le&C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x).\end{aligned}$$

For any $x,y\in B$, and $z\in(2B)^c$, we have $|y-z|\sim|x_0-z|$. Then by (2.17) we get,

$$\begin{aligned}M_2\le&\frac{1}{|B|}\int_B|T^{k,1}_{sm}M^{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,2}(f)(y)-T^{k,1}_{sm}M^{(b-b_{2B})\chi_{(2B)^c}}I_\alpha T^{k,2}(f)(x_0)|dy\\\le&C\frac{1}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}||H^{k,1}_{sm}(y-z)-H^{k,1}_{sm}(x_0-z)||I_\alpha T^{k,2}(f)(z)|dzdy\\\le&Cm^{n/2}\frac{1}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}|\frac{|x_0-y|}{|y-z|^{n+1}}|I_\alpha T^{k,2}(f)(z)|dzdy\\\le&Cm^{n/2}\sum_{j=1}^\infty\int_{2^{j+1}B\backslash 2^jB}|b(z)-b_{2B}|\frac{r}{|x_0-z|^{n+1}}|I_\alpha T^{k,2}(f)(z)|dz\\\le&Cm^{n/2}\sum_{j=1}^\infty\frac{r}{(2^jr)^{n+1}}\int_{2^{j+1}B}|b(z)-b_{2B}||I_\alpha T^{k,2}(f)(z)|dz\\\le&Cm^{n/2}\sum_{j=1}^\infty 2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|dz\end{aligned}$$

$$+Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||I_\alpha T^{k,2}(f)(z)|dz$$
$$=:M_{21}+M_{22}.$$

By (2.12),

$$\omega(2^{j+1}B)^{\beta/n}\le C\left(\inf_{x\in 2^{j+1}B}\omega(x)\right)^{-\alpha\beta r_0/n^2}\mu(2^{j+1}B)^{\beta/n}. \tag{3.5}$$

Since $1-\alpha\beta r_0/n^2>0$, we get

$$\begin{aligned}
|b_{2^{j+1}B}-b_{2B}|\le&\sum_{k=1}^{j}\frac{1}{|2^kB|}\int_{2^{k+1}B}|b(z)-b_{2^{k+1}B}|dz\\
\le& C\|b\|_{Lip_{\beta,\omega}}\sum_{k=1}^{j}\frac{\omega(2^{k+1}B)^{1+\beta/n}}{|2^kB|}\\
\le& C\|b\|_{Lip_{\beta,\omega}}\sum_{k=1}^{j}\inf_{x\in 2^{k+1}B}\omega(x)\omega(2^{k+1}B)^{\beta/n}\\
\le& C\|b\|_{Lip_{\beta,\omega}}\mu(2^{j+1}B)^{\beta/n}\sum_{k=1}^{j}\left(\inf_{x\in 2^{k+1}B}\omega(x)\right)^{1-\alpha\beta r_0/n^2}\\
\le& Cj\|b\|_{Lip_{\beta,\omega}}\left(\inf_{x\in B}\omega(x)\right)^{1-\alpha\beta r_0/n^2}\mu(2^{j+1}B)^{\beta/n}.
\end{aligned}$$

Thus

$$\begin{aligned}
M_{21}=&Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|dz\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}\\
&\times\sum_{j=1}^{\infty}j2^{-j}\frac{1}{\mu(2^{j+1}B)^{1-\beta/n}}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|\mu(z)dz\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,1}(I_\alpha T^{k,2}(f))(x)\sum_{j=1}^{\infty}j2^{-j}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\omega,1}(I_\alpha T^{k,2}(f))(x).
\end{aligned}$$

By Hölder's inequality, (2.11) and (3.5) we get

$$\begin{aligned}
M_{22}=&Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||I_\alpha T^{k,2}(f)(z)|dz\\
\le& Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^r\omega(z)dz\right)^{1/r}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\sum_{j=1}^{\infty}2^{-j}\frac{\omega(2^{j+1}B)^{1+\beta/n}}{|2^{j+1}B|}\left(\frac{1}{\mu(2^{j+1}B)}\int_{2^{j+1}B}|I_\alpha T^{k,2}(f)(z)|^r\mu(z)dz\right)^{1/r}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x)\sum_{j=1}^{\infty}2^{-j}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha\beta r_0/n^2}M_{\beta,\mu,r}(I_\alpha T^{k,2}(f))(x).
\end{aligned}
$$

Combining the above estimates for M_1, M_2, it finished the proof of (3.4).

Let $1/q=1/r_0-\beta/n$, and $1/r_0=1/p-\alpha/n$. Since $\omega^{q/p}\in A_1$, we have $\mu=\omega^{r_0/p}\in A_1$, and by (2.4) we have $\omega^{1/p}\in A_{r_0,p}$. Note that $(1-\alpha\beta r_0/n^2)q+1-(1-\alpha/n)q=r_0/p$. Thus, by Lemma 2.3, (3.4) and Lemma 2.2, we have

$$
\begin{aligned}
&\|T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\le& C\|M^\sharp T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\Big(\|M_{\beta,\mu,r}(I_\alpha T^{k,2}f)\|_{L^q(\mu)}+\|M_{\beta,\mu,1}(I_\alpha T^{k,2}f)\|_{L^q(\mu)}\\
&\quad+\|M_{\alpha+\beta,\omega,r}(T^{k,4}f)\|_{L^q(\omega)}+\|M_{\alpha+\beta,\omega,1}(T^{k,4}f)\|_{L^q(\omega)}\Big)\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\Big(\|I_\alpha T^{k,2}f\|_{L^{r_0}(\omega^{r_0/p})}+\|T^{k,4}f\|_{L^p(\omega)}\Big)\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^p(\omega)}.
\end{aligned}
$$

Choosing $l>(3n-2)/4$, then by (2.13) we get

$$
\begin{aligned}
&\Big\|\sum_{k=1}^{t}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}a^{k,1}_{sm}T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\Big\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\le&\sum_{k=1}^{t}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}\|a^{k,1}_{sm}\|_{L^\infty}\|T^{k,1}_{sm}M^{b-b_{2B}}I_\alpha T^{k,2}(f)\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\le& C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^p(\omega)}\sum_{k=1}^{t}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}m^{-2l+n/2}\\
\le& C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^p(\omega)}\sum_{m=1}^{\infty}m^{-2l+n/2+n-2}\le C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^p(\omega)}. \qquad (3.6)
\end{aligned}
$$

Next, we prove

$$
\begin{aligned}
&M^\sharp T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(x)\\
\le& C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha/n}(M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x)+M_{\alpha+\beta,\omega,1}(T^{k,4}(f))(x))\\
&+C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x). \qquad (3.7)
\end{aligned}
$$

Write $T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)$ as

$$\begin{aligned}&T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(y)\\ =&T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)+T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(y)\\ =:&V_1(y)+V_2(y).\end{aligned}$$

Taking $c=\mathrm{V}_2(x_0)$, then

$$\begin{aligned}&\frac{1}{|B|}\int_B|T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)(y)-c|dy\\ \le&\frac{1}{|B|}\int_B|\mathrm{V}_1(y)|dy+\frac{1}{|B|}\int_B|\mathrm{V}_2(y)-\mathrm{V}_2(x_0)|dy\\ =:&N_1+N_2.\end{aligned}$$

Since $T^{k,3}=\pm I$, by Lemma 2.5 and Hölder's inequality, then we deduce that

$$\begin{aligned}N_1=&\frac{1}{|B|}\int_B|I_\alpha M^{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)|dy\\ \le&\frac{C}{|B|^{1-\alpha/n}}\int_{R^n}|M^{(b-b_{2B})\chi_{2B}}T^{k,4}(f)(y)|dy\\ \le&\frac{C}{|B|^{1-\alpha/n}}\left(\int_{2B}|b(y)-b_{2B}|^{r'}\omega(y)^{1-r'}dy\right)^{1/r'}\left(\int_{2B}|T^{k,4}(f)(y)|^r\omega(y)dy\right)^{1/r}\\ \le&C\|b\|_{Lip_{\beta,\omega}}M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x)\left(\frac{\omega(B)}{|B|}\right)^{1-\alpha/n}\\ \le&C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,\omega,r}(T^{k,4}f)(x).\end{aligned}$$

For any $y\in B$, and $z\in(2B)^c$, we have $|y-z|\sim|x_0-z|$. Thus,

$$\begin{aligned}N_2\le&\frac{1}{|B|}\int_B|T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(y)-T^{k,3}I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}}T^{k,4}(f)(x_0)|dy\\ \le&C\frac{1}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}|\left|\frac{1}{|y-z|^{n-\alpha}}-\frac{1}{|x_0-z|^{n-\alpha}}\right||T^{k,2}(f)(z)|dzdy\\ \le&C\frac{1}{|B|}\int_B\int_{(2B)^c}|b(z)-b_{2B}\frac{|x_0-y|}{|x_0-z|^{n-\alpha+1}}|T^{k,2}(f)(z)|dzdy\\ \le&C\sum_{j=1}^\infty\frac{r}{(2^jr)^{n-\alpha+1}}\int_{2^{j+1}B}|b(z)-b_{2B}||T^{k,2}(f)(z)|dz\\ \le&C\sum_{j=1}^\infty2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,2}(f)(z)|dz\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||T^{k,2}(f)(z)|dz\\
=:&N_{21}+N_{22}.
\end{aligned}
$$

Note

$$
\begin{aligned}
|b_{2^{j+1}B}-b_{2B}|\leq&C\|b\|_{Lip_{\beta,\omega}}\sum_{k=1}^{j}\inf_{x\in 2^{k+1}B}\omega(x)\omega(2^{k+1}B)^{\beta/n}\\
\leq&Cj\|b\|_{Lip_{\beta,\omega}}\omega(x)\left(\inf_{x\in 2^{j+1}B}\omega(x)\right)^{\beta/n}|2^{j+1}B|^{\beta/n}\\
\leq&Cj\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}|2^{j+1}B|^{\beta/n},
\end{aligned}
$$

then

$$
\begin{aligned}
N_{21}=&C\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
\leq&C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}\sum_{j=1}^{\infty}j2^{-j}\frac{1}{|2^{j+1}B|^{1-(\alpha+\beta)/n}}\int_{2^{j+1}B}|T^{k,4}(f)(z)|dz\\
\leq&C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x)\sum_{j=1}^{\infty}j2^{-j}\\
\leq&C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))(x).
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
N_{22}\leq&C\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}B|^{1-\alpha/n}}\left(\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}|^{r'}\omega(z)^{1-r'}dz\right)^{1/r'}\\
&\times\left(\int_{2^{j+1}B}|T^{k,2}(f)(z)|^{r}\omega(z)dz\right)^{1/r}\\
\leq&\ C\|b\|_{Lip_{\beta,\omega}}\sum_{j=1}^{\infty}2^{-j}\left(\frac{\omega(2^{j+1}B)}{|2^{j+1}B|}\right)^{1-\alpha/n}\\
&\times\left(\frac{1}{\omega(2^{j+1}B)^{1-(\alpha+\beta)r/n}}\int_{2^{j+1}B}|T^{k,2}(f)(z)|^{r}\omega(z)dz\right)^{1/r}\\
\leq&C\|b\|_{Lip_{\beta,\omega}}\omega(x)^{1-\alpha/n}M_{\alpha+\beta,\omega,r}(T^{k,4}(f))(x).
\end{aligned}
$$

Combining the estimates for N_1 and N_2, the proof of (3.6) is completed.

Since $\omega^{q/p}\in A_1$, then $\omega^{1/p}\in A_{p,q}$. Thus by Lemma 2.2 we get

$$
\begin{aligned}
&\left\|\omega(\cdot)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))\right\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
=&C\left\|M_{\alpha+\beta}(T^{k,4}(f))\right\|_{L^q(\omega^{q/p})}\leq C\|T^{k,4}(f))\|_{L^p(\omega)}\leq C\|f\|_{L^p(\omega)}.
\end{aligned}
$$

By $\omega \in A_1$, we get

$$\begin{aligned}
&\left\|\omega(\cdot)^{1-\alpha/n}(M_{\alpha+\beta,\omega,r}(T^{k,4}f)+M_{\alpha+\beta,\omega,1}(T^{k,4}(f))\right\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
=&\left\|M_{\alpha+\beta,\omega,r}(T^{k,4}f)+M_{\alpha+\beta,\omega,1}(T^{k,4}(f))\right\|_{L^q(\omega)}\\
\leq& C\|T^{k,4}f\|_{L^p(\omega)}\leq C\|f\|_{L^p(\omega)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|\sum_{k=1}^{t}T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)\right\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\leq& C\left\|\sum_{k=1}^{t}M^\sharp T^{k,3}I_\alpha M^{b-b_{2B}}T^{k,4}(f)\right\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\leq& C\sum_{k=1}^{t}\left\|\omega(\cdot)^{1-\alpha/n}(M_{\alpha+\beta,\omega,r}(T^{k,4}f)+M_{\alpha+\beta,\omega,1}(T^{k,4}(f))\right\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
&\quad+C\sum_{k=1}^{t}\left\|\omega(\cdot)^{1+\beta/n}M_{\alpha+\beta}(T^{k,4}(f))\right\|_{L^q(\omega^{1-(1-\alpha/n)q})}\\
\leq& C\|f\|_{L^p(\omega)}.
\end{aligned}\tag{3.8}$$

Combined with (3.3), (3.6) and (3.8), it finishes the proof of Theorem 1.1. □

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