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On the Solution of Fractional Burgers' Equation and Its Optimal Control Problem

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Abstract. The main aim of this study is to prove that a class of fractional Burgers' equations has a unique solution under some special conditions. Then it is demonstrated that an optimal control problem for this class of fractional Burgers' equations has at least one optimal solution.

Key Words: Burgers' equation, fractional optimal control, linear functional.

AMS Subject Classifications: 26A33, 34A08, 35R11

1 Introduction

Burgers' equation is a fundamental partial differential equation occurring in various areas of applied mathematics such as fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow [4,11].

Fractional Burgers' equation [9] describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in the other systems such as shallow-water waves and waves in bubbly liquids.

Some researchers have worked on solving this equation analytically and numerically. In [2], the fractional derivatives in the sense of the Jumaries modified Riemann-Liouville derivative of order alpha and the fractional complex transform are used to obtain the most general solution for the fractional Burgers' equation. Saad and Al-Sharif [14] applied the variational iteration method (VIM) to solve the time and space-time fractional

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Burgers' equation for various initial conditions. In [11], the fractional reduced differential transform method (FRDTM) is proposed to solve nonlinear fractional partial differential equations such as the space-time fractional Burgers' equations and the time-fractional Cahn-Allen equation. N. Khan et al. [5] used the generalized differential transform method (GDTM) and homotopy perturbation method (HPM) for solving time-fractional Burgers' and coupled Burgers' equations. N. Khan et al. [6] are used the Haar wavelet method to be obtained the numerical solutions of the time-fractional Schrödinger equations. The main advantages of the Haar wavelet method are its simple application and no requirement of residual or product operational matrix. In [7], some properties of Taylor's series with the heuristic optimization techniques are investigated some classes of time fractional convection-diffusion equations. Esen and Tasbozan [3] presented some numerical examples for the time fractional Burgers' equation with various boundary and initial conditions obtained by collocation method using cubic B-spline base functions.

This paper investigates an optimal control problem for a class of fractional Burgers' equations. The researcher proves that this problem has at least one optimal solution. Optimal control problems governed by time-dependent fractional partial differential equations play an important role in many practical problems such as production factor mobility, technological dissemination, pollution spreading and others. In technical applications, the fractional partial differential equations are not solved for their own sake but the solution should fulfill some requirements. In some sense, the application asks for an optimal solution of the problem.

This paper is organized as follows. In Section 2, we give some preliminaries which is needed to prove the main results in this paper. In Section 3, we introduce an optimal control problem for a class of fractional Burgers' equations and give two important results related to this problem. Finally, the conclusion is presented in Section 4.

2 Preliminary results

Definition 2.1 ([13])**.** Let $\mathbf{X}$ be a normed linear space. A linear functional $\mathbb{T}$ on $\mathbf{X}$ is said to be bounded if there is an $M \geq 0$ such that

$$|\mathbb{T}(f)| \leq M\|f\| \quad \text{for any } f \in \mathbf{X}. \tag{2.1}$$

The infimum of all such M is called the norm of $\mathbb{T}$ and it is denoted by $\|\mathbb{T}\|_\star$. The collection of bounded linear functionals on $\mathbf{X}$ is denoted by $\mathbf{X}^\star$ and is called the dual space of $\mathbf{X}$ which is a linear space.

Definition 2.2 ([13])**.** The linear operator $\mathbb{J} : \mathbf{X} \to (\mathbf{X}^\star)^\star$ defined by

$$\mathbb{J}(x)[\psi] = \psi(x) \quad \text{for all } x \in \mathbf{X}, \ \psi \in \mathbf{X}^\star, \tag{2.2}$$

is called the natural embedding of $\mathbf{X}$ into $(\mathbf{X}^\star)^\star$. Also, the space $\mathbf{X}$ is said to be reflexive when $\mathbb{J}(\mathbf{X}) = (\mathbf{X}^\star)^\star$. It is customary to denote $(\mathbf{X}^\star)^\star$ by $\mathbf{X}^{\star\star}$ and call $\mathbf{X}^{\star\star}$ the bidual of $\mathbf{X}$.

Definition 2.3 ([13]). A normed linear space $\mathbf{X}$ is said to be separable when there is a countable subset of X that is dense in $\mathbf{X}$.

Remark 2.1 ([13]). If $\mathbf{E}$ is a measurable set, then the normed linear space $\mathbf{L}^p(\mathbf{E})$ is separable for all $1 \leq p < \infty$.

Definition 2.4 ([13]). A Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm.

Let Y be a real Banach space.

Definition 2.5 ([15]). An operator $B : Y \longrightarrow Y^\star$ is said to be bounded if it maps bounded sets of Y into bounded subsets of $Y^\star$.

Definition 2.6 ([15]). An operator $B : Y \longrightarrow Y^\star$ is said to be hemi-continuous if for each $z_1, z_2, w \in Y$, the function

$$f(\lambda) = \langle B(z_1 + \lambda z_2), w \rangle, \qquad \lambda \in \mathbb{R}, \tag{2.3}$$

is continuous.

Definition 2.7 ([15]). An operator $B : Y \longrightarrow Y^\star$ is said to be monotone if

$$\langle B(z_1) - B(z_2), z_1 - z_2 \rangle \geq 0, \tag{2.4}$$

for all $z_1, z_2 \in Y$. Also, if for $z_1, z_2 \in Y$, $z_1 \neq z_2$,

$$\langle B(z_1) - B(z_2), z_1 - z_2 \rangle > 0, \tag{2.5}$$

B is said to be strictly monotone.

Definition 2.8 ([16]). A sequence $\{x_n\}$ in a normed space X is said to be strongly convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0. \tag{2.6}$$

Definition 2.9 ([16]). A sequence $\{x_n\}$ in a normed space X is said to be weakly convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} T(x_n) = T(x) \tag{2.7}$$

for every continuous linear functional T in $X^\star$.

Definition 2.10 ([15]). A bounded operator $B : Y \longrightarrow Y^\star$ is said to be pseudo-monotone when

1. $\limsup_{n \to \infty} \langle B(u_n), u_n - u \rangle \leq 0$ implies $\lim_{n \to \infty} \langle B(u_n), u_n - u \rangle = 0$,
2. $B(u_n) \longrightarrow B(u)$, as $n \to \infty$, weakly convergent in $Y^\star$.

for all sequence $\{u_n\}$ in Y weakly convergance to u.

Theorem 2.1 ([15]). *Let Y be a reflexive Banach space. Assume that $B : Y \longrightarrow Y^\star$ is bounded, hemi-continuous and monotone. Then B is pseudo-monotone.*

Definition 2.11 ([15]). An operator $B : Y \longrightarrow Y^\star$ is said to be coercive if

$$\lim_{\|z\|\to\infty} \frac{\langle B(z), z\rangle}{\|z\|} = \infty. \tag{2.8}$$

Now, consider the operator $B : Y \longrightarrow Y^\star$ and functional $F \in Y^\star$ such that

$$B(z) = F. \tag{2.9}$$

Theorem 2.2 ([15]). *Let Y be a reflexive separable Banach space and assume that $B : Y \longrightarrow Y^\star$ is pseudo-monotone and coercive. Then for any arbitrary functional $F \in Y^\star$ there exists a solution $z \in Y$ for the equation* (2.9). *Furthermore, if B is strictly monotone, then the solution is unique.*

Lemma 2.1 (Mazur's lemma [12]). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in Y that converges weakly to some $u_0 \in Y$. Then there exists a function $N : \mathbb{N} \to \mathbb{N}$ such that the sequence $\{\bar{u}_n\}_{n\in\mathbb{N}}$ defined by*

$$\bar{u}_n = \sum_{k=n}^{N(n)} \lambda_k u_k \tag{2.10}$$

convergents strongly to u_0 in Y, where $\lambda_k \geq 0$, $(k = n, \cdots, N(n))$ and $\sum_{k=n}^{N(n)} \lambda_k = 1$.

Theorem 2.3 (Lebesgue's Dominated Convergence Theorem). *Suppose that $\{f_n\}$ is a sequence of measurable functions on a measurable set Λ, $f_n \to f$ pointwise almost everywhere as $n \to \infty$, and $|f_n| \leq g$ for all n, where g is a integrable function on Λ. Then f is integrable on Λ and*

$$\int_\Lambda f d\mu = \lim_{n\to\infty} \int_\Lambda f_n d\mu. \tag{2.11}$$

Definition 2.12 ([8,10]). Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a continuous function and $\alpha > 0$. Then the expression

$${}_0^R I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad t > 0, \tag{2.12}$$

is called the left Riemann-Liouville integral of order α, where $\Gamma(\cdot)$ is the gamma function and defined as follows

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt. \tag{2.13}$$

Definition 2.13 ([8,10]). Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$. The left Riemann-Liouville fractional derivative of order α of f is defined by

$${}_0^R D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \qquad t > 0, \tag{2.14}$$

where $\alpha \in (n-1, n)$, $n \in \mathbb{N}$.

Theorem 2.4 ([8,10]). *Let $T > 0$, $u \in C^m([0,T])$, $\alpha \in (m-1,m)$, $m \in \mathbb{N}$, $v \in C^1([0,T])$. Then for $t \in [0,T]$ the following properties hold*

1. ${}_0^R D_t^\alpha v(t) = \frac{d}{dt}\, {}_0^R I_t^{1-\alpha} v(t)$, $m = 1$,
2. ${}_0^R D_t^\alpha\, {}_0^R I_t^\alpha v(t) = v(t)$,
3. ${}_0^R I_t^\alpha\, {}_0^R D_t^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^k(0)$,
4. $\lim_{t\to 0+} {}_0^R D_t^\alpha u(t) = \lim_{t\to 0+} {}_0^R I_t^\alpha u(t) = 0$.

Definition 2.14 ([1]). Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $1 \le p \le \infty$, $k \in \mathbb{N}$. The Sobolev space $W^{k,p}(\Omega)$ is defined to be the set of all functions f defined on Ω such that for any multi-index α with $|\alpha| \le k$, the mixed partial derivatives

$$f^{(\alpha)} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \tag{2.15}$$

exists in the weak sense and

$$\left\| f^{(\alpha)} \right\|_{L^p(\Omega)} < \infty. \tag{2.16}$$

The natural number k is called the order of the Sobolev space $W^{k,p}(\Omega)$. A norm for $W^{k,p}(\Omega)$ is defined as follows.

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\le k} \left\| f^{(\alpha)} \right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{|\alpha|\le k} \left\| f^{(\alpha)} \right\|_{L^\infty(\Omega)}, & p = \infty. \end{cases} \tag{2.17}$$

3 Main results

Let $\Sigma = [0,T]$, $\Omega = [0,1]$, $Q = \Sigma \times \Omega$, $T \ge 1$ and $p > 1$. Assume that $\nu \ge 0$, $\beta \ge 0$, $\gamma \ge 0$, $a > 0$ are given constants. Also, suppose that $f \in L^\infty(Q)$ and $y_0 \in L^\infty(\Omega)$ such that $|y_0| \le a$ a.e. in Ω and $y_0(0) = y_0(1) = 0$. We consider the following optimal control problem associated to a class of fractional Burgers' equations:

Minimize

$$J(y,u) = \beta \|y - y_d\|_{L^p(Q)}^p + \gamma \|u\|_{L^p(Q)}^p, \tag{3.1}$$

subject to

$$\begin{cases} \frac{\partial y}{\partial t} + y \frac{\partial y}{\partial x} - \nu\, {}_0^R D_x^{1+\alpha} y = f + u & \text{in } Q, \\ y(t,0) = y(t,1) = 0, & t \in \Sigma, \\ y(0,x) = y_0(x), & x \in \Omega, \end{cases} \tag{3.2}$$

where $y \in W^{2,p}(Q)$ and $u \in L^\infty(\Omega)$ are the state and control variables, respectively, and $\alpha \in (0,1)$. We introduce the set

$$X = \{(y,u) : (y,u) \in \ (L^\infty(Q))^2,\ |u| \le b \ a.e. \text{ in } Q\}. \tag{3.3}$$

3.1 On the solution of the fractional Burgers' equation

In the following, we first show that the system (3.2) has a unique solution and then we prove that there exists at least one optimal solution for the optimal control problem (3.1)-(3.2).

Theorem 3.1. *Let $T > 0$, $p > 1$, $\alpha \in (0,1)$, $b > 0$ be given constants and ν, a and K be positive constants which satisfy*

$$\frac{\nu K}{\Gamma(2-\alpha)} - T^{\frac{p-2}{p}} - 2aT^{\frac{p-2}{p}} > 0. \tag{3.4}$$

Assume that $y_0 \in L^\infty(\Omega)$, $y_0(0) = y_0(1) = 0$ a.e. in Ω, y_0' and y_0'' exist a.e. in Ω, and $|y_0| \le a$, $|y_0'| \le a$, $|y_0''| \le a$ a.e. in Ω. Then, for any $u \in L^\infty(Q)$ which satisfies $|u| \le b$, the problem (3.2) *has a unique solution $y \in W^{2,p}(Q)$.*

Proof. Let $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Without loss of generality, we suppose that $y_0 = 0$ in Ω. Otherwise, we make the change $z = y - y_0$. Let V be a closed linear normed subspace of the Soboloev space $W^{2,p}(Q)$, equipped with $W^{2,p}(Q)$-norm, such that $0 \in V$ and for any $y_1, y_2 \in V$ we have

$$\begin{aligned}
&\Big|\int_0^T \int_0^1 \int_0^x (x-s)^{1-\alpha} \left(\frac{\partial y_1}{\partial s}(t,s) - \frac{\partial y_2}{\partial s}(t,s)\right) ds \left(\frac{\partial^2 y_1}{\partial x^2}(t,x) - \frac{\partial^2 y_2}{\partial x^2}(t,x)\right) dxdt\Big| \\
\ge& K\Big(\int_0^T \int_0^1 |y_1 - y_2|^p(t,x)dxdt + \int_0^T \int_0^1 \left|\frac{\partial}{\partial x}(y_1 - y_2)\right|^p (t,x)dxdt \\
&+ \int_0^T \int_0^1 \left|\frac{\partial}{\partial t}(y_1 - y_2)\right|^p (t,x)dxdt + \int_0^T \int_0^1 \left|\frac{\partial^2}{\partial x^2}(y_1 - y_2)\right|^p (t,x)dxdt \\
&+ \int_0^T \int_0^1 \left|\frac{\partial^2}{\partial t^2}(y_1 - y_2)\right|^p (t,x)dxdt + \int_0^T \int_0^1 \left|\frac{\partial^2}{\partial x \partial t}(y_1 - y_2)\right|^p (t,x)dxdt\Big)^{\frac{2}{p}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_0^1 \int_0^x (x-s)^{1-\alpha} \left(\frac{\partial y_1}{\partial s}(t,s) - \frac{\partial y_2}{\partial s}(t,s)\right) ds \left(\frac{\partial^2 y_1}{\partial x^2}(t,x) - \frac{\partial^2 y_2}{\partial x^2}(t,x)\right) dxdt \le 0, \\
&y_i(t,0) = y_i(t,1) = 0, \qquad t \in \Sigma, \\
&|y_i| \le a, \quad \left|\frac{\partial y_i}{\partial x}\right| \le a, \quad \left|\frac{\partial^2 y_i}{\partial x^2}\right| \le a \qquad a.e. \text{ in } Q, \\
&y_i(0,x) = y_i(T,x) = 0, \quad x \in \Omega, \quad i \in \{1,2\}.
\end{aligned}$$

In particular, for $y \in V$, we have

$$
\begin{aligned}
&\int_0^T \int_0^1 \left(\int_0^x (x-s)^{1-\alpha} \frac{\partial y}{\partial s}(t,s)ds \right) \frac{\partial^2 y}{\partial x^2}(t,x)dxdt \leq 0, \\
&\left| \int_0^T \int_0^1 \int_0^x (x-s)^{1-\alpha} \frac{\partial y}{\partial s}(t,s)ds \frac{\partial^2 y}{\partial x^2}(t,x)dxdt \right| \\
&\geq K \Big(\int_0^T \int_0^1 |y(t,x)|^p \, dxdt + \int_0^T \int_0^1 \left| \frac{\partial y}{\partial x}(t,x) \right|^p dxdt + \int_0^T \int_0^1 \left| \frac{\partial y}{\partial t}(t,x) \right|^p dxdt \\
&\quad + \int_0^T \int_0^1 \left| \frac{\partial^2 y}{\partial x^2}(t,x) \right|^p dxdt + \int_0^T \int_0^1 \left| \frac{\partial^2 y}{\partial t^2}(t,x) \right|^p dxdt \\
&\quad + \int_0^T \int_0^1 \left| \frac{\partial^2 y}{\partial x \partial t}(t,x) \right|^p dxdt \Big)^{\frac{2}{p}} = \|y\|_V^2.
\end{aligned}
$$

We will show that $V \neq \{0\}$. Really, let $T=1$, $p>1$, $\alpha \in (0,1)$ be arbitrarily chosen, $a = 100000$ and

$$
\begin{aligned}
&f(t,x) = \begin{cases} \dfrac{10}{(t+1)^2(x+1)^2}, & (t,x) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right], \\ 0, & (t,x) \in Q \backslash \left(\left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right] \right), \end{cases} \\
&L = \|f\|^2_{W^{2,p}(Q)}, \\
&M = 1200 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\int_0^x (x-s)^{1-\alpha} \frac{1}{(s+1)^3(t+1)^2} ds \right) \frac{1}{(t+1)^2(x+1)^4} dxdt.
\end{aligned}
$$

Note that $L>0$ and $M>0$. We take

$$
K = \frac{M}{2L}
$$

and we take $\nu > 0$ such that

$$
\nu > 600000 \frac{L\Gamma(2-\alpha)}{M}.
$$

We have

$$
\begin{aligned}
&\int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\int_0^x (x-s)^{1-\alpha} \frac{\partial f}{\partial s}(t,s)ds \right) \frac{\partial^2 f}{\partial x^2}(t,x)dxdt \\
&\quad = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\int_0^x (x-s)^{1-\alpha} \frac{-20}{(t+1)^2(s+1)^3} ds \right) \frac{60}{(t+1)^2(x+1)^4} dxdt \leq 0, \\
&\left| \int_0^1 \int_0^1 \int_0^x (x-s)^{1-\alpha} \frac{\partial f}{\partial s}(t,s)ds \frac{\partial^2 f}{\partial x^2}(t,x)dxdt \right| \\
&\quad = 1200 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\int_0^x (x-s)^{1-\alpha} \frac{1}{(t+1)^2(s+1)^3} ds \right) \frac{1}{(t+1)^2(x+1)^4} dxdt \\
&\quad = M = 2KL = 2K\|f\|^2_{W^{2,p}(Q)} \geq K\|f\|^2_{W^{2,p}(Q)},
\end{aligned}
$$

$$\begin{aligned}
&\frac{\partial f}{\partial x}(t,x)=\begin{cases} -\dfrac{20}{(t+1)^2(x+1)^3}, & (t,x)\in\left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4},\frac{3}{4}\right],\\ 0, & (t,x)\in Q\backslash\left(\left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4},\frac{3}{4}\right]\right),\end{cases}\\
&\frac{\partial^2 f}{\partial x^2}(t,x)=\begin{cases} \dfrac{60}{(t+1)^2(x+1)^4}, & (t,x)\in\left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4},\frac{3}{4}\right],\\ 0, & (t,x)\in Q\backslash\left(\left[\frac{1}{4},\frac{3}{4}\right]\times\left[\frac{1}{4},\frac{3}{4}\right]\right),\end{cases}\\
&|f(t,x)|\le a,\\
&\left|\frac{\partial f}{\partial x}(t,x)\right|\le a,\qquad \left|\frac{\partial^2 f}{\partial x^2}(t,x)\right|\le a,\\
&f(t,0)=0,\quad f(t,1)=0,\quad f(0,x)=0,\quad f(1,x)=0,\quad (t,x)\in Q.
\end{aligned}$$

Therefore $f\in V$. Also, $\frac{1}{n}f\in V$ for any $n\in\mathbb{N}$, and $-f\in V$.

Now, consider the following system

$$\frac{\partial y}{\partial t}+y\frac{\partial y}{\partial x}-\nu\,{}_0^RD_x^{1+\alpha}y=f+u,\qquad \text{in } Q, \tag{3.5a}$$
$$y(t,0)=y(t,1)=0,\qquad \text{on } \Sigma, \tag{3.5b}$$
$$y(0,x)=0,\qquad \text{in } \Omega. \tag{3.5c}$$

We take a function $v\in V$, then we multiply both sides of (3.5a) by v and we integrate over Q. We get

$$\begin{aligned}
&\int_Q\frac{\partial y}{\partial t}(t,x)v(t,x)dxdt+\int_Q y(t,x)\frac{\partial y}{\partial x}(t,x)v(t,x)dxdt\\
&\quad-\nu\int_Q({}_0^RD_x^{1+\alpha}y(t,x))v(t,x)dxdt\\
&=\int_Q(f(t,x)+u(t,x))v(t,x)dxdt.
\end{aligned}$$

Using integration by parts, the conditions (3.5b), (3.5c) and Theorem 2.4, we find

$$\begin{aligned}
&-\int_Q y(t,x)\frac{\partial v}{\partial t}(t,x)\,dxdt-\int_Q y(t,x)\left(\frac{\partial y}{\partial x}(t,x)v(t,x)+y(t,x)\frac{\partial v}{\partial x}(t,x)\right)dxdt\\
&\quad+\nu\int_Q\frac{\partial v}{\partial x}(t,x)\left(\frac{\partial}{\partial x}\,{}_0^RI_x^{1-\alpha}y(t,x)\right)dxdt\\
&=\int_Q(f(t,x)+u(t,x))v(t,x)dxdt,
\end{aligned}$$

or

$$
\begin{aligned}
&-\int_Q y(t,x)\frac{\partial v}{\partial t}(t,x)\,dxdt-\int_Q y(t,x)\Big(\frac{\partial y}{\partial x}(t,x)(t,x)v(t,x)\\
&\quad+y(t,x)\frac{\partial v}{\partial x}(t,x)\Big)dxdt-\nu\int_Q\left({}_0^RI_x^{1-\alpha}y(t,x)\right)\frac{\partial^2 v}{\partial x^2}(t,x)dxdt\\
&=\int_Q(f(t,x)+u(t,x))v(t,x)dxdt. \qquad (3.6)
\end{aligned}
$$

Note that using (3.5b), we have

$$
\begin{aligned}
&{}_0^RI_x^{1-\alpha}y(t,x)=\frac{1}{\Gamma(1-\alpha)}\int_0^x(x-s)^{-\alpha}y(t,s)ds\\
&=-\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\int_0^x\frac{\partial}{\partial s}(x-s)^{1-\alpha}y(t,s)ds\\
&=-\frac{1}{\Gamma(2-\alpha)}(x-s)^{1-\alpha}y(t,s)|_{s=0}^{s=x}+\frac{1}{\Gamma(2-\alpha)}\int_0^x(x-s)^{1-\alpha}\frac{\partial}{\partial s}y(t,s)ds\\
&=\frac{1}{\Gamma(2-\alpha)}\int_0^x(x-s)^{1-\alpha}\frac{\partial}{\partial s}y(t,s)ds.
\end{aligned}
$$

Hence and (3.6), we obtain

$$
\begin{aligned}
&-\int_Q y(t,x)\frac{\partial v}{\partial t}(t,x)dxdt-\int_Q y(t,x)\left(\frac{\partial y}{\partial x}(t,x)v(t,x)+y(t,x)\frac{\partial v}{\partial x}(t,x)\right)dxdt\\
&\quad-\nu\int_Q\frac{1}{\Gamma(2-\alpha)}\left(\int_0^x(x-s)^{1-\alpha}\frac{\partial y}{\partial s}(t,s)ds\right)\frac{\partial^2 v}{\partial x^2}(t,x)dxdt\\
&=\int_Q(f(t,x)+u(t,x))v(t,x)dxdt. \qquad (3.7)
\end{aligned}
$$

Now, let $F:V\to V^\star$ be a functional such that

$$
\langle F(u),v\rangle=\int_0^T\int_0^1(f(t,x)+u(t,x))v(t,x)dxdt,\qquad u,v\in V.
$$

Also, suppose that $A:V\to V^\star$ is a functional defined by

$$
\begin{aligned}
&\langle A(y),v\rangle\\
&=-\int_0^T\int_0^1 y(t,x)\frac{\partial v}{\partial t}(t,x)dxdt\\
&\quad-\int_0^T\int_0^1 y(t,x)\left(\frac{\partial y}{\partial x}(t,x)v(t,x)+y(t,x)\frac{\partial v}{\partial x}(t,x)\right)dxdt\\
&\quad-\frac{\nu}{\Gamma(2-\alpha)}\int_0^T\int_0^1\left(\int_0^x(x-s)^{1-\alpha}\frac{\partial y}{\partial s}(t,s)ds\right)\frac{\partial^2 v}{\partial x^2}(t,x)dxdt,\qquad y\in V.
\end{aligned}
$$

By (3.7) and the problem (3.5a)-(3.5c), to complete the proof, it is sufficient to show that $A(y)=F$ has a unique solution $y\in W^{2,p}(Q)$. We consider two cases $p>2$ and $1<p\le 2$, respectively.

Case I: Let $p>2$**.** We will show that A is a bounded linear operator, hemi-continuous, monotone and coercive.

1. $A:V\to V^\star$ **is a bounded linear operator.** Let $y\in V$. By using Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_0^T\int_0^1 y(t,x)\frac{\partial v}{\partial t}(t,x)dxdt\right| \le \int_0^T\int_0^1 |y(t,x)|\left|\frac{\partial v}{\partial t}(t,x)\right| dxdt\\
&\quad\le \left(\int_0^T\int_0^1 |y(t,x)|^q\ dxdt\right)^{\frac{1}{q}}\left(\int_0^T\int_0^1 \left|\frac{\partial v}{\partial t}(t,x)\right|^p\ dxdt\right)^{\frac{1}{p}}\\
&\quad\le \|v\|_V\left(\int_0^T\int_0^1 |y(t,x)|^q\ dxdt\right)^{\frac{1}{q}}\\
&\quad\le \|v\|_V\left(\int_0^T\int_0^1 dxdt\right)^{\frac{p-2}{(p-1)q}}\left(\int_0^T\int_0^1 |y(t,x)|^{q(p-1)}\ dxdt\right)^{\frac{1}{q(p-1)}}\\
&\quad\le T^{\frac{p-2}{p}}\|v\|_V\|y\|_V, \qquad\qquad (3.8a)\\
&\left|\int_0^T\int_0^1 y(t,x)\left(\frac{\partial v}{\partial x}(t,x)y(t,x)+v(t,x)\frac{\partial y}{\partial x}(t,x)\right)dxdt\right|\\
&\quad\le \int_0^T\int_0^1 |y(t,x)|\,|v(t,x)|\left|\frac{\partial y}{\partial x}(t,x)\right|dxdt+\int_0^T\int_0^1 |y(t,x)|\,|y(t,x)|\left|\frac{\partial v}{\partial x}(t,x)\right|dxdt\\
&\quad\le a\left(\int_0^T\int_0^1 |v(t,x)|\left|\frac{\partial y}{\partial x}(t,x)\right|\ dxdt\right)+a\left(\int_0^T\int_0^1 |y(t,x)|\left|\frac{\partial v}{\partial x}(t,x)\right|dxdt\right)\\
&\quad\le a\left(\int_0^T\int_0^1 \left|\frac{\partial y}{\partial x}(t,x)\right|^q\ dxdt\right)^{\frac{1}{q}}\left(\int_0^T\int_0^1 |v(t,x)|^p\ dxdt\right)^{\frac{1}{p}}\\
&\qquad\quad + a\left(\int_0^T\int_0^1 |y(t,x)|^q\ dxdt\right)^{\frac{1}{q}}\left(\int_0^T\int_0^1 \left|\frac{\partial v}{\partial x}(t,x)\right|^p\ dxdt\right)^{\frac{1}{p}}\\
&\quad\le a\|v\|_V\left(\int_0^T\int_0^1 dxdt\right)^{\frac{p-2}{(p-1)q}}\left(\int_0^T\int_0^1 \left|\frac{\partial y}{\partial x}(t,x)\right|^{q(p-1)}\ dxdt\right)^{\frac{1}{q(p-1)}}\\
&\qquad\quad + a\|v\|_V\left(\int_0^T\int_0^1 dxdt\right)^{\frac{p-2}{(p-1)q}}\left(\int_0^T\int_0^1 |y(t,x)|^{q(p-1)}\ dxdt\right)^{\frac{1}{q(p-1)}}\\
&\quad\le 2aT^{\frac{p-2}{p}}\|y\|_V\|v\|_V, \qquad\qquad (3.8b)\\
&\left|\int_0^T\int_0^1\left(\int_0^x (x-s)^{1-\alpha}\frac{\partial y}{\partial s}(t,s)ds\right)\frac{\partial^2 v}{\partial x^2}(t,x)dxdt\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T \int_0^1 \left| \int_0^x (x-s)^{1-\alpha} \frac{\partial y}{\partial s}(t,s)ds \right| \left| \frac{\partial^2 v}{\partial x^2}(t,x) \right| dxdt \\
&\leq \left(\int_0^T \int_0^1 \left| \int_0^x (x-s)^{1-\alpha} \frac{\partial y}{\partial s}(t,s)ds \right|^q dxdt \right)^{\frac{1}{q}} \left(\int_0^T \int_0^1 \left| \frac{\partial^2 v}{\partial x^2}(t,x) \right|^p dxdt \right)^{\frac{1}{p}} \\
&\leq \|v\|_V \left(\int_0^T \int_0^1 \left| \int_0^1 \frac{\partial y}{\partial s}(t,s)ds \right|^q dxdt \right)^{\frac{1}{q}} \\
&\leq \|v\|_V \left(\int_0^T \left(\int_0^1 \left| \frac{\partial y}{\partial s}(t,s) \right| ds \right)^q dt \right)^{\frac{1}{q}} \\
&\leq \|v\|_V \left(\int_0^T \left(\int_0^1 \left| \frac{\partial y}{\partial s}(t,s) \right|^q ds \right) \left(\int_0^1 ds \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\
&= \|v\|_V \left(\int_0^T \int_0^1 \left| \frac{\partial y}{\partial s}(t,s) \right|^q dsdt \right)^{\frac{1}{q}} \\
&\leq \|v\|_V \left(\int_0^T \int_0^1 \left| \frac{\partial y}{\partial s}(t,s) \right|^{q(p-1)} dsdt \right)^{\frac{1}{q(p-1)}} \left(\int_0^T \int_0^1 dsdt \right)^{\frac{p-2}{(p-1)q}} \\
&\leq T^{\frac{p-2}{p}} \|v\|_V \|y\|_V.
\end{aligned} \tag{3.8c}
$$

Consequently, using (3.8a), (3.8b) and (3.8c) we get

$$
\begin{aligned}
|\langle A(y), v \rangle| =& \Big| - \int_0^T \int_0^1 y(t,x) \frac{\partial v}{\partial t}(t,x) dxdt \\
& - \int_0^T \int_0^1 y(t,x) \left(\frac{\partial y}{\partial x}(t,x) v(t,x) + y(t,x) \frac{\partial v}{\partial x}(t,x) \right) dxdt \\
& - \frac{\nu}{\Gamma(2-\alpha)} \int_0^T \int_0^1 \left(\int_0^x (x-s)^{1-\alpha} \frac{\partial y}{\partial s}(t,s) ds \right) \frac{\partial^2 v}{\partial x^2}(t,x) dxdt \Big| \\
\leq& \Big| \int_0^T \int_0^1 y(t,x) \frac{\partial v}{\partial t}(t,x) dxdt \Big| \\
& + \Big| \int_0^T \int_0^1 y(t,x) \left(\frac{\partial y}{\partial x}(t,x) v(t,x) + y(t,x) \frac{\partial v}{\partial x}(t,x) \right) dxdt \Big| \\
& + \frac{\nu}{\Gamma(2-\alpha)} \Big| \int_0^T \int_0^1 \left(\int_0^x (x-s)^{1-\alpha} \frac{\partial y}{\partial s}(t,s) ds \right) \frac{\partial^2 v}{\partial x^2}(t,x) dxdt \Big| \\
\leq& T^{\frac{p-2}{p}} \|y\|_V \|v\|_V + 2a T^{\frac{p-2}{p}} \|y\|_V \|v\|_V + \frac{\nu}{\Gamma(2-\alpha)} T^{\frac{p-2}{p}} \|y\|_V \|v\|_V \\
=& \left(1 + 2a + \frac{\nu}{\Gamma(2-\alpha)} \right) T^{\frac{p-2}{p}} \|y\|_V \|v\|_V.
\end{aligned}
$$

Hence,

$$\|A(y)\|_{V^\star} \le \left(1+2a+\frac{\nu}{\Gamma(2-\alpha)}\right) T^{\frac{p-2}{p}} \|y\|_V.$$

Thus $A: V \to V^\star$ is bounded.

2. $A: V \to V^\star$ **is hemi-continuous.** Let $y_1, y_2, v \in V$ and consider the function

$$\phi(\lambda) = \langle A(y_1+\lambda y_2), v\rangle, \qquad \lambda \in \mathbb{R}.$$

Suppose that $\{\lambda_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that $\lambda_k \to \lambda$, as $k \to \infty$. We have

$$\begin{aligned}
&\langle A(y_1+\lambda_k y_2), v\rangle \\
=& -\int_0^T \int_0^1 (y_1(t,x)+\lambda_k y_2(t,x)) \frac{\partial v}{\partial t}(t,x)dxdt \\
&- \int_0^T \int_0^1 (y_1(t,x)+\lambda_k y_2(t,x)) \left(\frac{\partial}{\partial x}(y_1(t,x)+\lambda_k y_2(t,x)) v(t,x)\right) dxdt \\
&- \int_0^T \int_0^1 (y_1(t,x)+\lambda_k y_2(t,x))^2 \frac{\partial v}{\partial x}(t,x)dxdt \\
&- \frac{\nu}{\Gamma(2-\alpha)} \int_0^T \int_0^1 \left(\int_0^x (x-s)^{1-\alpha} \frac{\partial}{\partial s}(y_1(t,s)+\lambda_k y_2(t,s))ds\right) \frac{\partial^2 v}{\partial x^2}(t,x)dxdt.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^x (x-s)^{1-\alpha} \frac{\partial}{\partial s}\left(y_1(t,s)+\lambda_k y_2(t,s)\right) ds \longrightarrow \\
&\qquad \int_0^x (x-s)^{1-\alpha} \frac{\partial}{\partial s}\left(y_1(t,s)+\lambda y_2(t,s)\right) ds, \\
&y_1+\lambda_k y_2 \longrightarrow y_1+\lambda y_2, \\
&\frac{\partial y_1}{\partial x}+\lambda_k \frac{\partial y_2}{\partial x} \longrightarrow \frac{\partial y_1}{\partial x}+\lambda \frac{\partial y_2}{\partial x} \quad \text{as } k\to\infty, \quad \text{in } Q.
\end{aligned}$$

Since $\{\lambda_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, there exists a positive constant c such that

$$\begin{aligned}
&\left|\int_0^x (x-s)^{1-\alpha} \frac{\partial}{\partial s}\left(y_1(t,s)+\lambda_k y_2(t,s)\right) ds\right|^q \\
&\quad \le c\left(\left|\int_0^x (x-s)^{1-\alpha}\frac{\partial}{\partial s} y_1(t,s)ds\right|^q + \left|\int_0^x (x-s)^{1-\alpha}\frac{\partial}{\partial s} y_2(t,s)ds\right|^q\right), \\
&|y_1+\lambda_k y_2|^q \le c\left(|y_1|^q+|y_2|^q\right), \\
&|y_1+\lambda_k y_2|^{2q} \le c\left(|y_1|^{2q}+|y_2|^{2q}\right), \\
&|y_1+\lambda_k y_2|^q \left|\frac{\partial(y_1+\lambda_k y_2)}{\partial x}\right|^q \le c\left(|y_1|^q+|y_2|^q\right)\left(\left|\frac{\partial y_1}{\partial x}\right|^q+\left|\frac{\partial y_2}{\partial x}\right|^q\right) \quad \text{in } Q.
\end{aligned}$$

By Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{k\to\infty}\langle A(y_1+\lambda_k y_2),v\rangle=\langle A(y_1+\lambda y_2),v\rangle.$$

3. $A:V\to V^\star$ **is monotone.** Let $y_1,y_2\in V$, $y_1\neq y_2$. Then, for $y=v=y_1-y_2$, using (3.8a), (3.8b), (3.8c), we get

$$\begin{aligned}
&\langle A(y_1-y_2),y_1-y_2\rangle\\
=&-\int_0^T\int_0^1(y_1(t,x)-y_2(t,x))\frac{\partial}{\partial t}(y_1(t,x)-y_2(t,x))\,dxdt\\
&-2\int_0^T\int_0^1(y_1(t,x)-y_2(t,x))^2\frac{\partial}{\partial x}(y_1(t,x)-y_2(t,x))\,dxdt\\
&-\frac{\nu}{\Gamma(2-\alpha)}\int_0^T\int_0^1\left(\int_0^x(x-s)^{1-\alpha}\frac{\partial}{\partial s}(y_1(t,s)-y_2(t,s))\,ds\right)\\
&\times\frac{\partial^2}{\partial x^2}(y_1(t,x)-y_2(t,x))\,dxdt\\
\geq&-\left|\int_0^T\int_0^1(y_1(t,x)-y_2(t,x))\frac{\partial}{\partial t}(y_1(t,x)-y_2(t,x))\,dxdt\right|\\
&-2\left|\int_0^T\int_0^1(y_1(t,x)-y_2(t,x))^2\frac{\partial}{\partial x}(y_1(t,x)-y_2(t,x))\,dxdt\right|\\
&-\frac{\nu}{\Gamma(2-\alpha)}\int_0^T\int_0^1\int_0^x(x-s)^{1-\alpha}\frac{\partial}{\partial s}(y_1(t,s)-y_2(t,s))\,ds\\
&\times\frac{\partial^2}{\partial x^2}(y_1(t,x)-y_2(t,x))\,dxdt\\
\geq&-T^{\frac{p-2}{p}}\|y_1-y_2\|_V^2-2aT^{\frac{p-2}{p}}\|y_1-y_2\|_V^2+\frac{\nu K}{\Gamma(2-\alpha)}\|y_1-y_2\|_V^2\\
=&\left(-T^{\frac{p-2}{p}}-2aT^{\frac{p-2}{p}}+\frac{\nu K}{\Gamma(2-\alpha)}\right)\|y_1-y_2\|_V^2.
\end{aligned}$$

Hence, by the condition (3.4) it follows that $A:V\to V^\star$ is strictly monotone. Moreover, since $A:V\to V^\star$ is bounded, hemi-continuous and monotone using Theorem 2.1, we conclude that A is pseudo-monotone.

4. $A:V\to V^\star$ **is coercive.** For $y\in V$ we have

$$\langle A(y),y\rangle\geq\left(-T^{\frac{p-2}{p}}-2aT^{\frac{p-2}{p}}+\frac{\nu K}{\Gamma(2-\alpha)}\right)\|y\|_V^2.$$

So,

$$\frac{\langle A(y),y\rangle}{\|y\|_V}\geq\left(-T^{\frac{p-2}{p}}-2aT^{\frac{p-2}{p}}+\frac{\nu K}{\Gamma(2-\alpha)}\right)\|y\|_V,$$

and

$$\lim_{\|y\|_V \to \infty} \frac{\langle A(y), y \rangle}{\|y\|_V} = \infty.$$

Thus, since $A : V \to V^\star$ is strictly monotone, pseudo-monotone and coercive, by Theorem 2.2, it follows that the equation

$$A(y) = F$$

has a unique solution $y \in V$. We have that $y \in W^{2,p}(Q)$.

Case II: Let $1 < p \le 2$. Since Q is bounded and $W^{2,r_1}(Q) \hookrightarrow W^{2,r_2}(Q)$ for $r_1 \ge 2$ and $r_2 \in (1,2]$, it follows that $y \in W^{2,p}(Q)$ for any $p > 1$.

This completes the proof. □

3.2 On the solution of FOC problem

Now, in the following theorem we prove that there exist at least one optimal solution for the problem (3.1)-(3.2).

Theorem 3.2. *Let $T > 0$, $p > 1$, $b > 0$, $\gamma \ge 0$, and $\alpha \in (0,1)$ be arbitrarily chosen, ν, a and K be positive constants which satisfy* (3.4). *Assume that $y_0 \in L^\infty(Q)$, $y_0(0) = y_0(1) = 0$, y_0' and y_0'' exist a.e. in Ω, and $|y_0| \le a$, $|y_0'| \le a$ and $|y_0''| \le a$ a.e. in Ω. Then the problem* (3.1)-(3.2) *has at least one optimal solution.*

Proof. Since $J(y,u) \ge 0$ on X, where X is defined by (3.3), then $J(y,u)$ is bounded below. Therefore $\inf(J)$ exists. We will prove that $\inf(J)$ is achieved at a point $(\bar{y}, \bar{u}) \in X$. Let $\{(y_k, u_k)\}_{k \in \mathbf{N}}$ be a minimizing sequence of (3.1), i.e.,

$$J(y_k, u_k) \longrightarrow \inf(J) \quad \text{as } k \to \infty.$$

We take a subsequence, again denoted in the same way, converging weakly in $(L^\infty(Q))^2$ to an element $(\bar{y}, \bar{u}) \in X$. Let $\overline{y}_d$ and y_{d_l}, $l \in \mathbb{N}$, be the unique solutions of (3.1)-(3.2) for $u = \bar{u}$ and $u = u_l$, respectively. We have $y_{d_l} \to \overline{y}_d$ in $L^p(Q)$, as $l \to \infty$. By Mazur's lemma, there exist sequences $\{z_k\}_{k=1}^{N_n}$, $\{v_k\}_{k=1}^{N_n}$ such that

$$z_k = \sum_{l=k}^{N_k} \lambda_l y_l, \qquad v_k = \sum_{l=k}^{N_k} \mu_l u_l,$$

where $\sum_{l=k}^{N_k} \lambda_l = 1$, $\sum_{l=k}^{N_k} \mu_l = 1$, $\lambda_l \ge 0$, $\mu_l \ge 0$, $l \in \{k, \cdots, N_k\}$, $k \in \mathbb{N}$, and $z_k \to \bar{y}$, $v_k \to \bar{u}$, as $k \to \infty$, strongly in $L^p(Q)$. Then

$$
\begin{aligned}
&\inf(J) \leq J(\bar{y},\bar{u}) = \beta \left\| \bar{y} - \bar{y}_d \right\|^p_{L^p(Q)} + \gamma \left\| \bar{u} \right\|^p_{L^p(Q)} \\
=&\beta \limsup_{k\to\infty} \left\| z_k - \bar{y}_d \right\|^p_{L^p(Q)} + \gamma \limsup_{k\to\infty} \left\| v_k \right\|^p_{L^p(Q)} \\
=&\beta \limsup_{k\to\infty} \left\| \sum_{l=k}^{N_k} \lambda_l y_l - \bar{y}_d \right\|^p_{L^p(Q)} + \gamma \limsup_{k\to\infty} \left\| \sum_{l=k}^{N_k} \mu_l u_l \right\|^p_{L^p(Q)} \\
\leq&\beta \limsup_{k\to\infty} \left\| \sum_{l=k}^{N_k} \lambda_l (y_l - \bar{y}_d) \right\|^p_{L^p(Q)} + \gamma \limsup_{k\to\infty} \sum_{l=k}^{N_k} \mu_l \left\| u_l \right\|^p_{L^p(Q)} \\
\leq&\beta \limsup_{k\to\infty} \sum_{l=k}^{N_k} \lambda_l \left\| y_l - \bar{y}_d \right\|^p_{L^p(Q)} + \gamma \limsup_{k\to\infty} \sum_{l=k}^{N_k} \mu_l \left\| u_l \right\|^p_{L^p(Q)} \\
=&\beta \limsup_{k\to\infty} \sum_{l=k}^{N_k} \lambda_l \left\| y_l - y_{d_l} + y_{d_l} - \bar{y}_d \right\|^p_{L^p(Q)} + \gamma \limsup_{k\to\infty} \sum_{l=k}^{N_k} \mu_l \left\| u_l \right\|^p_{L^p(Q)} \\
\leq&\beta \limsup_{k\to\infty} \sum_{l=k}^{N_k} \lambda_l \left(\left\| y_l - y_{d_l} \right\|_{L^p(Q)} + \left\| y_{d_l} - \bar{y}_d \right\|_{L^p(Q)} \right)^p \\
&\quad + \gamma \limsup_{k\to\infty} \sum_{l=k}^{N_k} \mu_l \left\| u_l \right\|^p_{L^p(Q)} \\
=&\inf(J).
\end{aligned}
$$

Hence,

$$\inf_{(y,u)\in X} (J(y,u)) = J(\bar{y},\bar{u})$$

and this completes the proof. □

4 Conclusions

In this paper, we prove that a class of fractional Burgers' equations under some special conditions has a unique solution. Also, the optimal control problem governed by fractional Burgers' equation has at least one optimal solution.

Using the results in this paper, as a future work, we will investigate the considered Burgers' equation use of Haar wavelet method for obtaining numerical solutions and it will be obtained an estimate of the error between the exact solution and the numerical solutions.

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References

[1] R. Adams, J. Fournier, Sobolev Spaces, Pure and Applied Mathematics Series, Boston, 2003.

[2] A. Cesar and S. Gomez, A note on the exact solution for the fractional Burgers' equation, Int. J. Pure Appl. Math., 93 (2014), 229–232.

[3] A. Esen and O. Tasbozan, Numerical solution of time fractional Burgers' equation by cubic B-spline finite elements, Mediterr. J. Math., 13 (2016), 1325–1337.

[4] M. Inc, The approximate and exact solutions of the space- and time-fractional Burgers' equations with initial conditions by variational iteration method, J. Math. Anal. Appl., 345 (2008), 476–484.

[5] N. Khan, A. Ara and A. Mahmood, Numerical solutions of time-fractional Burgers' equations: A comparison between generalized differential transformation technique and homotopy perturbation method, Int. J. Numer. Methods Heat Fluid Flow., 22 (2012), 175–193.

[6] N. Khan and T. Hameed, An implementation of Haar wavelet based method for numerical treatment of time-fractional Schrödinger and coupled Schrödinger systems, IEEE/CAA J. Automatica Sinica, (2016), 1–10.

[7] N. Khan, T. Hameed, N. Khan and M. Raja, A heuristic optimization method of fractional cenvection reaction: An application to diffusion process, Thermal Sci., 22 (2018), S243–S252.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Application of Fractional Differential Equations, Elsevier, 2006.

[9] S. Momani, Non-perturbative analytical solutions of the space- and time-fractional Burgers' equations, Chaos Solitons Fractals, 28 (2006), 930–937.

[10] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering, 1999.

[11] M. Rawashdeh, A reliable method for the space-time fractional Burgers' and time-fractional Cahn-Allen equations via the FRDTM, Adv. Difference Equations, 99 (2017).

[12] M. Renardy and R. Rogers, An Introduction to Partial Differential Equations, Springer, 2004.

[13] H. Royden and P. Fitzpatrick, Real Analaysis, Pearson Modern Classics for Advanced Mathematics Series, 2010.

[14] K. M. Saad and E. HF. Al-Sharif, Analytical study for time and time-space fractional Burger equation, Adv. Difference Equ., 300 (2017).

[15] A. L. Skubachevskii, Elliptic Functional Differential Equation and Applications, Birkhäuser Verlag, 1997.

[16] F. Troltzsch, Optimal Control of Partial Differential Equations: Theory, Methods and Applications, American Mathematical Society Providence, Rhode Island, 2010.