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Some Estimates of the Maximum Modulus for Polynomials with Gaps

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Abstract. Let $p(z)$ be a polynomial of degree n having some zeros at a point $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and the rest of the zeros lying on or outside the boundary of a prescribed disk. In this brief note, we consider this class of polynomials and obtain some bounds for $\left(\max_{|z|=R} |p(z)|\right)^s$ in terms of $\left(\max_{|z|=1} |p(z)|\right)^s$ for any $R \geq 1$ and $s \in \mathbb{N}$.

Key Words: Polynomials, maximum modulus, zeros, prescribed disk.

AMS Subject Classifications: 30C10, 30C80, 30D15, 26C10, 26D10

1 Introduction

Let $p(z)$ be a polynomial of degree n. For effective management of space, we shall adopt the following notations:

$$D(0,k) := \{z : |z| < k\}, \qquad S(0,k) := \{z : |z| = k\}, \qquad M(p,R) := \max_{|z|=R} |p(z)|,$$
$$m(p,k) := \min_{|z|=k} |p(z)|, \qquad ||p|| := \max_{|z|=1} |p(z)|,$$

where k and R are positive real numbers.

By using the maximum modulus principle, one obtains that for $R \geq 1$,

$$M(p,R) \geq ||p||.$$

The general problem of interest, however, is the following:

$$(\mathrm{P}): \quad \text{Find a factor } (*) \text{ such that } M(p,R) \leq (*)\, ||p|| \text{ for any } R \geq 1.$$

In view of (P), S. Bernstein [6, pp. 442] observed that for $R \geq 1$,

$$M(p,R) \leq R^n ||p||. \tag{1.1}$$

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The above result is best possible with equality holding for $p(z) = \lambda z^n$, λ being a complex number. Since the extremal polynomial $p(z) = \lambda z^n$ in (1.1) has all its zeros at the origin. It should be possible to improve upon the bound in (1.1) for polynomial not vanishing at the origin. For this, Ankeny and Rivlin [1] proved that if $p(z)$ has no zero in $D(0,1)$, then for $R \geq 1$,

$$M(p,R) \leq \frac{R^n+1}{2}||p||. \tag{1.2}$$

As a sharpening of the above result, Aziz and Dawood [3] proved that for $R \geq 1$,

$$M(p,R) \leq \frac{R^n+1}{2}\,||p|| - \frac{R^n-1}{2}m(p,1). \tag{1.3}$$

Now, for the class of polynomials not vanishing in the disk $D(0,k)$, $k \geq 1$, Shah [10] proved that if $p(z)$ is a polynomial of degree n having no zero in $D(0,k)$, $k \geq 1$, then for every real number $R > k$,

$$M(p,R) \leq \frac{R^n+1}{1+k}\,||p|| - \frac{R^n-1}{1+k}\,m(p,k). \tag{1.4}$$

Several research articles have been written on this subject of inequalities (see for example Govil and Mohapatra [4], Rahman and Schmeisser [9], and recent article of Govil and Nwaeze [5].)

Inspired by the work in [8], we consider polynomials having some zeros at a point $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and the rest of the zeros lying on or outside the boundary of a prescribed disk. For this, we estimate $(M(p,R)/||p||)^s$ for any $R \geq 1$ and any natural number s. The paper is organized as follows: we present two lemmas in Section 2 which will be used in the proof of our results. In Section 3, the results are formulated and proved and then followed by a short conclusion in Section 4.

2 Lemmas

For the proof of our theorems, we will need the following lemmas due to Nakprasit and Somsuwan [7].

Lemma 2.1. *Let*

$$p(z) = (z-z_0)^m\left(a_0 + \sum_{j=\mu}^{n-m} a_j z^j\right), \quad 1 \leq \mu \leq n-m, \quad 0 \leq m \leq n-1,$$

be a polynomial of degree n having zero of order m at z_0 *with* $|z_0| < 1$ *and the remaining* $n-m$ *zeros are outside* $D(0,k)$, $k \geq 1$. *Then*

$$\max_{|z|=1}|p'(z)| \leq \left[\frac{m}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^m}\right]||p|| - \frac{A}{(k+|z_0|)^m}m(p,k),$$

where

$$A=\frac{(1+|z_0|)^{m+1}(n-m)}{(1+k^{\mu})(1-|z_0|)}.$$

Lemma 2.2. *Let*

$$p(z)=(z-z_0)^m\left(a_0+\sum_{j=\mu}^{n-m}a_jz^j\right),\quad 1\le\mu\le n-m,\quad 0\le m\le n-1,$$

be a polynomial of degree n having zero of order m at z_0 with $|z_0|<1$ and the remaining $n-m$ zeros are on $S(0,k)$, $k\ge 1$. Then

$$\max_{|z|=1}|p'(z)|\le\left[\frac{m}{(1-|z_0|)}+\frac{(1+|z_0|)^{m+1}(n-m)}{(k^{n-m-2\mu+1}+k^{n-m-\mu+1})(1-|z_0|)^{m+1}}\right]||p||.$$

3 Main results

Theorem 3.1. *Let*

$$p(z)=z^m\left(a_0+\sum_{j=\mu}^{n-m}a_jz^j\right),\quad 1\le\mu\le n-m,\quad 0\le m\le n-1,$$

be a polynomial of degree n having zero of order m at the origin and the remaining $n-m$ zeros are outside $D(0,k)$, $k\ge 1$. Then for $R\ge 1$ and every natural number s,

$$\begin{aligned}[M(p,R)]^s\le&\left[\frac{n+nk^{\mu}+(n+mk^{\mu})(R^{ns}-1)}{n(1+k^{\mu})}\right]||p||^s\\&-\left[\frac{(n-m)(R^{ns}-1)m(p,k)}{nk^m(1+k^{\mu})}\right]||p||^{s-1}.\end{aligned}$$

If we take $m=0$ and $s=1$, we obtain

Corollary 3.1. *Let*

$$p(z)=a_0+\sum_{j=\mu}^{n}a_jz^j,\qquad 1\le\mu\le n,$$

be a polynomial of degree n having all its zeros outside $D(0,k)$, $k\ge 1$. Then for $R\ge 1$,

$$M(p,R)\le\frac{k^{\mu}+R^n}{1+k^{\mu}}||p||-\frac{R^n-1}{1+k^{\mu}}m(p,k).$$

The above corollary is a generalization of a result due to Aziz [2, Theorem 4]. Setting $k=1$ in Corollary 3.1, we obtain the result of Aziz and Dawood given in inequality (1.3). Instead of proving Theorem 3.1, we will prove the following more general result.

Theorem 3.2. *Let*

$$p(z)=(z-z_0)^m\left(a_0+\sum_{j=\mu}^{n-m}a_jz^j\right),\quad 1\le\mu\le n-m,\quad 0\le m\le n-1,$$

be a polynomial of degree n having zero of order m at z_0 *with* $|z_0|<1$ *and the remaining* $n-m$ *zeros are outside* $D(0,k)$, $k\ge 1$. *Then for* $R\ge 1$ *and every natural number s,*

$$[M(p,R)]^s\le\left[1+\frac{m(R^{ns}-1)}{n(1-|z_0|)}+\frac{A(R^{ns}-1)}{n(1-|z_0|)^m}\right]||p||^s-\left[\frac{A(R^{ns}-1)m(p,k)}{n(k+|z_0|)^m}\right]||p||^{s-1},$$

where

$$A=\frac{(1+|z_0|)^{m+1}(n-m)}{(1+k^\mu)(1-|z_0|)}.$$

Proof. Applying inequality (1.1) and Lemma 2.1 to the polynomial $p'(z)$ which is of degree $n-1$, it follows that for $R\ge1$ and $\theta\in[0,2\pi)$, we have

$$\begin{aligned}|p'(Re^{i\theta})|\le&\max_{|z|=R}|p'(z)|\le R^{n-1}\max_{|z|=1}|p'(z)|\\ \le& R^{n-1}\left[\frac{m}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^m}\right]||p||-\frac{AR^{n-1}}{(k+|z_0|)^m}m(p,k).\end{aligned}\tag{3.1}$$

From the fundamental principle of calculus, we obtain that

$$\begin{aligned}&[p(Re^{i\theta})]^s-[p(e^{i\theta})]^s\\ =&\int_1^R\frac{d[p(te^{i\theta})]^s}{dt}\,dt\\ =&\int_1^R s[p(te^{i\theta})]^{s-1}p'(te^{i\theta})e^{i\theta}\,dt.\end{aligned}\tag{3.2}$$

This implies that

$$|p(Re^{i\theta})|^s\le|p(e^{i\theta})|^s+s\int_1^R|p(te^{i\theta})|^{s-1}|p'(te^{i\theta})|\,dt.\tag{3.3}$$

Hence, using (2.1) and (3.1) together with the above inequality, we get

$$\begin{aligned}|M(p,R)|^s\le&||p||^s+s\int_1^R t^{ns-n}||p||^{s-1}|p'(te^{i\theta})|\,dt\\ \le&||p||^s+s||p||^s\left[\frac{m}{(1-|z_0|)}+\frac{A}{(1-|z_0|)^m}\right]\int_1^R t^{ns-1}\,dt\\ &-s\frac{A||p||^{s-1}}{(k+|z_0|)^m}m(p,k)\int_1^R t^{ns-1}\,dt\\ =&\left[1+\frac{m(R^{ns}-1)}{n(1-|z_0|)}+\frac{A(R^{ns}-1)}{n(1-|z_0|)^m}\right]||p||^s-\left[\frac{A(R^{ns}-1)m(p,k)}{n(k+|z_0|)^m}\right]||p||^{s-1}.\end{aligned}$$

That proves our result. □

Theorem 3.3. *Let*

$$p(z)=(z-z_0)^m\left(a_0+\sum_{j=\mu}^{n-m}a_jz^j\right),\quad 1\le\mu\le n-m,\quad 0\le m\le n-1,$$

be a polynomial of degree n having zero of order m at z_0 with $|z_0|<1$ and the remaining $n-m$ zeros are on $S(0,k)$, $k\ge 1$. Then for $R\ge 1$ and every natural number s,

$$[M(p,R)]^s\le\left[1+\frac{m(R^{ns}-1)}{n(1-|z_0|)}+\frac{(n-m)(1+|z_0|)^{m+1}(R^{ns}-1)}{n(1-|z_0|)^{m+1}\left(k^{n-m-2\mu+1}+k^{n-m-\mu+1}\right)}\right]||p||^s.$$

By choosing $s=1$ in Theorem 3.3 above, we obtain:

Corollary 3.2. *Let*

$$p(z)=(z-z_0)^m\left(a_0+\sum_{j=\mu}^{n-m}a_jz^j\right),\quad 1\le\mu\le n-m,\quad 0\le m\le n-1,$$

be a polynomial of degree n having zero of order m at z_0 with $|z_0|<1$ and the remaining $n-m$ zeros are on $S(0,k)$, $k\ge 1$. Then for $R\ge 1$,

$$M(p,R)\le\left[1+\frac{m(R^{n}-1)}{n(1-|z_0|)}+\frac{(n-m)(1+|z_0|)^{m+1}(R^{n}-1)}{n(1-|z_0|)^{m+1}\left(k^{n-m-2\mu+1}+k^{n-m-\mu+1}\right)}\right]||p||.$$

The next corollary follows by setting $z_0=m=0$ in the above corollary.

Corollary 3.3. *Let*

$$p(z)=a_0+\sum_{j=\mu}^{n}a_jz^j,\qquad 1\le\mu\le n,$$

be a polynomial of degree n having all its zeros on $S(0,k)$, $k\ge 1$. Then for $R\ge 1$,

$$M(p,R)\le\left[1+\frac{R^n-1}{k^{n-2\mu+1}+k^{n-\mu+1}}\right]||p||.$$

Proof. We now present the proof of Theorem 3.3 by following a similar fashion as in the proof of Theorem 3.2. Using Lemma 2.2, we have that for $R\ge 1$, and $\theta\in[0,2\pi)$

$$\begin{aligned}|M(p,R)|^s\le&||p||^s+s\int_1^R t^{ns-n}||p||^{s-1}|p'(te^{i\theta})|\,dt\\ \le&||p||^s+s\left[\frac{m}{(1-|z_0|)}+\frac{(1+|z_0|)^{m+1}(n-m)}{\left(k^{n-m-2\mu+1}+k^{n-m-\mu+1}\right)(1-|z_0|)^{m+1}}\right]\\ &\times||p||\int_1^R t^{ns-1}\,dt,\end{aligned}$$

hence, Theorem 3.3 follows. □

4 Conclusions

Much have not been done for polynomials having some of their zeros at a point and the rest on or outside a prescribed disk. This work investigates such class of polynomials to see how $[M(p,R)]^s$ compares with $||p||^s$ for any given $R \geq 1$ and $s \in \mathbb{N}$.

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