

Long Time Well-Posedness of the MHD Boundary Layer Equation in Sobolev Space

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Abstract. In this paper, we study the long time well-posedness of 2-D MHD boundary layer equation. It was proved that if the initial data satisfies

$$\|(u_0, h_0 - 1)\|_{H_\mu^{3,0} \cap H_\mu^{1,2}} \leq \varepsilon,$$

then the life span of the solution is at least of order $\varepsilon^{2-\eta}$ for $\eta > 0$.

Key Words: MHD boundary layer equation, Sobolev space, well-posedness.

AMS Subject Classifications: 35Q30, 76D03

1 Introduction

In this paper, we study the well-posedness of the MHD boundary layer equation in \mathbf{R}_+^2 :

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - h\partial_x h - g\partial_y h = \kappa\partial_y^2 u - \partial_x p, \\ \partial_t h + \partial_y(vh - ug) = v\partial_y^2 h, \\ \partial_x u + \partial_y v = 0, \quad \partial_x h + \partial_y g = 0, \\ (u, v, \partial_y h, g)|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} (u, h) = (U(t, x), H(t, x)), \\ (u, h)|_{t=0} = (u_0, h_0), \end{cases} \quad (1.1)$$

where (u, v) denotes the velocity field of the boundary layer flow, (h, g) denotes the magnetic field, and $(U(t, x), H(t, x), p(t, x))$ denotes the outflow of velocity, magnetic and

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pressure, which satisfies the Bernoulli's law:

$$\partial_t U + U\partial_x U - H\partial_x H + \partial_x p = 0, \quad \partial_t H + U\partial_x H - H\partial_x U = 0.$$

This system is a boundary layer model, which describes the behaviour of the solution of the viscous MHD equations when the viscosity and the resistivity tend to zero [6, 11].

When $h = 0$, the system (1.1) is reduced to the classical Prandtl equation:

$$\partial_t u + u\partial_x u + v\partial_y u = \kappa\partial_y^2 u - \partial_x p, \quad \partial_x u + \partial_y v = 0.$$

The well-posedness theory of the 2-D Prandtl equation was well understood. For the monotonic data, Oleinik [14] proved the local existence and uniqueness of classical solutions. With the additional favorable pressure, Xin and Zhang [16] proved the global existence of weak solutions of the Prandtl equation. Sammartino and Califisch [15] established the local well-posedness of the Prandtl equation for the analytic data. Recently, Alexandre et al. [1] and Masmoudi-Wong [13] independently developed direct energy method to prove the well-posedness of the Prandtl equation for monotonic data in Sobolev spaces. Without monotonicity, Gérard-Varet and Dormy [7] proved the ill-posedness of the Prandtl equation in Sobolev space. However, the Prandtl equation is well-posed in Gevrey class 2 for a class of non-monotone data with non-degenerate critical points [4, 8, 12]. On the other hand, E and Engquist [5] proved that the analytic solution can blow up in a finite time [5]. See [9] for the extension to van Dommelen-Shen type singularity. For small analytic initial data, Zhang and the fourth author [18] proved the long time well-posedness of the Prandtl equation: if the initial data satisfies

$$\|e^{\frac{1+\nu^2}{8}} e^{|D_x|} u_0\|_{B^{\frac{1}{2},0}} \leq \varepsilon,$$

then the lifespan of the solution is greater than $\varepsilon^{-\frac{4}{3}}$. In [10], Ignatova and Vicol obtained a larger lifespan $\exp \frac{\varepsilon^{-1}}{\ln \varepsilon^{-1}}$ with small analytical data of size $\mathcal{O}(\varepsilon)$, whose analytical width $\tau_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

In two recent interesting works [6, 11], the authors showed that the tangential magnetic field has stabilization effect on the boundary layer of the fluid. In particular, they proved the well-posedness of the system (1.1) for the data without monotonicity under an uniform tangential magnetic field.

The goal of this paper is two folds: (1) present a simple proof of well-posedness based on the parilinearization method developed in [3]; (2) study the long time well-posedness of the system (1.1) for small data in Sobolev space. In [17], Xu and Zhang proved a long time existence of the Prandtl equation for the data close a monotonic shear flow. However, it is unclear how the lifespan of the solution depends on the data. Here we would like to give the explicit lifespan of the solution of the system (1.1).

For simplicity, we consider a uniform outflow $(U, H) = (0, 1)$ and take $\kappa = \nu = 1$. Let

$h(t, x, y) = 1 + \tilde{h}(t, x, y)$. Then (u, \tilde{h}) satisfies the following system

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - h\partial_x \tilde{h} - g\partial_y \tilde{h} - \partial_y^2 u = 0, \\ \partial_t \tilde{h} + u\partial_x \tilde{h} + v\partial_y \tilde{h} - h\partial_x u - g\partial_y u - \partial_y^2 \tilde{h} = 0, \\ \partial_x u + \partial_y v = 0, \quad \partial_x \tilde{h} + \partial_y g = 0, \\ (u, v, \partial_y \tilde{h}, g)|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow +\infty} (u, \tilde{h}) = (0, 0), \\ (u, \tilde{h})|_{t=0} = (u_0, \tilde{h}_0). \end{cases} \tag{1.2}$$

To state our result, we introduce the following weighted Sobolev space. For $k, \ell \in \mathbf{N}$, the space $H_\omega^{k,\ell}(\mathbf{R}_+^2)$ consists of all functions $f \in L_\omega^2$ satisfying

$$\|f\|_{H_\omega^{k,\ell}}^2 \stackrel{\text{def}}{=} \sum_{\alpha=0}^k \sum_{\beta=0}^\ell \|\partial_x^\alpha \partial_y^\beta f\|_{L_\omega^2}^2 < +\infty,$$

where $\|f\|_{L_\omega^p} = \|\omega(y)f(x, y)\|_{L^p}$ with $\omega(y)$ a positive weight function.

Our main result is stated as follows.

Theorem 1.1. *Let $\mu = \exp(\frac{1+y^2}{8\langle t \rangle})$ with $\langle t \rangle = 1 + t$. For any $\eta \in (0, 1)$, there exists $\varepsilon > 0$ so that if the initial data (u_0, \tilde{h}_0) satisfies*

$$\|(u_0, \tilde{h}_0)\|_{H_\mu^{3,0}} + \|(u_0, \tilde{h}_0)\|_{H_\mu^{1,2}} \leq \varepsilon, \tag{1.3}$$

then there exists a time $T_\varepsilon \geq \varepsilon^{-(2-\eta)}$ so that the system (1.2) has a solution (u, \tilde{h}) on $[0, T_\varepsilon]$, which satisfies

$$(u, \tilde{h}) \in L^\infty\left([0, T_\varepsilon]; H_\mu^{3,0}(\mathbf{R}_+^2) \cap H_\mu^{1,2}(\mathbf{R}_+^2)\right) \cap L^2\left([0, T_\varepsilon]; H_\mu^{3,1}(\mathbf{R}_+^2) \cap H_\mu^{1,3}(\mathbf{R}_+^2)\right).$$

Remark 1.1. It is unclear whether the lifespan of the solution obtained in Theorem 1.1 is sharp. It remains open whether the solution is global in time for small data.

2 Littlewood-Paley decomposition and paraproduct

We first introduce the Littlewood-Paley decomposition in the horizontal direction $x \in \mathbf{R}$. Choose two smooth functions $\chi(\tau)$ and $\varphi(\tau)$, which satisfy

$$\text{supp}\varphi \subset \left\{ \tau \in \mathbf{R} : \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\}, \quad \text{supp}\chi \subset \left\{ \tau \in \mathbf{R} : |\tau| \leq \frac{4}{3} \right\},$$

and for any $\tau \in \mathbf{R}$,

$$\chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.$$

Then we define

$$\begin{aligned}\Delta_j f &= \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\widehat{f}), \quad S_j f = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\widehat{f}) \quad \text{for } j \geq 0, \\ \Delta_{-1} f &= S_0 f, \quad S_j f = S_0 f \quad \text{for } j < 0.\end{aligned}$$

The Bony's paraproduct $T_f g$ is defined by

$$T_f g = \sum_{j \geq -1} S_{j-1} f \Delta_j g.$$

Then we have the following Bony's decomposition

$$fg = T_f g + R_g f, \quad (2.1)$$

where the remainder term $R_g f$ is defined by

$$R_g f = \sum_{j \geq 0} \Delta_j f S_1 g + \sum_{j \geq 1, j' \geq j-1} \Delta_{j'} f \Delta_j g.$$

We denote by $W^{s,p}$ the usual Sobolev spaces in \mathbf{R} and denote $W^{s,2}$ by H^s . Let us recall classical paraproduct estimates and paraproduct calculus.

Lemma 2.1. *Let $s \in \mathbf{R}$. It holds that*

$$\|T_f g\|_{H^s} \leq C \|f\|_{L^\infty} \|g\|_{H^s}.$$

If $s > 0$, then we have

$$\|R(f, g)\|_{H^s} \leq C \min(\|f\|_{L^\infty} \|g\|_{H^s}, \|f\|_{H^s} \|g\|_{L^\infty}).$$

Lemma 2.2. *Let $s \in \mathbf{R}$ and $\sigma \in (0, 1]$. It holds that*

$$\|(T_a T_b - T_{ab})f\|_{H^s} \leq C(\|a\|_{W^{\sigma,\infty}} \|b\|_{L^\infty} + \|a\|_{L^\infty} \|b\|_{W^{\sigma,\infty}}) \|f\|_{H^{s-\sigma}}.$$

Especially, we have

$$\begin{aligned}\| [T_a, T_b] f \|_{H^s} &\leq C(\|a\|_{W^{\sigma,\infty}} \|b\|_{L^\infty} + \|a\|_{L^\infty} \|b\|_{W^{\sigma,\infty}}) \|f\|_{H^{s-\sigma}}, \\ \| (T_a - T_a^*) f \|_{H^s} &\leq C \|a\|_{W^{\sigma,\infty}} \|f\|_{H^{s-\sigma}}.\end{aligned}$$

Here T_a^* is the adjoint of T_a .

Lemma 2.3. *Let $s \in \mathbf{N}$. It holds that*

$$\|[\partial^s, T_a]f\|_{L^2} \leq C \|\partial_x a\|_{L^\infty} \|f\|_{H^{s-1}}.$$

Let us refer to [2] for more introduction.

3 Parilinearization and symmetrization

As in the Prandtl equation, an essential difficulty in solving the MHD boundary layer equation is the loss of one derivative in the horizontal direction induced by the terms like $v\partial_y u, v\partial_y h, g\partial_y u, g\partial_y h$. To overcome this difficulty, motivated by [3], we will first parilinearize the system (1.2), and then introduce good unknowns to symmetrize the system following the idea in [11].

Using Bony’s decomposition (2.1), we can rewrite the system (1.2) as

$$\begin{cases} \partial_t u + T_u \partial_x u + T_{\partial_y u} v - T_h \partial_x \tilde{h} - T_{\partial_y \tilde{h}} g - \partial_y^2 u = f_1, \\ \partial_t \tilde{h} + T_u \partial_x \tilde{h} + T_{\partial_y \tilde{h}} v - T_h \partial_x u - T_{\partial_y u} g - \partial_y^2 \tilde{h} = f_2, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f_1 &= -R_{\partial_x u} u - R_v \partial_y u + R_{\partial_x \tilde{h}} h + R_g \partial_y \tilde{h}, \\ f_2 &= -R_v \partial_y \tilde{h} - R_{\partial_x h} u + R_{\partial_x u} \tilde{h} + R_g \partial_y u. \end{aligned}$$

Let us introduce

$$h_1(t, x, y) = \int_0^y \tilde{h}(t, x, y') dy'.$$

From the second equation of (3.1), we deduce that

$$\partial_t h_1 + T_h v - T_u g - \partial_y^2 h_1 = \int_0^y f_2(y') dy'.$$

Motivated by [11], we introduce two good unknowns

$$\begin{cases} u_\beta = u - T_{\frac{\partial_y u}{h}} h_1, \\ \tilde{h}_\beta = \tilde{h} - T_{\frac{\partial_y \tilde{h}}{h}} h_1. \end{cases} \tag{3.2}$$

It is easy to check that

$$\begin{cases} \partial_t u_\beta + T_u \partial_x u_\beta - T_h \partial_x \tilde{h}_\beta - \partial_y^2 u_\beta = G_1, \\ \partial_t \tilde{h}_\beta - T_h \partial_x u_\beta + T_u \partial_x \tilde{h}_\beta - \partial_y^2 \tilde{h}_\beta = G_2, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} G_1 &= [T_{\frac{\partial_y u}{h}} T_h - T_{\partial_y u}] v - [T_{\frac{\partial_y u}{h}}, T_u] g - [T_h T_{\frac{\partial_y h}{h}} - T_{\partial_y h}] g - T_{(\partial_t - \partial_y^2)(\frac{\partial_y u}{h})} h_1 \\ &\quad + 2T_{\partial_y(\frac{\partial_y u}{h})} \tilde{h} - T_u T_{\partial_x \frac{\partial_y u}{h}} h_1 + T_h T_{\partial_x \frac{\partial_y h}{h}} h_1 - T_{\frac{\partial_y u}{h}} \int_0^y f_2 dy' + f_1 \\ &=: G_{11} + \dots + G_{19}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} G_2 &= [T_{\frac{\partial y h}{h}} T_h - T_{\partial_y h}]v - [T_h T_{\frac{\partial y u}{h}} - T_{\partial_y u}]g - [T_{\frac{\partial y h}{h}}, T_u]g + T_{(\partial_t - \partial_y^2)(\frac{\partial y h}{h})} h_1 \\ &\quad - 2T_{\partial_y(\frac{\partial y h}{h})} \tilde{h} - T_u T_{\partial_x \frac{\partial y h}{h}} h_1 + T_h T_{\partial_x \frac{\partial y h}{h}} h_1 - T_{\frac{\partial y h}{h}} \int_0^y f_2 dy' + f_2 \\ &=: G_{21} + \cdots + G_{29}. \end{aligned} \quad (3.5)$$

Moreover, it is easy to see that $(u_\beta, \tilde{h}_\beta)$ satisfies the following boundary condition:

$$(u_\beta, \partial_y \tilde{h}_\beta)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} (u_\beta, \tilde{h}_\beta) = (0, 0). \quad (3.6)$$

4 Sobolev estimate in horizontal direction

Let us first introduce the energy functional

$$\begin{aligned} E(t) &= \|u_\beta\|_{H_\mu^{3,0}}^2 + \|u\|_{H_\mu^{1,2}}^2 + \|\tilde{h}_\beta\|_{H_\mu^{3,0}}^2 + \|\tilde{h}\|_{H_\mu^{1,2}}^2, \\ D(t) &= \|\partial_y u_\beta\|_{H_\mu^{3,0}}^2 + \|\partial_y u\|_{H_\mu^{1,2}}^2 + \|\partial_y \tilde{h}_\beta\|_{H_\mu^{3,0}}^2 + \|\partial_y \tilde{h}\|_{H_\mu^{1,2}}^2. \end{aligned}$$

In this section, we always assume that (u, \tilde{h}) is a smooth solution of (1.2) on $[0, T]$ and

$$\sup_{t \leq T} E(t) \leq C_1 \varepsilon^2, \quad T < C_1 \varepsilon^{-2}, \quad (4.1)$$

for some $C_1 > 0$.

Proposition 4.1. *It holds that for any $t \in [0, T]$,*

$$\begin{aligned} &\frac{d}{dt} \|(u_\beta, \tilde{h}_\beta)\|_{H_\mu^{3,0}}^2 + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|_{H_\mu^{3,0}}^2 \\ &\leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + C\langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}}. \end{aligned}$$

4.1 Some technical lemmas

Lemma 4.1. *There exists $\varepsilon_0 > 0$ so that if $\varepsilon \in (0, \varepsilon_0)$, then*

$$h(t, x, y) \geq \frac{1}{2} \quad \text{for } (t, x, y) \in [0, T] \times \mathbf{R}_+^2.$$

Proof. As $h = \tilde{h} + 1$, we get by Sobolev embedding that

$$\|\tilde{h}\|_{L^\infty} \leq C \|\tilde{h}\|_{H_\mu^{1,1}} \leq CE(t)^{\frac{1}{2}} \leq C\varepsilon \leq \frac{1}{2}$$

by taking ε_0 small enough. □

The following lemma is a direct consequence of Hölder inequality.

Lemma 4.2. *It holds that*

$$\left\| \int_0^y f dy' \right\|_{L^\infty} \leq C \langle t \rangle^{\frac{1}{4}} \|f\|_{L^2_{y,\mu}}.$$

In particular, thanks to $\partial_x u + \partial_y v = 0$, $\partial_x \tilde{h} + \partial_y g = 0$, it holds that for $k \in \mathbf{N}$,

$$\|v\|_{H^k_x L^\infty_y} \leq C \langle t \rangle^{\frac{1}{4}} \|u\|_{H^{k+1,0}}, \quad \|g\|_{H^k_x L^\infty_y} \leq C \langle t \rangle^{\frac{1}{4}} \|\tilde{h}\|_{H^{k+1,0}}.$$

The following lemma gives the relationship of norm between good unknown $(u_\beta, \tilde{h}_\beta)$ and (u, \tilde{h}) .

Lemma 4.3. *There exists $\varepsilon_0 > 0$ so that if $\varepsilon \in (0, \varepsilon_0)$, then for $t \in [0, T]$,*

$$\|u(t)\|_{H^{3,0}_\mu} + \|\tilde{h}(t)\|_{H^{3,0}_\mu} \leq 2E(t)^{\frac{1}{2}}, \quad \|\partial_y u(t)\|_{H^{3,0}_\mu} + \|\partial_y \tilde{h}(t)\|_{H^{3,0}_\mu} \leq 4D(t)^{\frac{1}{2}}.$$

Proof. By Lemma 2.1, Lemma 4.1 and Lemma 4.2, we obtain

$$\begin{aligned} \|u\|_{H^{3,0}_\mu} &\leq \|u_\beta\|_{H^{3,0}_\mu} + \|T_{\frac{\partial_y u}{h}} h_1\|_{H^{3,0}_\mu} \leq \|u_\beta\|_{H^{3,0}_\mu} + C \langle t \rangle^{\frac{1}{4}} \|\partial_y u\|_{L^\infty_x L^2_{y,\mu}} \|\tilde{h}\|_{H^{3,0}_\mu} \\ &\leq \|u_\beta\|_{H^{3,0}_\mu} + C \langle t \rangle^{\frac{1}{4}} \|u\|_{H^{1,1}_\mu} \|\tilde{h}\|_{H^{3,0}_\mu} \leq \|u_\beta\|_{H^{3,0}_\mu} + C \langle t \rangle^{\frac{1}{4}} E(t)^{\frac{1}{2}} \|\tilde{h}\|_{H^{3,0}_\mu} \\ &\leq \|u_\beta\|_{H^{3,0}_\mu} + C\varepsilon^{\frac{1}{2}} \|\tilde{h}\|_{H^{3,0}_\mu}. \end{aligned}$$

Similarly, we have

$$\|\tilde{h}\|_{H^{3,0}_\mu} \leq \|\tilde{h}_\beta\|_{H^{3,0}_\mu} + C\varepsilon^{\frac{1}{2}} \|\tilde{h}\|_{H^{3,0}_\mu}.$$

Thus, we deduce by taking ε_0 small enough that

$$\|\tilde{h}\|_{H^{3,0}_\mu} + \|u\|_{H^{3,0}_\mu} \leq 2(\|u_\beta\|_{H^{3,0}_\mu} + \|\tilde{h}_\beta\|_{H^{3,0}_\mu}) \leq 4E(t)^{\frac{1}{2}}.$$

By Lemma 2.1, Lemma 4.1 and Lemma 4.2 again, we get

$$\begin{aligned} \|\partial_y u\|_{H^{3,0}_\mu} &\leq \|\partial_y u_\beta\|_{H^{3,0}_\mu} + \|\partial_y T_{\frac{\partial_y u}{h}} h_1\|_{H^{3,0}_\mu} \\ &\leq \|\partial_y u_\beta\|_{H^{3,0}_\mu} + C \langle t \rangle^{\frac{1}{4}} \|\partial_y^2 u\|_{L^\infty_x L^2_{y,\mu}} + \langle t \rangle^{\frac{1}{4}} \|\partial_y u\|_{L^\infty_x L^2_{y,\mu}} \|\partial_y h\|_{L^\infty} + \|\partial_y u\|_{L^\infty} \|\tilde{h}\|_{H^{3,0}_\mu} \\ &\leq \|\partial_y u_\beta\|_{H^{3,0}_\mu} + C \langle t \rangle^{\frac{1}{4}} \|\partial_y u\|_{H^{1,1}_\mu} + \langle t \rangle^{\frac{1}{4}} \|\partial_y u\|_{H^{1,0}_\mu} \|\tilde{h}\|_{H^{1,2}_\mu} + \|\partial_y u\|_{H^{1,1}_\mu} \|\tilde{h}\|_{H^{3,0}_\mu} \\ &\leq \|\partial_y u_\beta\|_{H^{3,0}_\mu} + C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} \\ &\leq \|\partial_y u_\beta\|_{H^{3,0}_\mu} + C\varepsilon^{\frac{1}{2}} D(t)^{\frac{1}{2}} \leq 2D(t)^{\frac{1}{2}}. \end{aligned}$$

In the same way, we have

$$\|\partial_y \tilde{h}\|_{H^{3,0}_\mu} \leq 2D^{\frac{1}{2}}(t).$$

This proves the lemma. □

4.2 Nonlinear estimates

Let us now estimate the nonlinear terms G_1 and G_2 .

Lemma 4.4. *It holds that*

$$\|G_1\|_{H_\mu^{3,0}} + \|G_2\|_{H_\mu^{3,0}} \leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + C\langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t).$$

Proof. Let us only present the estimate of G_1 . The estimate of G_2 is similar. By Lemma 2.2, Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} \|G_{12}\|_{H_\mu^{3,0}} &= \|[T_{\frac{\partial_y u}{h}}, T_u]g\|_{H_\mu^{3,0}} \\ &\leq C\left(\|\partial_y u\|_{L_{y,\mu}^2(W_x^{1,\infty})}\|u\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|u\|_{L_{y,\mu}^2(W_x^{1,\infty})}\right)\|g\|_{L_y H_x^2} \\ &\leq C\langle t \rangle^{\frac{1}{4}}\|\partial_y u\|_{H_\mu^{2,0}}\|u\|_{H_\mu^{1,1}}\|\tilde{h}\|_{H_\mu^{3,0}} + C\langle t \rangle^{\frac{1}{4}}\|\partial_y u\|_{H_\mu^{1,1}}\|u\|_{H_\mu^{2,0}}\|\tilde{h}\|_{H_\mu^{3,0}} \\ &\leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned}$$

Using the facts that

$$\left[T_{\frac{\partial_y u}{h}} T_h - T_{\frac{\partial_y u}{h} h}\right] = \left[T_{\frac{\partial_y u}{h}} T_{\tilde{h}} - T_{\frac{\partial_y u}{h} \tilde{h}}\right], \quad \left[T_{\frac{\partial_y h}{h}} T_h - T_{\frac{\partial_y h}{h} h}\right] = \left[T_{\frac{\partial_y \tilde{h}}{h}} T_{\tilde{h}} - T_{\frac{\partial_y \tilde{h}}{h} \tilde{h}}\right],$$

we can deduce from Lemmas 2.2, 4.1 and 4.2 that

$$\begin{aligned} \|G_{11}\|_{H_\mu^{3,0}} &= \|[T_{\frac{\partial_y u}{h}} T_{\tilde{h}} - T_{\frac{\partial_y u}{h} \tilde{h}}]v\|_{H_\mu^{3,0}} \\ &\leq C\left(\|\partial_y u\|_{L_{y,\mu}^2(W_x^{1,\infty})}\|\tilde{h}\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|\tilde{h}\|_{L_{y,\mu}^2(W_x^{1,\infty})}\right)\|v\|_{L_y H_x^2} \\ &\leq C\left(\langle t \rangle^{\frac{1}{4}}\|\partial_y u\|_{H_\mu^{2,0}}\|\tilde{h}\|_{H_\mu^{1,1}} + \langle t \rangle^{\frac{1}{4}}\|\tilde{h}\|_{H_\mu^{2,0}}\|\partial_y u\|_{H_\mu^{1,1}}\right)\|u\|_{H_\mu^{3,0}} \\ &\leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|G_{13}\|_{H_\mu^{3,0}} &= \|[T_{\tilde{h}} T_{\frac{\partial_y \tilde{h}}{h}}, T_{\frac{\partial_y \tilde{h}}{h} \tilde{h}}]g\|_{H_\mu^{3,0}} \\ &\leq C\left(\langle t \rangle^{\frac{1}{4}}\|\partial_y \tilde{h}\|_{H_\mu^{2,0}}\|\tilde{h}\|_{H_\mu^{1,1}} + \langle t \rangle^{\frac{1}{4}}\|\tilde{h}\|_{H_\mu^{2,0}}\|\partial_y \tilde{h}\|_{H_\mu^{1,1}}\right)\|\tilde{h}\|_{H_\mu^{3,0}} \\ &\leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned}$$

Thanks to Lemma 2.1 and Lemma 4.1, we have

$$\begin{aligned} \|G_{15}\|_{H_\mu^{3,0}} &= \|2T_{\frac{\partial_y u}{h}} \tilde{h}\|_{H_\mu^{3,0}} \leq 2\|T_{\frac{\partial_y u}{h}} \tilde{h}\|_{H_\mu^{3,0}} + 2\|T_{\frac{\partial_y u \partial_y \tilde{h}}{h}} \tilde{h}\|_{H_\mu^{3,0}} \\ &\leq C\|\partial_y u\|_{H_\mu^{1,2}}\|\tilde{h}\|_{H_\mu^{3,0}} + C\|\partial_y u\|_{H_\mu^{1,1}}\|\partial_y \tilde{h}\|_{H_\mu^{1,1}}\|\tilde{h}\|_{H_\mu^{3,0}} \leq CD(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2.1, Lemma 4.1 and Lemma 4.2, we get

$$\begin{aligned} \|G_{17}\|_{H_\mu^{3,0}} &= \|T_h T_{\partial_x \frac{\partial_y h}{h}} h_1\|_{H_\mu^{3,0}} \leq \|h\|_{L^\infty} \|(\partial_x \partial_y \tilde{h} + \partial_x \tilde{h} \partial_y \tilde{h})\|_{L_x^\infty L_{y,\mu}^2} \|h_1\|_{H_x^3 L_y^\infty} \\ &\leq C(\|\partial_y \tilde{h}\|_{H_\mu^{2,0}} + \|\partial_y \tilde{h}\|_{H_\mu^{1,1}} \|\tilde{h}\|_{H_\mu^{2,0}}) \langle t \rangle^{\frac{1}{4}} \|\tilde{h}\|_{H_\mu^{3,0}} \leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}}. \end{aligned}$$

Similarly, we have

$$\|G_{16}\|_{H_\mu^{3,0}} = \|T_u T_{\partial_x \frac{\partial_y u}{h}} h_1\|_{H_\mu^{3,0}} \leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}}.$$

For G_{14} , we use Eq. (1.2) to find that

$$\begin{aligned} (\partial_t - \partial_y^2) \left(\frac{\partial_y u}{h} \right) &= \frac{(\partial_t - \partial_y^2) \partial_y u}{h} - \frac{\partial_y u (\partial_t - \partial_y^2) h}{h^2} + \frac{2h \partial_y h \partial_y^2 u - 2\partial_y u (\partial_y h)^2}{h^3} \\ &= \frac{h \partial_{xy}^2 \tilde{h} + g \partial_y^2 \tilde{h} - u \partial_{xy}^2 u - v \partial_y^2 u}{h} - \frac{\partial_y u \partial_y (u g - v h)}{h^2} + \frac{2h \partial_y \tilde{h} \partial_y^2 u - 2\partial_y u (\partial_y \tilde{h})^2}{h^3} \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

Using Lemmas 2.1, 4.1 and 4.2, we deduce that

$$\begin{aligned} \|T_{A_1} h_1\|_{H_\mu^{3,0}} &\leq \|T_{\partial_{xy}^2 \tilde{h}} h_1\|_{H_\mu^{3,0}} + \|T_{\frac{g \partial_y^2 \tilde{h}}{h}} h_1\|_{H_\mu^{3,0}} + \|T_{\frac{u \partial_{xy}^2 u}{h}} h_1\|_{H_\mu^{3,0}} + \|T_{\frac{v \partial_y^2 u}{h}} h_1\|_{H_\mu^{3,0}} \\ &\leq C \langle t \rangle^{\frac{1}{4}} \|\partial_y \tilde{h}\|_{H_\mu^{2,0}} \|\tilde{h}\|_{H_\mu^{3,0}} + C \langle t \rangle^{\frac{1}{2}} \|\tilde{h}\|_{H_\mu^{2,0}} \|\partial_y \tilde{h}\|_{H_\mu^{1,1}} \|\tilde{h}\|_{H_\mu^{3,0}} \\ &\quad + C \langle t \rangle^{\frac{1}{4}} \|u\|_{H_\mu^{1,1}} \|\partial_y u\|_{H_\mu^{1,1}} \|\tilde{h}\|_{H_\mu^{3,0}} + C \langle t \rangle^{\frac{1}{2}} \|u\|_{H_\mu^{2,0}} \|\partial_y u\|_{H_\mu^{1,1}} \|\tilde{h}\|_{H_\mu^{3,0}} \\ &\leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + C \langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t), \\ \|T_{A_2} h_1\|_{H_\mu^{3,0}} &\leq \|T_{\frac{g(\partial_y u)^2}{h^2}} h_1\|_{H_\mu^{3,0}} + \|T_{\frac{u \partial_y u \partial_x u}{h^2}} h_1\|_{H_\mu^{3,0}} + \|T_{\frac{\partial_y u \partial_x u}{h}} h_1\|_{H_\mu^{3,0}} + \|T_{\frac{v \partial_y u \partial_y \tilde{h}}{h^2}} h_1\|_{H_\mu^{3,0}} \\ &\leq C \langle t \rangle^{\frac{1}{2}} \|\tilde{h}\|_{H_\mu^{2,0}} \|\partial_y u\|_{H_\mu^{1,1}}^2 \|\tilde{h}\|_{H_\mu^{3,0}} + C \langle t \rangle^{\frac{1}{4}} \|u\|_{H_\mu^{1,1}} \|\partial_y u\|_{H_\mu^{1,1}} \|u\|_{H_\mu^{2,0}} \|\tilde{h}\|_{H_\mu^{3,0}} \\ &\quad + C \langle t \rangle^{\frac{1}{4}} \|\partial_y u\|_{H_\mu^{1,1}} \|u\|_{H_\mu^{2,0}} \|\tilde{h}\|_{H_\mu^{3,0}} + C \langle t \rangle^{\frac{1}{2}} \|u\|_{H_\mu^{2,0}} \|\partial_y u\|_{H_\mu^{1,1}} \|\partial_y \tilde{h}\|_{H_\mu^{1,0}} \|\tilde{h}\|_{H_\mu^{3,0}} \\ &\leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + C \langle t \rangle^{\frac{1}{2}} E(t)^{\frac{3}{2}} D(t)^{\frac{1}{2}}, \\ \|T_{A_3} h_1\|_{H_\mu^{3,0}} &\leq C \|T_{\frac{\partial_y \tilde{h} \partial_y^2 u}{h^2}} h_1\|_{H_\mu^{3,0}} + C \|T_{\frac{\partial_y u (\partial_y \tilde{h})^2}{h^3}} h_1\|_{H_\mu^{3,0}} \\ &\leq C \langle t \rangle^{\frac{1}{4}} \|\partial_y \tilde{h}\|_{H_\mu^{1,1}} \|\partial_y u\|_{H_\mu^{1,1}} \|\tilde{h}\|_{H_\mu^{3,0}} + C \langle t \rangle^{\frac{1}{4}} \|\partial_y u\|_{H_\mu^{1,1}} \|\partial_y \tilde{h}\|_{H_\mu^{1,1}}^2 \|\tilde{h}\|_{H_\mu^{3,0}} \\ &\leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned}$$

Thus, we conclude that

$$\|G_{14}\|_{H_\mu^{3,0}} = \|T_{(\partial_t - \partial_y^2) \frac{\partial_y u}{h}} h_1\|_{H_\mu^{3,0}} \leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + C \langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t).$$

Using

$$\|f\|_{L_y^\infty} \leq C\|f\|_{L_y^2}^{\frac{1}{2}}\|\partial_y f\|_{L_y^2}^{\frac{1}{2}}$$

and Lemma 2.1, we have

$$\begin{aligned} \|f_2\|_{H_\mu^{3,0}} &\leq \|R_{\partial_x u}\tilde{h}\|_{H_\mu^{3,0}} + \|R_{\partial_y}\tilde{h}\|_{H_\mu^{3,0}} + \|R_{\partial_x}\tilde{h}u\|_{H_\mu^{3,0}} + \|R_g\partial_y u\|_{H_\mu^{3,0}} \\ &\leq C\|u\|_{H_\mu^{2,0}}^{\frac{1}{2}}\|\partial_y u\|_{H_\mu^{2,0}}^{\frac{1}{2}}\|\tilde{h}\|_{H_\mu^{3,0}} + C\langle t\rangle^{\frac{1}{4}}\|u\|_{H_\mu^{2,0}}\|\partial_y\tilde{h}\|_{H_\mu^{3,0}} \\ &\quad + C\|\tilde{h}\|_{H_\mu^{2,0}}^{\frac{1}{2}}\|\partial_y\tilde{h}\|_{H_\mu^{2,0}}^{\frac{1}{2}}\|u\|_{H_\mu^{3,0}} + C\langle t\rangle^{\frac{1}{4}}\|\tilde{h}\|_{H_\mu^{2,0}}\|\partial_y u\|_{H_\mu^{3,0}} \\ &\leq CD(t)^{\frac{1}{4}}E(t)^{\frac{3}{4}} + C\langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}}, \end{aligned}$$

which gives

$$\begin{aligned} \|G_{18}\|_{H_\mu^{3,0}} &= \left\| T_{\frac{\partial_y u}{h}} \int_0^y f_2 dy' \right\|_{H_\mu^{3,0}} \leq C\langle t\rangle^{\frac{1}{4}}\|\partial_y u\|_{H_\mu^{1,0}}\|f_2\|_{H_\mu^{3,0}} \\ &\leq C\langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t) + C\langle t\rangle^{\frac{1}{2}}D(t)^{\frac{1}{2}}E(t). \end{aligned}$$

Similarly, we have

$$\|G_{19}\|_{H_\mu^{3,0}} \leq C\langle t\rangle^{\frac{1}{4}}D(t)^{\frac{1}{2}}E(t) + C\langle t\rangle^{\frac{1}{2}}D(t)^{\frac{1}{2}}E(t).$$

Putting all the above estimates together leads to the estimate of $\|G_1\|_{H_\mu^{3,0}}$. □

4.3 Tangential energy estimate

In this subsection, we prove Proposition 4.1.

Proof. Making $H_\mu^{3,0}$ energy estimate to (3.3), we obtain

$$\begin{aligned} &(\partial_t u_\beta, u_\beta)_{H_\mu^{3,0}} + (\partial_t \tilde{h}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} - (\partial_y^2 u_\beta, u_\beta)_{H_\mu^{3,0}} - (\partial_y^2 \tilde{h}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} \\ &+ (T_u \partial_x u_\beta, u_\beta)_{H_\mu^{3,0}} - (T_h \partial_x \tilde{h}_\beta, u_\beta)_{H_\mu^{3,0}} - (T_h \partial_x \tilde{u}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} + (T_u \partial_x \tilde{h}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} \\ &= (G_1, u_\beta)_{H_\mu^{3,0}} + (G_2, \tilde{h}_\beta)_{H_\mu^{3,0}}. \end{aligned}$$

First of all, we have

$$\begin{aligned} (\partial_t u_\beta, u_\beta)_{H_\mu^{3,0}} &= \frac{1}{2} \frac{d}{dt} \|u_\beta\|_{H_\mu^{3,0}}^2 - \int_{\mathbf{R}_+} \partial_t \theta \|e^\theta u_\beta\|_{H_x^3}^2 dy, \\ (\partial_t \tilde{h}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} &= \frac{1}{2} \frac{d}{dt} \|\tilde{h}_\beta\|_{H_\mu^{3,0}}^2 - \int_{\mathbf{R}_+} \partial_t \theta \|e^\theta \tilde{h}_\beta\|_{H_x^3}^2 dy, \end{aligned}$$

where we denote $\theta(t, y) = \frac{1+y^2}{8\langle t\rangle}$. Thanks to

$$u_\beta|_{y=0} = \partial_y \tilde{h}_\beta|_{y=0} = 0,$$

we get by integration by parts that

$$\begin{aligned} -(\partial_y^2 u_\beta, u_\beta)_{H_\mu^{3,0}} &= \|\partial_y u_\beta\|_{H_\mu^{3,0}}^2 + 2 \int_{\mathbf{R}_+} \partial_y \theta (e^\theta \partial_y u_\beta, e^\theta u_\beta)_{H_x^3} dy \\ &\geq \frac{1}{2} \|\partial_y u_\beta\|_{H_\mu^{3,0}}^2 - 2 \int_{\mathbf{R}_+} (\partial_y \theta)^2 \|e^\theta u_\beta\|_{H_x^3}^2 dy. \end{aligned}$$

Similarly, we have

$$\begin{aligned} -(\partial_y^2 \tilde{h}_\beta, u_\beta)_{H_\mu^{3,0}} &= \|\partial_y \tilde{h}_\beta\|_{H_\mu^{3,0}}^2 + 2 \int_{\mathbf{R}_+} \partial_y \theta (e^\theta \partial_y \tilde{h}_\beta, e^\theta u_\beta)_{H_x^3} dy \\ &\geq \frac{1}{2} \|\partial_y \tilde{h}_\beta\|_{H_\mu^{3,0}}^2 - 2 \int_{\mathbf{R}_+} (\partial_y \theta)^2 \|e^\theta \tilde{h}_\beta\|_{H_x^3}^2 dy. \end{aligned}$$

We write

$$\begin{aligned} (T_h \partial_x u_\beta, u_\beta)_{H_\mu^{2,0}} &= \sum_{k=0}^3 (\partial_x^k T_h \partial_x u_\beta, \partial_x^k u_\beta)_{L_\mu^2} \\ &= \sum_{k=0}^3 (T_h \partial_x^k \partial_x u_\beta, \partial_x^k u_\beta)_{L_\mu^2} + \sum_{k=1}^3 ([\partial_x^k, T_{\tilde{h}}] \partial_x u_\beta, \partial_x^k u_\beta)_{L_\mu^2} \\ &=: D + \sum_{k=1}^3 ([\partial_x^k, T_{\tilde{h}}] \partial_x u_\beta, \partial_x^k u_\beta)_{L_\mu^2}, \end{aligned}$$

where

$$\begin{aligned} D &= -\frac{1}{2} \sum_{k=0}^2 (T_{\partial_x \tilde{h}} \partial_x^k u_\beta, \partial_x^k u_\beta)_{L_\mu^2} + \frac{1}{2} \sum_{k=0}^3 ((T_{\tilde{h}} - T_{\tilde{h}}^*) \partial_x^k \partial_x u_\beta, \partial_x^k u_\beta)_{L_\mu^2} \\ &=: D_1 + D_2. \end{aligned}$$

By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} |D_1| &\leq \sum_{k=0}^3 \|T_{\partial_x \tilde{h}} \partial_x^k u_\beta\|_{L_\mu^2} \|\partial_x^k u_\beta\|_{L_\mu^2} \leq C \|\partial_x \tilde{h}\|_{L^\infty} \|u_\beta\|_{H_\mu^{3,0}}^2 \\ &\leq C \|\tilde{h}\|_{H_\mu^{2,0}}^{\frac{1}{2}} \|\partial_y \tilde{h}\|_{H_\mu^{2,0}}^{\frac{1}{2}} \|u_\beta\|_{H_\mu^{3,0}}^2 \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}, \\ |D_2| &\leq \sum_{k=0}^3 \|(T_{\tilde{h}} - T_{\tilde{h}}^*) \partial_x^k \partial_x u_\beta\|_{L_\mu^2} \|\partial_x^k u_\beta\|_{L_\mu^2} \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}, \end{aligned}$$

and by Lemma 2.3,

$$\sum_{k=1}^3 ([\partial_x^k, T_{\tilde{h}}] \partial_x u_\beta, \partial_x^k u_\beta)_{L_\mu^2} \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}.$$

This shows that

$$(T_u \partial_x u_\beta, u_\beta)_{H_\mu^{3,0}} \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}.$$

Similarly, we have

$$(T_u \partial_x \tilde{h}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}.$$

On the other hand, we write

$$\begin{aligned} & (T_h \partial_x \tilde{h}_\beta, u_\beta)_{H_\mu^{3,0}} + (T_h \partial_x u_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} \\ &= (T_h \partial_x (\tilde{h}_\beta + u_\beta), u_\beta + \tilde{h}_\beta)_{H_\mu^{3,0}} - (T_h \partial_x \tilde{h}_\beta, \tilde{h}_\beta)_{H_\mu^{3,0}} - (T_h \partial_x u_\beta, u_\beta)_{H_\mu^{3,0}}. \end{aligned}$$

Thus, we also have

$$(T_h \partial_x \tilde{h}_\beta, u_\beta)_{H_\mu^{3,0}} + (T_h \partial_x \tilde{h}_\beta, u_\beta)_{H_\mu^{3,0}} \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}.$$

It follows from Lemma 4.4 that

$$\begin{aligned} & (G_1, u_\beta)_{H_\mu^{3,0}} + (G_2, \tilde{h}_\beta)_{H_\mu^{3,0}} \\ & \leq C(\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + \langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)) \|(u_\beta, \tilde{h}_\beta)\|_{H_\mu^{3,0}} \\ & \leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + C\langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}}. \end{aligned}$$

Summing up all the estimates, we conclude that

$$\begin{aligned} & \frac{d}{dt} \|(u_\beta, \tilde{h}_\beta)\|_{H_\mu^{3,0}}^2 + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|_{H_\mu^{3,0}}^2 \\ & \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + C\langle t \rangle^{\frac{1}{2}} D(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}}, \end{aligned}$$

where we used $\partial_t \theta + 2(\partial_y \theta)^2 < 0$. □

5 Sobolev estimate in vertical direction

To close the energy estimates, we need to derive high order derivative estimates in the vertical variable y . We again assume that (u, \tilde{h}) is a smooth solution of (1.2) on $[0, T]$ satisfying (4.1).

Proposition 5.1. *It holds that for any $t \in [0, T]$,*

$$\frac{d}{dt} \|(u, \tilde{h})\|_{H_\mu^{1,2}}^2 + \|(\partial_y u, \partial_y \tilde{h})\|_{H_\mu^{1,2}}^2 \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t).$$

Proof. We split the proof into the following three steps.

Step 1. $H_\mu^{1,0}$ estimate.

Taking $H_\mu^{1,0}$ -inner product between (1.2) and (u, \tilde{h}) , we obtain

$$\begin{aligned} & (\partial_t u, u)_{H_\mu^{1,0}} + (\partial_t \tilde{h}, \tilde{h})_{H_\mu^{1,0}} - (\partial_y^2 u, u)_{H_\mu^{1,0}} - (\partial_y^2 \tilde{h}, \tilde{h})_{H_\mu^{1,0}} \\ &= - \left((u \partial_x u, u)_{H_\mu^{1,0}} + (u \partial_x \tilde{h}, \tilde{h})_{H_\mu^{1,0}} - (h \partial_x u, \tilde{h})_{H_\mu^{1,0}} - (h \partial_x \tilde{h}, u)_{H_\mu^{1,0}} \right) \\ & \quad + \left(- (v \partial_y u, u)_{H_\mu^{1,0}} + (g \partial_y \tilde{h}, u)_{H_\mu^{1,0}} - (v \partial_y \tilde{h}, \tilde{h})_{H_\mu^{1,0}} + (g \partial_y u, \tilde{h})_{H_\mu^{1,0}} \right) \\ &=: A + B. \end{aligned} \tag{5.1}$$

By integrating by parts and thanks to $\partial_t \theta + 2(\partial_y \theta)^2 \leq 0$, we have

$$\begin{aligned} & (\partial_t u, u)_{H_\mu^{1,0}} + (\partial_t \tilde{h}, \tilde{h})_{H_\mu^{1,0}} - (\partial_y^2 u, u)_{H_\mu^{1,0}} - (\partial_y^2 \tilde{h}, \tilde{h})_{H_\mu^{1,0}} \\ & \geq \frac{1}{2} \frac{d}{dt} (\| (u, \tilde{h}) \|_{H_\mu^{1,0}}^2 + \frac{1}{2} \| (\partial_y u, \partial_y \tilde{h}) \|_{H_\mu^{1,0}}^2 - \int_{\mathbf{R}_+} (\partial_t \theta + 2(\partial_y \theta)^2) \| e^\theta u \|_{H_x^1}^2 dy) \\ & \geq \frac{1}{2} \frac{d}{dt} (\| (u, \tilde{h}) \|_{H_\mu^{1,0}}^2 + \frac{1}{2} \| (\partial_y u, \partial_y \tilde{h}) \|_{H_\mu^{1,0}}^2). \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} (u \partial_x u, u)_{H_\mu^{1,0}} &= (u \partial_x u, u)_{L_\mu^2} + \frac{1}{2} (\partial_x u \partial_x u, \partial_x u)_{L_\mu^2} \\ &\leq C \| \partial_x u \|_{L^\infty} \| u \|_{H_\mu^{1,0}}^2 \leq C \| \partial_y u \|_{H_\mu^{2,0}}^{\frac{1}{2}} \| u \|_{H_\mu^{2,0}}^{\frac{1}{2}} \| u \|_{H_\mu^{1,0}}^2, \end{aligned}$$

and

$$\begin{aligned} & (h \partial_x u, \tilde{h})_{H_\mu^{1,0}} + (h \partial_x \tilde{h}, u)_{H_\mu^{1,0}} \\ &= (h \partial_x u, \tilde{h})_{L_\mu^2} + (h \partial_x \tilde{h}, u)_{L_\mu^2} + (\partial_x h \partial_x \tilde{h}, \partial_x u)_{L_\mu^2} \\ &\leq C \| \partial_y u \|_{H_\mu^{2,0}}^{\frac{1}{2}} \| u \|_{H_\mu^{2,0}}^{\frac{1}{2}} \| h \|_{H_\mu^{1,0}}^2. \end{aligned}$$

Similarly, we have

$$(u \partial_x \tilde{h}, \tilde{h})_{H_\mu^{1,0}} \leq C \| \partial_y u \|_{H_\mu^{2,0}}^{\frac{1}{2}} \| u \|_{H_\mu^{2,0}}^{\frac{1}{2}} \| \tilde{h} \|_{H_\mu^{1,0}}^2.$$

This shows that

$$A \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}.$$

Due to $v = -\int_0^y \partial_x u dy'$, we get by Lemma 4.2 that

$$\begin{aligned} (v\partial_y u, u)_{H_\mu^{1,0}} &\leq \|v\|_{H_x^1 L_y^\infty} \|\partial_y u\|_{H_\mu^{1,0}} \|u\|_{H_\mu^{1,0}} \\ &\leq C\langle t \rangle^{\frac{1}{4}} \|u\|_{H_\mu^{2,0}} \|\partial_y u\|_{H_\mu^{1,0}} \|u\|_{H_\mu^{1,0}} \leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned}$$

The other terms in B could be estimated in a similar way. Then we have

$$B \leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t).$$

Thus, we deduce that

$$\frac{d}{dt} \|(u, \tilde{h})\|_{H_\mu^{1,0}}^2 + \|(\partial_y u, \partial_y \tilde{h})\|_{H_\mu^{1,0}}^2 \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \quad (5.2)$$

Step 2. $H_\mu^{1,1}$ estimate.

Taking ∂_y to (1.2) and then taking $H_\mu^{1,0}$ -inner product with $(\partial_y u, \partial_y \tilde{h})$, we obtain

$$\begin{aligned} &\frac{d}{dt} \|(\partial_y \tilde{h}, \partial_y u)\|_{H_\mu^{1,0}}^2 + \|(\partial_y^2 u, \partial_y^2 \tilde{h})\|_{H_\mu^{1,0}}^2 \\ &\leq \left(- (u\partial_x \partial_y u, \partial_y u)_{H_\mu^{1,0}} - (u\partial_x \partial_y \tilde{h}, \partial_y \tilde{h})_{H_\mu^{1,0}} + (h\partial_x \partial_y \tilde{h}, \partial_y u)_{H_\mu^{1,0}} + (h\partial_x \partial_y u, \partial_y \tilde{h})_{H_\mu^{1,0}} \right) \\ &\quad + \left(- (v\partial_y^2 u, \partial_y u)_{H_\mu^{1,0}} + (g\partial_y^2 \tilde{h}, \partial_y u)_{H_\mu^{1,0}} + (g\partial_y^2 u, \partial_y \tilde{h})_{H_\mu^{1,0}} - (v\partial_y^2 \tilde{h}, \partial_y \tilde{h})_{H_\mu^{1,0}} \right) \\ &\quad - 2(\partial_x \tilde{h} \partial_y u - \partial_x u \partial_y \tilde{h}, \partial_y \tilde{h})_{H_\mu^{1,0}} \\ &=: A_1 + B_1 + C_1. \end{aligned}$$

For A_1 , we have

$$\begin{aligned} A_1 &\leq C \left(\|(u, \tilde{h})\|_{L^\infty} + \|(\partial_x u, \partial_x \tilde{h})\|_{L^\infty} \right) \left(\|\partial_y u\|_{H_\mu^{1,0}}^2 + \|\partial_y \tilde{h}\|_{H_\mu^{1,0}}^2 \right) \\ &\leq C \|\partial_y u\|_{H_\mu^{2,0}}^{\frac{1}{2}} \|u\|_{H_\mu^{2,0}}^{\frac{1}{2}} \left(\|\partial_y u\|_{H_\mu^{1,0}}^2 + \|\partial_y \tilde{h}\|_{H_\mu^{1,0}}^2 \right) \\ &\leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}, \end{aligned}$$

and for B_1 , we have

$$\begin{aligned} B_1 &\leq C \left(\|v\|_{H_x^1 L_y^\infty} + \|g\|_{H_x^1 L_y^\infty} \right) \left(\|\partial_y^2 u\|_{H_\mu^{1,0}} + \|\partial_y^2 \tilde{h}\|_{H_\mu^{1,0}} \right) \left(\|\partial_y u\|_{H_\mu^{1,0}} + \|\partial_y \tilde{h}\|_{H_\mu^{1,0}} \right) \\ &\leq C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned}$$

and for C_1 , we have

$$\begin{aligned} C_1 &\leq C \|\partial_x \tilde{h} \partial_y u - \partial_x u \partial_y \tilde{h}\|_{H_\mu^{1,0}} \|\partial_y \tilde{h}\|_{H_\mu^{1,0}} \\ &\leq C \left(\|\tilde{h}\|_{H_\mu^{2,0}} \|\partial_y u\|_{H_\mu^{1,0}}^{\frac{1}{2}} \|\partial_y^2 u\|_{H_\mu^{1,0}}^{\frac{1}{2}} + \|u\|_{H_\mu^{2,0}} \|\partial_y \tilde{h}\|_{H_\mu^{1,0}}^{\frac{1}{2}} \|\partial_y^2 \tilde{h}\|_{H_\mu^{1,0}}^{\frac{1}{2}} \right) \|\partial_y \tilde{h}\|_{H_\mu^{1,0}} \\ &\leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}. \end{aligned}$$

Thus, we deduce that

$$\frac{d}{dt} \|(\partial_y \tilde{h}, \partial_y u)\|_{H_\mu^{1,0}}^2 + \|(\partial_y^2 u, \partial_y^2 \tilde{h})\|_{H_\mu^{1,0}}^2 \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \tag{5.3}$$

Step 3. $H_\mu^{1,2}$ estimate.

Taking ∂_y^2 to (1.2) and taking $H_\mu^{1,0}$ -inner product with $(\partial_y^2 u, \partial_y^2 \tilde{h})$, we obtain

$$\begin{aligned} & \frac{d}{dt} \|(\partial_y^2 \tilde{h}, \partial_y^2 u)\|_{H_\mu^{1,0}}^2 + \|(\partial_y^3 u, \partial_y^3 \tilde{h})\|_{H_\mu^{1,0}}^2 \\ & \leq \left(- (u \partial_x \partial_y^2 u, \partial_y^2 u)_{H_\mu^{1,0}} - (u \partial_x \partial_y^2 \tilde{h}, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} + (h \partial_x \partial_y^2 \tilde{h}, \partial_y^2 u)_{H_\mu^{1,0}} + (h \partial_x \partial_y^2 u, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} \right) \\ & \quad + \left((g \partial_y^3 \tilde{h}, \partial_y^2 u)_{H_\mu^{1,0}} + (g \partial_y^3 u, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} - (v \partial_y^3 u, \partial_y^2 u)_{H_\mu^{1,0}} - (v \partial_y^3 \tilde{h}, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} \right) \\ & \quad + \left(- (\partial_y u \partial_x \partial_y u, \partial_y^2 u)_{H_\mu^{1,0}} - (\partial_y u \partial_x \partial_y \tilde{h}, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} + (\partial_y \tilde{h} \partial_x \partial_y \tilde{h}, \partial_y^2 u)_{H_\mu^{1,0}} \right) \\ & \quad + (\partial_y \tilde{h} \partial_x \partial_y u, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} + \left(- (\partial_y v \partial_y^2 u, \partial_y^2 u)_{H_\mu^{1,0}} + (\partial_y g \partial_y^2 \tilde{h}, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} \right) \\ & \quad - (\partial_y v \partial_y^2 \tilde{h}, \partial_y^2 u)_{H_\mu^{1,0}} + (\partial_y g \partial_y^2 u, \partial_y^2 \tilde{h})_{H_\mu^{1,0}} - 2(\partial_y (\partial_x \tilde{h} \partial_y u - \partial_x u \partial_y \tilde{h}), \partial_y^2 \tilde{h})_{H_\mu^{1,0}} \\ & =: A_2 + B_2 + C_2 + D_2 + E_2. \end{aligned}$$

We estimate nonlinear terms as follows:

$$\begin{aligned} A_2 & \leq C \left(\| (u, \tilde{h}) \|_{L^\infty} + \| (\partial_x u, \partial_x \tilde{h}) \|_{L^\infty} \right) \left(\| \partial_y^2 u \|_{H_\mu^{1,0}}^2 + \| \partial_y^2 \tilde{h} \|_{H_\mu^{1,0}}^2 \right) \\ & \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}, \end{aligned}$$

and

$$\begin{aligned} B_2 & \leq C \left(\| v \|_{H_x^1 L_y^\infty} + \| g \|_{H_x^1 L_y^\infty} \right) \left(\| \partial_y^3 u \|_{H_\mu^{1,0}} + \| \partial_y^3 \tilde{h} \|_{H_\mu^{1,0}} \right) \left(\| \partial_y^2 u \|_{H_\mu^{1,0}} + \| \partial_y^2 \tilde{h} \|_{H_\mu^{1,0}} \right) \\ & \leq C \langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t), \\ C_2 & \leq \left(\| \partial_y u \|_{L^\infty} + \| \partial_y \tilde{h} \|_{L^\infty} \right) \left(\| \partial_x \partial_y u \|_{H_\mu^{1,0}} + \| \partial_x \partial_y \tilde{h} \|_{H_\mu^{1,0}} \right) \left(\| \partial_y^2 u \|_{H_\mu^{1,0}} + \| \partial_y^2 \tilde{h} \|_{H_\mu^{1,0}} \right) \\ & \quad + \left(\| \partial_x \partial_y u \|_{L_x^2 L_y^\infty} + \| \partial_x \partial_y \tilde{h} \|_{L_x^2 L_y^\infty} \right) \left(\| \partial_x \partial_y u \|_{L_x^\infty L_{y,\mu}^2} \right. \\ & \quad \left. + \| \partial_x \partial_y \tilde{h} \|_{L_x^\infty L_{y,\mu}^2} \right) \left(\| \partial_x \partial_y^2 u \|_{L_\mu^2} + \| \partial_x \partial_y^2 \tilde{h} \|_{L_\mu^2} \right) \\ & \leq CD(t)^{\frac{1}{2}} E(t), \\ D_2 & \leq \left(\| \partial_x u \|_{L^\infty} + \| \partial_x \tilde{h} \|_{L^\infty} \right) \left(\| \partial_y^2 u \|_{H_\mu^{1,0}}^2 + \| \partial_y^2 \tilde{h} \|_{H_\mu^{1,0}}^2 \right) + \left(\| \partial_y^2 u \|_{L_x^2 L_y^\infty} + \| \partial_y^2 \tilde{h} \|_{L_x^2 L_y^\infty} \right) \\ & \quad \cdot \left(\| \partial_x^2 u \|_{L_x^\infty L_{y,\mu}^2} + \| \partial_x^2 \tilde{h} \|_{L_x^\infty L_{y,\mu}^2} \right) \left(\| \partial_x \partial_y^2 u \|_{L_\mu^2} + \| \partial_x \partial_y^2 \tilde{h} \|_{L_\mu^2} \right) \\ & \leq CD(t)^{\frac{1}{2}} E(t) + CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}}, \end{aligned}$$

and

$$\begin{aligned} E_2 &\leq C \left(\|\partial_y \partial_x \tilde{h}\|_{H_\mu^{1,0}} \|\partial_y u\|_{H_\mu^{1,1}} + \|\partial_x \tilde{h}\|_{H_\mu^{1,0}} \|\partial_y^2 u\|_{H_\mu^{1,1}} + \|\partial_y \partial_x u\|_{H_\mu^{1,0}} \|\partial_y \tilde{h}\|_{H_\mu^{1,1}} \right. \\ &\quad \left. + \|\partial_x u\|_{H_\mu^{1,0}} \|\partial_y^2 \tilde{h}\|_{H_\mu^{1,1}} \right) \|\partial_y^2 \tilde{h}\|_{H_\mu^{1,0}}^2 \\ &\leq CD(t)^{\frac{1}{2}} E(t). \end{aligned}$$

This shows that

$$\begin{aligned} &\frac{d}{dt} \|(\partial_y^2 \tilde{h}, \partial_y^2 u)\|_{H_\mu^{1,0}}^2 + \|(\partial_y^3 u, \partial_y^3 \tilde{h})\|_{H_\mu^{1,0}}^2 \\ &\leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t). \end{aligned} \tag{5.4}$$

Then the proposition follows from (5.2)-(5.4). □

6 Proof of Theorem 1.1

We first introduce the following Poincaré type inequality.

Lemma 6.1. *Let $\alpha \in (0, 1]$, $\mu_\alpha = e^{\frac{1+y^2}{8\langle t \rangle} \alpha}$. Then there holds*

$$\|\partial_y f\|_{L_{y,\mu_\alpha}^2}^2 \geq \frac{\alpha}{2\langle t \rangle} \|f\|_{L_{y,\mu_\alpha}^2}^2.$$

Proof. A direct calculation gives

$$\begin{aligned} \|\partial_y f\|_{L_{y,\mu_\alpha}^2}^2 &= \int_{\mathbf{R}_+} (\partial_y(\mu_\alpha f) + \partial_y \mu_\alpha f)^2 dy - 4 \int_{\mathbf{R}_+} \partial_y(\mu_\alpha f) \partial_y \mu_\alpha f dy \\ &\geq -4 \int_{\mathbf{R}_+} \frac{\alpha y}{4\langle t \rangle} \partial_y(\mu_\alpha f) \mu_\alpha f dy = -2 \int_{\mathbf{R}_+} \frac{\alpha y}{4\langle t \rangle} \partial_y((\mu_\alpha f)^2) dy \\ &= \frac{\alpha}{2\langle t \rangle} \int_{\mathbf{R}_+} (\mu_\alpha f)^2 dy = \frac{\alpha}{2\langle t \rangle} \|f\|_{L_{y,\mu_\alpha}^2}^2. \end{aligned}$$

Thus, we complete the proof. □

Now we prove Theorem 1.1.

Proof. As in [11], the approximate solution can be constructed by adding the viscosity term $\kappa \partial_x^2 u, \nu \partial_x^2 \tilde{h}$ to the system (1.2). Thus, we only present the uniform estimates of smooth solution. With the uniform estimates, the existence and uniqueness of the solution can be obtained by showing that the approximate sequence is a Cauchy sequence in lower order Sobolev spaces.

Thanks to the initial condition and Lemma 4.3, we have

$$\begin{aligned} \|u_0\|_{H_\mu^{2,1}}^2 + \|\tilde{h}_0\|_{H_\mu^{2,1}}^2 &\leq \varepsilon^2, \\ \|u_\beta(0)\|_{H_\mu^{3,0}}^2 + \|\tilde{h}_\beta(0)\|_{H_\mu^{3,0}}^2 &\leq 4(\|u_0\|_{H_\mu^{3,0}}^2 + \|\tilde{h}_0\|_{H_\mu^{3,0}}^2) \leq 4\varepsilon^2. \end{aligned}$$

So, $E(0) \leq 5\varepsilon^2$. Let $M(t) = \langle t \rangle^{\frac{1-\delta}{2}} E(t)$, then $M(0) \leq 5\varepsilon^2$.

The uniform estimate is based on a bootstrap argument. Let us first assume that $[0, T^*)$ is the maximal time interval so that

$$M(t) \leq C_1 \varepsilon^2, \tag{6.1}$$

where $C_1 > 0$ is a fixed constant. Let us also assume $T^* < \varepsilon^{-2}$.

Thanks to $E(t) \leq C_1 \varepsilon^2$, it follows from Proposition 4.1 and Proposition 5.1 that

$$\frac{d}{dt} E(t) + D(t) \leq CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + C\langle t \rangle^{\frac{1}{4}} D(t)^{\frac{1}{2}} E(t) + C\langle t \rangle^{\frac{1}{2}} E(t)^{\frac{3}{2}} D(t)^{\frac{1}{2}}.$$

Let $\delta \in (0, 1)$ be determined later. Then we have

$$\frac{d}{dt} E(t) + \left(1 - \frac{1}{2}\delta\right) D(t) \leq C\delta^{-1} E(t)^{\frac{5}{3}} + C\delta^{-1} \langle t \rangle^{\frac{1}{2}} E(t)^2 + C\delta^{-1} \langle t \rangle E(t)^3.$$

Thanks to Lemma 6.1 with $\alpha = 1$, we get

$$\frac{d}{dt} E(t) + \frac{1-\delta}{2\langle t \rangle} E(t) + \frac{\delta}{2} D(t) \leq C\delta^{-1} E(t)^{\frac{5}{3}} + C\delta^{-1} \langle t \rangle^{\frac{1}{2}} E(t)^2 + C\delta^{-1} \langle t \rangle E(t)^3,$$

which gives

$$\frac{d}{dt} M(t) \leq C\delta^{-1} \langle t \rangle^{-\frac{1-\delta}{3}} M(t)^{\frac{5}{3}} + C\delta^{-1} \langle t \rangle^{\frac{\delta}{2}} M(t)^2 + C\delta^{-1} \langle t \rangle^\delta M(t)^3.$$

Thus, for given $\eta \in (0, 1)$, there exists $\delta > 0$ so that for $t \leq \varepsilon^{-2+\eta}$,

$$M(t) \leq C_\delta M(0) \leq C_\delta \varepsilon^2.$$

Taking $C_1 = 2C_\delta$, the theorem follows by a bootstrap argument. □

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