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Multiple Axially Asymmetric Solutions to a Mean Field Equation on $\mathbb{S}^2$

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Abstract. We study the following mean field equation

$$\Delta_g u + \rho\left(\frac{e^u}{\int_{\mathbb{S}^2} e^u d\mu} - \frac{1}{4\pi}\right) = 0 \quad \text{in } \mathbb{S}^2,$$

where ρ is a real parameter. We obtain the existence of multiple axially asymmetric solutions bifurcating from $u = 0$ at the values $\rho = 4n(n+1)\pi$ for any odd integer $n \geq 3$.

Key Words: Mean field equation, axially asymmetric solutions, bifurcation.

AMS Subject Classifications: 35B32, 35J61, 58J55

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1 Introduction

In this paper, we consider the mean field equation on the unit sphere

$$\Delta_g u + \rho\left(\frac{e^u}{\int_{\mathbb{S}^2} e^u d\mu} - \frac{1}{4\pi}\right) = 0 \quad \text{in } \mathbb{S}^2, \tag{1.1}$$

where ρ is a real parameter, Δ_g stands for Laplace-Beltrami operator on $\mathbb{S}^2$ associated to the metric g inherited from the ambient Euclidean metric and $d\mu$ is the volume form with respect to g. Since the above equation is invariant by adding a constant to a solution, we introduce

$$\mathcal{H} = \left\{ u \in H^2(\mathbb{S}^2) \Big| \int_{\mathbb{S}^2} u d\mu = 0 \right\},$$

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where $H^2(\mathbb{S}^2)$ is the classical sobolev space. Note that $\mathcal{H}$, equipped with the $H^2(\mathbb{S}^2)$ norm, is a Hilbert space. We write $\mathbb{S}^2$ in the following coordinate system

$$\mathbb{S}^2 = \left\{ \left(\sqrt{1-z^2}\cos\theta,\ \sqrt{1-z^2}\sin\theta,\ z \right) \Big|\ z \in [-1,1],\ \theta \in [0,2\pi] \right\}.$$

Note that $\int_{\mathbb{S}^2} d\mu = 4\pi$. Clearly equation (1.1) admits solution $u \equiv 0$, so in what follows mentioned existence of solutions of (1.1) means existence of non-trivial solution. The corresponding energy functional of (1.1) is

$$J_\rho(u) = \frac{1}{2}\int_{\mathbb{S}^2} |\nabla_g u|^2 d\mu - \rho \log\left(\int_{\mathbb{S}^2} e^u d\mu \right). \tag{1.2}$$

The study of existence of solution for mean field equations possess a long history and huge literature. This kind equations arise from Onsager's vortex model for turbulent Euler flows, see [20]. They also arise from the Chern-Simons-Higgs model vortex when some parameter tends to zero, we refer the reader to [3,25–27].

For $\rho < 8\pi$, by the Moser-Truding inequality, J_ρ is bounded below and coercive, so the proof of the existence of a minimizer of J_ρ is standard. For $\rho = 8\pi$, the existence of a minimizer of (1.2) had been proved in [21] and [7]. For $\rho > 8\pi$, J_ρ is not bounded below. Topological degree theory plays an important role in the solvability of (1.1). Starting with the work of Li in [16], one knows that the solutions of (1.1) are uniformly bounded on any compact subset of $\cup_{m=0}^{\infty}(8m\pi, 8(m+1)\pi)$, and the Leray-Schauder topological degree $d_\rho = 1$ for $\rho < 8\pi$. Due to the result of Li and the homotopy invariance of the degree, it is readily checked that d_ρ is constant in each interval $\rho \in (8m\pi, 8(m+1)\pi)$. Further Lin in [17] proved that $d_\rho = -1$ for $8\pi < \rho < 16\pi$, and $d_\rho = 0$ for $16\pi < \rho < 24\pi$. Subsequently Chen and Lin in [4] obtained apriori bound for a sequence ρ_n with $\rho = \rho_n$. Using this apriori bound, they were able to calculate the degree in [5] $d_\rho = 0$ for $\rho \in (8m\pi, 8(m+1)\pi), m \in \mathbb{N}$ with $m \geq 2$. By a more precise topological argument, the author of [8] proved that in this case (1.1) admits a solution for any $\rho \in \mathbb{R}\backslash 8\pi\mathbb{N}$. Dolbeault, Esteban and Tarantello in [9] proved that for all $k \geq 2$ and $\rho > 4k(k+1)\pi$ (so $\rho > 24\pi$), (1.1) admits at least $2(k-2)+1$ distinct axially symmetric solutions by using bifurcation method. Indeed, they proved that for any $k \geq 2$ there are two continuous unbounded half-branches of solutions of (1.1) bifurcating from the trivial solution at points $\rho = 4k(k+1)\pi$. Note that blow-up solutions could appear only when $\rho \to 8m\pi$, $m \in \mathbb{N}$. Lin in [17] establish the existence of the blow-up solutions to (1.1) when ρ approaches 16π from above. Recently Gui and Hu [13] proved the existence of a family of blow-up solutions for ρ approaches 32π by using Lyapunov reduction method. As far as uniqueness is concerned, Lin in [18] showed that the solution to (1.1) is unique for $0 < \rho < 8\pi$, namely (1.1) only admits the trivial solution. Lin in [19] showed that the axially symmetric solution to (1.1) is unique for $8\pi < \rho \leq 16\pi$, namely $u \equiv 0$ is the only axially symmetric solution of (1.1). By developing a "sphere covering inequality", Gui and Moradifam in [14] extend the uniqueness result to a broader parameter range $\rho \in (0,8\pi) \cup (8\pi,16\pi)$ for any solutions of (1.1). By applying the "sphere covering inequality", Shi, Sun, Tian and Wei [24] proved that even solution to (1.1) is unique for $\rho = 24\pi$. More results under various conditions are also obtained in [12].

In this paper, for Eq. (1.1), we will generalize the axially symmetric solutions case in [9] to axially asymmetric solutions case. Precisely, we will use the bifurcation method to obtain the multiplicity result for axially asymmetric solutions, which are bifurcating from some points ρ with the form of $8\pi\mathbb{N}$. However, the applicability is not all trivial. The main difficulty is that the kernel of the nature associated operator is not one dimensional. In order to overcome this obstacle, we follow the method from [2] by searching for spaces with some symmetry. Multiple two-dimensional solutions of mean field equation on flat tori bifurcating from trivial solution can be seen in [10].

An important role in the procedure is played by the associated Legendre polynomial $P_n^m(z)$ which is defined by

$$P_n^m(z) = \frac{(-1)^m}{2^n n!}(1-z^2)^{\frac{m}{2}} \frac{d^{n+m}}{dz^{n+m}}(z^2-1)^n, \tag{1.3}$$

where $m, n \in \mathbb{N}$, $m \le n$. It is known that $P_n^m(z)$ satisfies the equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left(n(n+1) - \frac{m^2}{1-z^2}\right) y = 0. \tag{1.4}$$

We note that $P_n^m(z)$ behaves like $(1-z^2)^{\frac{m}{2}}$ near $z = \pm 1$ and has $n-m$ simple zeros in $(-1,1)$ for $0 \le m \le n$, $m, n \in \mathbb{N}$.

Our main results in this paper are the existence of multiple bifurcation curves and their local convexities near bifurcation points as follows.

Theorem 1.1. *Let $3 \le n \in \mathbb{N}$ be odd and $\rho_n = 4n(n+1)\pi$. The points $(\rho_n, 0)$ are axially asymmetric bifurcation points for the curve of solutions $(\rho, 0)$. In particular, for any odd integer $m \in (\frac{n}{2}, n]$ there exists $\varepsilon_0 > 0$, and for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, a C^1-family of solutions $(\rho_n(\varepsilon), u_\varepsilon)$ of* (1.1), *which satisfy*

$$\begin{cases} \rho_n(0) = \rho_n, \\ u_\varepsilon(\theta, z) = \varepsilon P_n^m(z)\cos(m\theta) + \varepsilon Z_\varepsilon(\theta, z), \end{cases}$$

where $Z_\varepsilon(\theta, z)$ is uniformly bounded in $L^\infty(\mathbb{S}^2)$ and $Z_0 = 0$. Moreover the bifurcation is global and the Rabinowitz alternative holds true, i.e., a global continuum of solutions to (1.1) *either goes to infinity or meets the trivial solution curve $(\rho, 0)$ at $\rho = \rho_k$, $k \ne n$.*

The following Legendre functions are solutions to (1.4) when $m, n \in \mathbb{N}$, $m > n$ and play a similar role as the Legendre polynomials (1.3)

$$\tilde{P}_n^m(z) = \left(\frac{1-z}{1+z}\right)^{\frac{m}{2}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{k+n}{k}}{\binom{k+m}{k}} \left(\frac{1-z}{2}\right)^k, \quad \forall n, m \in \mathbb{N}.$$

We note that $\tilde{P}_n^m(z)$ behaves like $\left(\frac{1-z}{1+z}\right)^{\frac{m}{2}}$ near $z = \pm 1$ and is singular at $z = -1$ for $m > n, m, n \in \mathbb{N}$. It has no zero in $(-1, 1)$.

Theorem 1.2. *The parameter function $\rho_n(\varepsilon)$ in Theorem 1.1 satisfies $\rho_n'(0) = 0$ and*

$$\begin{aligned}\rho_n''(0) = &- b_{m,n}\Bigg\{ \left(\int_{-1}^{1} (P_n^m(z))^4 \, dz - \left(\int_{-1}^{1} (P_n^m(z))^2 \, dz \right)^2 \right) \\ &+ n(n+1) \Bigg[2 \int_{-1}^{1} (P_n^m(z))^2 \, P_n^0(z) \int_{-1}^{z} \frac{1}{(1-y^2)(P_n^0(y))^2} \\ &\times \int_{y}^{1} P_n^0(x) \left((P_n^m(x))^2 - \frac{1}{2} \int_{-1}^{1} (P_n^m(\tau))^2 d\tau \right) dxdydz \\ &+ \int_{-1}^{1} (P_n^m(z))^2 \, \tilde{P}_n^{2m}(z) \int_{-1}^{z} \frac{1}{(1-y^2)(\tilde{P}_n^{2m}(y))^2} \int_{y}^{1} \tilde{P}_n^{2m}(x)(P_n^m(x))^2 dxdydz \Bigg] \Bigg\}, \end{aligned} \tag{1.5}$$

where

$$b_{m,n} = \frac{n(n+1)}{4} \left(\frac{(2n+1)(n-m)!}{(n+m)!} \right)^2 > 0.$$

We note that here the inner improper integrals with respect to y is well defined at $y = -1$ due to the behavior of $P_n^m(y)$ and $\tilde{P}_n^m(y)$ at $y = \pm 1$ and

$$\int_{-1}^{1} (P_n^m(z))^2 \, P_n^0(z) dz = 0.$$

Also the integrands of the inner integrals with respect to z are continuously defined even though the singular integrals with respect to y are not proper at the finite number of simple zeros of $P_n^0(y)$ in $(-1, 1]$.

Remark 1.1. We give numerical results of the sign of $\rho_n''(0)$ for some small n at the end of this paper.

2 Preliminaries

This section contains several properties which will be used to prove Theorems 1.1 and 1.2.

Proposition 2.1. *The following eigenvalue problem*

$$-\Delta_g u = \lambda u \quad in \ \mathbb{S}^2,$$

has a nontrivial bounded solution if and only if $\lambda = \lambda_n := n(n+1)$. Each eigenvalue λ_n has multiplicity $2n+1$ and a basis for its eigenspace is given by

$$\left\{ P_n^0(z), P_n^m(z) \cos m\theta, P_n^m(z) \sin m\theta \right\}_{m=1,\cdots,n}.$$

Let $F(t,x)$ be an operator mapping from $\mathbb{R}\times X$ to Y. Denote $\partial_x F$ and $\partial_t F$ as the Fréchet partial derivatives of F with respect to x and t respectively.

Proposition 2.2 ([6]). *Let X,Y be Banach spaces, $V\subset X$ a neighborhood of 0 and $F:\mathbb{R}\times V\to Y$ a map with the following properties*

(1) $F(t,0)=0$ for any $t\in\mathbb{R}$,

(2) $\partial_t F$,$\partial_x F$ and $\partial^2_{t,x}F$ exist and are continuous,

(3) $\ker(\partial_x F(t^*,0))=span\{w_0\}$ and $\mathcal{R}(\partial_x F(t^*,0))^\perp$ has dimension 1,

(4) $\partial^2_{t,x}F(t^*,0)w_0\notin\mathcal{R}(\partial_x F(t^*,0))$.

If Z is any complement of $\ker(\partial_x F(t^*,0))$ in X, then there exists $\varepsilon_0>0$, a neighborhood $U\subset\mathbb{R}\times X$ of $(t^*,0)$, and continuously differentiable maps $\eta:(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}$ and $z:(-\varepsilon_0,\varepsilon_0)\to Z$ such that

$$\begin{cases}\eta(0)=t^*,\\ z(0)=0,\\ F^{-1}(0)\cap U\setminus(\mathbb{R}\times\{0\})=\{(\eta(\varepsilon),\varepsilon w_0+\varepsilon z(\varepsilon))|\varepsilon\in(-\varepsilon_0,\varepsilon_0)\setminus\{0\}\}.\end{cases}$$

Proposition 2.3 ([15]). *Assume all the hypotheses of Proposition 2.2 are satisfied. Select $\psi\neq 0\in Y^*$, where Y^* is the dual space of Y, such that $\mathcal{R}(\partial_x F(t^*,0))=\{y\in Y|\langle\psi,y\rangle=0\}$, then the derivative $\eta'(0)$ of $\eta(\varepsilon)$ at $\varepsilon=0$ is given by*

$$\eta'(0)=-\frac{\langle\partial^2_{x,x}F(t^*,0)[w_0,w_0],\psi\rangle}{2\|w_0\|\langle\partial^2_{t,x}F(t^*,0)w_0,\psi\rangle}.$$

Moreover, if $\eta'(0)=0$ and F is of class C^3, then we have

$$\begin{aligned}&\eta''(0)\\ =&-\frac{\langle\partial^3_{x,x,x}F(t^*,0)[w_0]^3-3\partial^2_{x,x}F(t^*,0)\left[w_0,(\partial_x F(t^*,0))^{-1}(I-Q)\partial^2_{x,x}F(t^*,0)[w_0]^2\right],\psi\rangle}{3\|w_0\|^2\langle\partial^2_{t,x}F(t^*,0)w_0,\psi\rangle},\end{aligned}$$

where $Q:y\to\frac{\langle y,\psi\rangle}{\|\psi\|^2}\psi$ is the projection from Y to $\mathcal{R}(\partial_x F(t^*,0))^\perp$ and $(\partial_x F(t^*,0))^{-1}:\mathcal{R}(\partial_x F(t^*,0))\to\ker(\partial_x F(t^*,0))^\perp$ is the inverse of $\partial_x F(t^*,0)$ restricted to the complementary of its kernel.

3 Axially asymmetric solutions

In this section we first apply Proposition 2.2 to prove Theorem 1.1. Define an operator $T:\mathbb{R}\times\mathcal{H}\to L^2(\mathbb{S}^2)$ as

$$T:(\rho,u)\to\Delta_g u+\rho\left(\frac{e^u}{\int_{\mathbb{S}^2}e^u}-\frac{1}{4\pi}\right).$$

Direct computation shows that

$$\partial_u T(\rho,0)\phi = \Delta_g \phi + \frac{\rho}{4\pi}\phi.$$

We set

$$\mathcal{L} = \left\{ u \in L^2(\mathbb{S}^2) \Big| \int_{\mathbb{S}^2} u d\mu = 0 \right\}.$$

Clearly T maps $T : \mathbb{R} \times \mathcal{H}$ into $\mathcal{L}$.

Lemma 3.1. *Assume $\rho = \rho_n = 4n(n+1)\pi$, $n \in \mathbb{N}$, then the dimension of the space of bounded solutions to $\partial_u T(\rho,0)\phi = 0$ is $2n+1$, and the kernel is*

$$\{\phi(\theta,z)\} = span \left\{ P_n^0(z), P_n^m(z)\cos(m\theta),\ P_n^m(z)\sin(m\theta) \right\}_{m=1,\cdots,n},$$

where $P_n^m(z)$ is defined by (1.3).

The proof of this lemma can be directly derived from Proposition 2.1, we omit it.

The multiplicity in the above lemma implies that we need to select a subspace of $\mathcal{H}$ such that the kernel of T restricted on it is one-dimensional. To this end, let us introduce the following three isometries

$$\sigma:\ z \to -z, \quad t_m:\ \theta \to \theta + \frac{2\pi}{m}, \quad s_m:\ \theta \to -\theta + \frac{2\pi}{m},$$

where $m > 1$ is an odd integer. We also introduce the following spaces

$$\begin{aligned} \mathcal{X}_m &:= \{\phi \in \mathcal{H} | \phi \circ \sigma = \phi,\ \phi \circ t_m = \phi,\ \phi \circ s_m = \phi\}, \\ \mathcal{Y}_m &:= \{\psi \in \mathcal{L} | \psi \circ \sigma = \psi,\ \psi \circ t_m = \psi,\ \psi \circ s_m = \psi\}. \end{aligned}$$

Lemma 3.2. *The restriction $T_m := T|_{\mathbb{R}\times\mathcal{X}_m}$ maps its domain into $\mathcal{Y}_m$. Moreover, if n is odd and the odd integer $m \in (\frac{n}{2}, n]$, then* $\dim\{\ker(\partial_u T_m(\rho_n,0))\} = 1$ *and the basis is*

$$\{P_n^m(z)\cos(m\theta)\}.$$

Proof. First, we need to prove that the range of operator T_m is $\mathcal{Y}_m$. Laplace-Beltrami operator can be written as

$$\Delta_g u(\theta,z) = (1-z^2)\frac{\partial^2 u}{\partial z^2} - 2z\frac{\partial u}{\partial z} + \frac{1}{1-z^2}\frac{\partial^2 u}{\partial \theta^2}.$$

On one hand

$$\begin{aligned} T_m(\rho,\phi\circ\sigma) =& (1-z^2)\frac{\partial^2(\phi\circ\sigma)}{\partial(\sigma z)^2} \\ &+ 2z\frac{\partial(\phi\circ\sigma)}{\partial \sigma z} + \frac{1}{1-z^2}\frac{\partial^2(\phi\circ\sigma)}{\partial\theta^2} + \rho\left(\frac{e^{\phi\circ\sigma}}{\int_{\mathbb{S}^2} e^{\phi\circ\sigma}} - \frac{1}{4\pi}\right). \end{aligned}$$

On the other hand

$$\begin{aligned}T_m(\rho,\phi)\circ\sigma =&(1-(\sigma z)^2)\frac{\partial^2(\phi\circ\sigma)}{\partial(\sigma z)^2}-2(\sigma z)\frac{\partial(\phi\circ\sigma)}{\partial\sigma z}\\&+\frac{1}{1-(\sigma z)^2}\frac{\partial^2(\phi\circ\sigma)}{\partial\theta^2}+\rho\left(\frac{e^{\phi\circ\sigma}}{\int_{\mathbb{S}^2}e^{\phi\circ\sigma}}-\frac{1}{4\pi}\right)\\=&(1-z^2)\frac{\partial^2(\phi\circ\sigma)}{\partial(\sigma z)^2}+2z\frac{\partial(\phi\circ\sigma)}{\partial\sigma z}+\frac{1}{1-z^2}\frac{\partial^2(\phi\circ\sigma)}{\partial\theta^2}\\&+\rho\left(\frac{e^{\phi\circ\sigma}}{\int_{\mathbb{S}^2}e^{\phi\circ\sigma}}-\frac{1}{4\pi}\right),\end{aligned}$$

which yields $T_m(\rho,u\circ\sigma)=(T_m(\rho,u))\circ\sigma$. Similarly, we may prove $T_m(\rho,u\circ t_m)=(T_m(\rho,u))\circ t_m$ and $T_m(\rho,u\circ s_m)=(T_m(\rho,u))\circ s_m$. Hence we know that $T_m(\rho,\cdot)$ maps $\mathcal{X}_m$ to $\mathcal{Y}_m$.

Next we prove $\ker(\partial_u T_m(\rho_n,0))=\mathrm{span}\{P_n^m(z)\cos(m\theta)\}$.

We set

$$\phi=A_0P_n^0(z)+\sum_{l=0}^{n}P_n^l(z)[A_l\cos(l\theta)+B_l\sin(l\theta)].$$

Due to $\phi\circ t_m=\phi$, for any $l\in\{1,\cdots,n\}\backslash\{m\}$, one has

$$\begin{cases}A_l=A_l\cos\left(\frac{2l}{m}\pi\right)+B_l\sin\left(\frac{2l}{m}\pi\right),\\ B_l=-A_l\sin\left(\frac{2l}{m}\pi\right)+B_l\cos\left(\frac{2l}{m}\pi\right).\end{cases}$$

We have $A_l\neq 0$ or $B_l\neq 0$ if and only if $\frac{2l}{m}$ is an even integer. Since $0<\frac{2l}{m}\leq\frac{2n}{m}<4$ and $l\neq m$, so it is impossible for $\frac{2l}{m}$ equalling to some even integer, which shows that $A_l=B_l=0$. So

$$\phi=A_0P_n^0(z)+P_n^m(z)[A_m\cos(m\theta)+B_m\sin(m\theta)].$$

By the property $\phi\circ\sigma=\phi$, we deduce $A_0=0$. Hence

$$\phi=P_n^m(z)\left(A_m\cos(m\theta)+B_m\sin(m\theta)\right). \tag{3.1}$$

By the property of $\phi\circ s_m=\phi$, we get

$$\begin{aligned}&P_n^m(z)\left(A_m\cos(m\theta)+B_m\sin(m\theta)\right)=\phi\\=&\phi\circ s_m=P_n^m(z)\left(A_m\cos\left(m\Big(-\theta+\frac{2\pi}{m}\Big)\right)+B_m\sin\left(m\Big(-\theta+\frac{2\pi}{m}\Big)\right)\right)\\=&P_n^m(z)\left(A_m\cos(m\theta)-B_m\sin(m\theta)\right),\end{aligned}$$

which implies $B_m=0$. Now (3.1) gives the desired result. □

Remark 3.1. For any $\psi \in [0,2\pi)$ one can construct similar subspaces $\mathcal{X}_{m,\psi} \subset \mathcal{H}$ and $\mathcal{Y}_{m,\psi} \subset \mathcal{L}$ such that $T(\mathbb{R}\times\mathcal{X}_{m,\psi}) \subset \mathcal{Y}_{m,\psi}$ and the kernel of its derivative in $(\rho_n,0)$ is generated by $P_n^m(z)\cos(m\theta+\psi)$. To prove this fact, just argue as in Lemma 3.2 replacing the reflection s_m with $s_{m,\psi}:\theta\to-\theta+\frac{2\pi-2\psi}{m}$. In particular, for $\psi=\frac{\pi}{2}$, we get $-P_n^m(z)\sin(m\theta)$. Notice that, in general

$$P_n^m(z)\cos(m\theta+\psi)=\cos(\psi)P_n^m(z)\cos(m\theta)-\sin(\psi)P_n^m(z)\sin(m\theta)$$

is a combination of the two generators of its kernel.

Of course, since Eq. (1.1) is rotationally invariant, in general this argument does not give rise to new solutions different from the previous ones.

Lemma 3.3. *The range of the operator $\partial_u T_m(\rho_n,0)$ has co-dimension one and is given by*

$$\mathcal{R}(\partial_u T_m(\rho_n,0))=\left\{\phi\in L^2(\mathbb{S}^2)\,\Big|\,\int_{\mathbb{S}^2}\phi(\theta,z)P_n^m(z)\cos(m\theta)d\theta dz=0\right\}. \tag{3.2}$$

Proof. By the definition of the operator T and the well-known spectral properties of $-\Delta_g$ on $\mathbb{S}^2$, the range of $\partial_u T_m(\rho_n,0)$ coincides with the orthogonal of its kernel. This and the result of Lemma 3.2 yield the desired results of this lemma. $\square$

Lemma 3.4. $\partial^2_{\rho,u}T_m(\rho_n,0)\{P_n^m(z)\cos(m\theta)\}\notin\mathcal{R}(\partial_u T_m(\rho_n,0))$.

Proof. Differentiating $\partial_u T_m$ with respect to ρ on the point $(\rho_n,0)$, we get

$$\partial^2_{\rho,u}T_m(\rho_n,0)\phi=\frac{\phi}{4\pi}.$$

Therefore, owing to

$$\frac{1}{4\pi}\int_{\mathbb{S}^2}\left(P_n^m(z)\cos(m\theta)\right)^2\neq 0,$$

we know that

$$\partial^2_{\rho,u}T(\rho_n,0)\,[P_n^m(z)\cos(m\theta)]=\frac{P_n^m(z)\cos(m\theta)}{4\pi}\notin\mathcal{R}(\partial_u T_m(\rho_n,0)),$$

where we used the results in Lemmas 3.2 and 3.3. $\square$

Proof of Theorem 1.1. We apply Proposition 2.2 with $T_m:\mathbb{R}\times\mathcal{X}_m\to\mathcal{Y}_m$. Then the existence of a local branch follows from Lemmas 3.2-3.4, namely there exists a branch of non-trivial solutions $(\rho_n(\varepsilon),u_\varepsilon)$, where u_ε satisfies

$$u_\varepsilon(\theta,z)=\varepsilon P_n^m(z)\cos(m\theta)+\varepsilon Z_\varepsilon(\theta,z)$$

with Z_ε satisfying $Z_0=0$. Moreover, since $u_\varepsilon\in\mathcal{X}_m$, then $Z_\varepsilon\in\mathcal{X}_m$, so it satisfies

$$Z_\varepsilon(\theta,-z)=Z_\varepsilon(\theta,z),\quad Z_\varepsilon\Big(\theta+\frac{2\pi}{m},z\Big)=Z_\varepsilon(\theta,z),\quad Z_\varepsilon\Big(-\theta+\frac{2\pi}{m},z\Big)=Z_\varepsilon(\theta,z).$$

In order to show that the bifurcation is global we use a degree argument. We introduce operators F and $\mathcal{G}$ as follows

$$F:(\rho,u)\to\frac{u}{\rho}-(-\Delta_g)^{-1}\left(\frac{e^u}{\int_{\mathbb{S}^2}e^u}-\frac{1}{4\pi}\right)=:\frac{u}{\rho}-\mathcal{G}(u).$$

Note that $F=-\rho^{-1}(-\Delta_g)^{-1}T$. Then it is immediate to check that Lemma 3.2 holds true for the operator F, namely 0 is the simple eigenvalue of the operator

$$\partial_u F|_{\rho=\rho_n,u=0}=\frac{1}{\rho_n}-\mathcal{G}'(0).$$

Note that the operator $\mathcal{G}$ is a compact operator from $\mathcal{H}$ to itself. Hence classical results in bifurcation theory (see [22] or [23]) ensure the existence of a global continuum of solutions to (1.1) satisfying the Rabinowitz alternative, i.e., a global continuum of solutions to (1.1) either goes to infinity or meets the trivial solution curve $(\rho,0)$ at $\rho=\rho_k$, $k\neq n$. We note that it is a challenging task to exclude the second possibility here. $\square$

To prove Theorem 1.2, we need to introduce the following lemma, which is slightly different from Lemma A.3 in [2].

Lemma 3.5. *For any $\zeta\in\mathcal{R}(\partial_u T(\rho_n,0))$, the only solution $\phi\in\ker(\partial_u T(\rho_n,0))^{\perp}$ of*

$$\partial_u T(\rho_n,0)\phi=\Delta\phi+\frac{\rho_n}{4\pi}\phi=\zeta$$

is given by

$$\phi(\theta,z)=\phi_0(z)+\sum_{i=1}^{+\infty}[\phi_i^1(z)\cos(i\theta)+\phi_i^2(z)\sin(i\theta)],$$

where

$$\begin{aligned}
\phi_0(z)&=P_n^0(z)\left(C_0-\int_{-1}^z\frac{1}{(1-y^2)(P_n^0(y))^2}\int_y^1 P_n^0(x)\zeta_0(x)dxdy\right),\\
\phi_i^j(z)&=P_n^i(z)\left(C_i^j-\int_0^z\frac{1}{(1-y^2)(P_n^i(y))^2}\int_y^1 P_n^i(x)\zeta_i^j(x)dxdy\right),\quad i\le n,\\
\phi_i^j(z)&=-\tilde{P}_n^i(z)\int_{-1}^z\frac{1}{(1-y^2)(\tilde{P}_n^i(y))^2}\int_y^1\tilde{P}_n^i(x)\zeta_i^j(x)dxdy,\quad i>n,
\end{aligned}$$

with C_0, C_i uniquely determined by

$$\begin{aligned}
C_0&=\frac{\int_{-1}^1(P_n^0(z))^2\int_{-1}^z\frac{1}{(1-y^2)(P_n^0(y))^2}\int_y^1 P_n^0(x)\zeta_0(x)dxdydz}{\int_{-1}^1(P_n^0(z))^2dz},\\
C_i^j&=\frac{\int_{-1}^1(P_n^i(z))^2\int_0^z\frac{1}{(1-y^2)(P_n^i(y))^2}\int_y^1 P_n^i(x)\zeta_i^j(x)dxdydz}{\int_{-1}^1(P_n^i(z))^2dz},
\end{aligned}$$

and ζ_0, ζ_i defined by the Fourier decomposition in θ

$$\zeta(\theta, z) = \zeta_0(z) + \sum_{i=1}^{+\infty}[\zeta_i^1(z)\cos(i\theta) + \zeta_i^2(z)\sin(i\theta)].$$

We note that all the improper integrals are well defined due to the behavior of $P_n^m(y)$ and $\tilde{P}_n^m(y)$ at $y = \pm 1$ and the simplicity of the zeros of $P_n^m(y)$ in $(-1,1)$ and the oddness of $P_n^0(y)$ and evenness of $P_n^m(y)$ when n, m are odd. See Lemma A.3 of [2] for a detailed explanation.

Proof of Theorem 1.2.

From Proposition 2.3 we have

$$\rho_n'(0) = -\frac{\langle \partial_{u,u}^2 T(\rho_n, 0)[w_0, w_0], \psi\rangle}{2\|w_0\|\langle \partial_{\rho,u}^2 T(\rho_n, 0)w_0, \psi\rangle}.$$

Simple computation shows that

$$\partial_{u,u}^2 T(\rho, 0)[\varphi, \varsigma] = \rho\left(\frac{\varphi\varsigma}{4\pi} - \frac{\int_{\mathbb{S}^2}\varphi\varsigma}{(4\pi)^2}\right).$$

Take $\psi = w_0 = P_n^m(z)\cos(m\theta)$. Hence

$$\begin{aligned}\langle \partial_{u,u}^2 T(\rho_n, 0)[w_0, w_0], \psi\rangle &= \int_{\mathbb{S}^2} w_0\rho_n\left(\frac{w_0^2}{4\pi} - \frac{\int_{\mathbb{S}^2} w_0^2}{(4\pi)^2}\right) \\ =&\frac{\rho_n}{4\pi}\int_{\mathbb{S}^2} w_0^3 - \frac{\rho_n}{(4\pi)^2}\int_{\mathbb{S}^2} w_0\int_{\mathbb{S}^2} w_0^2 = 0,\end{aligned} \tag{3.3}$$

which follows from that

$$\int_0^{2\pi}\cos(m\theta)d\theta = \int_0^{2\pi}\cos^3(m\theta)d\theta = 0.$$

We have

$$\begin{aligned}\|w_0\|^2\langle \partial_{\rho,u}^2 T(\rho_n, 0)w_0, \psi\rangle &= \|w_0\|^2\frac{1}{4\pi}\int_{\mathbb{S}^2} w_0^2 \\ =&\frac{\pi}{4}\left(\int_{-1}^1 (P_n^m(z))^2\right)^2 = \frac{\pi}{4}\left(\frac{2(n+m)!}{(2n+1)(n-m)!}\right)^2.\end{aligned} \tag{3.4}$$

It is not difficult to compute that

$$\begin{aligned}\langle \partial_{u,u,u}^3 T(\rho_n, 0)[w_0, w_0, w_0], \psi\rangle &= \int_{\mathbb{S}^2} w_0\rho_n\left(\frac{w_0^3}{4\pi} - 3\frac{w_0\int_{\mathbb{S}^2} w_0^2}{(4\pi)^2} - \frac{\int_{\mathbb{S}^2} w_0^3}{(4\pi)^2}\right) \\ =&\frac{\rho_n}{4\pi}\int_{\mathbb{S}^2} w_0^4 - \frac{3\rho_n}{(4\pi)^2}\left(\int_{\mathbb{S}^2} w_0^2\right)^2 = \frac{3\rho_n}{16}\left(\int_{-1}^1 (P_n^m(z))^4 - \left(\int_{-1}^1 (P_n^m(z))^2\right)^2\right).\end{aligned} \tag{3.5}$$

From (3.3) we have that

$$Q\partial^2_{u,u}T(\rho_n,0)[w_0,w_0]=0.$$

Hence

$$(\partial_u T(\rho_n,0))^{-1}(I-Q)\partial^2_{u,u}T(\rho_n,0)[w_0,w_0]=(\partial_u T(\rho_n,0))^{-1}\partial^2_{u,u}T(\rho_n,0)[w_0,w_0],$$

and we denote this term as ϕ. Correspondingly we denote

$$\zeta:=\partial^2_{u,u}T(\rho_n,0)[w_0,w_0]=\rho_n\left(\frac{w_0^2}{4\pi}-\frac{\int_{\mathbb{S}^2}w_0^2}{(4\pi)^2}\right),$$

then

$$\begin{aligned}\zeta=&\frac{n(n+1)}{2}\left[\left((P_n^m(z))^2-\frac{1}{2}\int_{-1}^1(P_n^m(z))^2dz\right)+(P_n^m(z))^2\cos(2m\theta)\right]\\ :=&\zeta_0(z)+\zeta_{2m}(z)\cos(2m\theta).\end{aligned}$$

From Lemma 3.5 we have

$$\begin{aligned}\phi=&P_n^0(z)[C_0-\frac{n(n+1)}{2}\int_{-1}^z\frac{1}{(1-y^2)(P_n^0(y))^2}\\ &\times\int_y^1P_n^0(x)\left((P_n^m(x))^2-\frac{1}{2}\int_{-1}^1(P_n^m(\tau))^2d\tau\right)dxdy]\\ &-\frac{n(n+1)}{2}\tilde{P}_n^{2m}(z)\int_{-1}^z\frac{1}{(1-y^2)(\tilde{P}_n^{2m}(y))^2}\int_y^1\tilde{P}_n^{2m}(x)(P_n^m(x))^2dxdy\cos(2m\theta).\end{aligned}$$

So

$$\begin{aligned}&\left\langle\partial^2_{u,u}T(\rho_n,0)\left[w_0,(\partial_uT(\rho_n,0))^{-1}(I-Q)\partial^2_{u,u}T(\rho_n,0)[w_0,w_0]\right],\psi\right\rangle\\ =&\langle\partial^2_{u,u}T(\rho_n,0)[w_0,\phi],\psi\rangle=\int_{\mathbb{S}^2}w_0\rho_n\left(\frac{w_0\phi}{4\pi}-\int_{\mathbb{S}^2}\frac{w_0\phi}{(4\pi)^2}\right)\\ =&n(n+1)\int_{\mathbb{S}^2}w_0^2\phi\\ =&-\frac{n^2(n+1)^2}{2}\int_0^{2\pi}\cos^2(m\theta)d\theta\int_{-1}^1(P_n^m(z))^2P_n^0(z)\int_{-1}^z\frac{1}{(1-y^2)(P_n^0(y))^2}\\ &\quad\times\int_y^1P_n^0(x)\left((P_n^m(x))^2-\frac{1}{2}\int_{-1}^1(P_n^m(\tau))^2d\tau\right)dxdydz\\ &-\frac{n^2(n+1)^2}{2}\int_0^{2\pi}\cos^2(m\theta)\cos(2m\theta)d\theta\int_{-1}^1(P_n^m(z))^2\tilde{P}_n^{2m}(z)\\ &\quad\times\int_{-1}^z\frac{1}{(1-y^2)(\tilde{P}_n^{2m}(y))^2}\int_y^1\tilde{P}_n^{2m}(x)(P_n^m(x))^2dxdydz\end{aligned}$$

$$
\begin{aligned}
=& -\frac{n^2(n+1)^2\pi}{4}\Big[2\int_{-1}^{1}\left(P_n^m(z)\right)^2 P_n^0(z)\int_{-1}^{z}\frac{1}{(1-y^2)(P_n^0(y))^2} \\
&\times\int_y^1 P_n^0(x)\left((P_n^m(x))^2-\frac{1}{2}\int_{-1}^{1}(P_n^m(\tau))^2 d\tau\right)dxdydz \\
&+\int_{-1}^{1}\left(P_n^m(z)\right)^2\tilde{P}_n^{2m}(z)\int_{-1}^{z}\frac{1}{(1-y^2)(\tilde{P}_n^{2m}(y))^2}\int_y^1\tilde{P}_n^{2m}(x)(P_n^m(x))^2 dxdydz\Big], \qquad (3.6)
\end{aligned}
$$

where we used the equality

$$\int_{-1}^{1}\left(P_n^m(z)\right)^2 P_n^0(z)dz=0.$$

From (3.4), (3.5), (3.6), we obtain the desired result (1.5) in Theorem 1.2. ☐

Remark 3.2. For the cases that odd integers m,n with $\frac{n}{2}<m\le n$, we have the following numerical results.

n	$\rho_n''(0)>0$	$\rho_n''(0)<0$
3	$m=3$	
5	$m=5$	$m=3$
7	$m=7$	$m=5$
9	$m=9$	$m=5,7$
11	$m=11$	$m=7,9$
13	$m=9,11,13$	$m=7$
15	$m=9,11,13,15$	
17	$m=15,17$	$m=9,11,13$
19	$m=17,19$	$m=11,13,15$
21	$m=19,21$	$m=11,13,15,17$
23	$m=17,21,23$	$m=13,15,19$
25	$m=15,17,19,21,23,25$	$m=13$
27	$m=17,19,21,23,25,27$	$m=15$
29	$m=23,25,27,29$	$m=15,17,19,21$
31	$m=27,29,31$	$m=17,19,21,23,25$
33	$m=29,31,33$	$m=17,19,21,23,25,27$
35	$m=23,25,27,31,33,35$	$m=19,21,29$
37	$m=23,25,27,29,33,35,37$	$m=19,21,31$
39	$m=23,25,27,29,31,33,35,37,39$	$m=21$
41	$m=23,33,35,37,39,41$	$m=21,25,27,29,31$
43	$m=37,39,41,43$	$m=23,25,27,29,31,33,35$
45	$m=33,39,41,43,45$	$m=23,25,27,29,31,35,37$
47	$m=31,33,35,37,41,43,45,47$	$m=25,27,29,39$
49	$m=29,31,33,35,37,39,43,45,47,49$	$m=25,27,41$

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