Hölder Continuity of Spectral Measures for the Finitely Differentiable Quasi-Periodic Schrödinger Operators

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Abstract. In the present paper, we prove the $\frac{1}{2}$-Hölder continuity of spectral measures for the $C^k$ Schrödinger operators. This result is based on the quantitative almost reducibility and an estimate for the growth of the Schrödinger cocycles in [5].

Key Words: Schrödinger operator, quasi-periodic, almost reducibility, finitely differentiable.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

In this paper, we consider the Schrödinger operators defined on $\ell^2(\mathbb{Z})$

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n,$$

where $V : T^d \to \mathbb{R}$ is the potential, $\theta \in T^d = (\mathbb{R}/\mathbb{Z})^d$ is the phase, and $\alpha \in T^d$ is the frequency.

These operators have been extensively and thoroughly studied for the deep connection with quasi-crystal and quantum Hall effects [11,18]. This paper concerns the regularity of the spectral measure of the quasi-periodic Schrödinger operators. For the analytic potential $V \in C^\omega(T^d, \mathbb{R})$, there is some significant progress [5,21,24]. However for the smooth potential $V \in C^k(T^d, \mathbb{R})$, there is no similar result as far as we know, so we will give a supplementary answer to this situation.

Let us review some results on the Hölder continuity of the integrated density of states (IDS) and the individual spectral measures.

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1.1 Hölder continuity of IDS

Let $\Sigma_{V,\alpha,\beta}$ be the spectrum of $H_{V,\alpha,\beta}$, then $\Sigma_{V,\alpha,\beta} \subset \mathbb{R}$ since $H_{V,\alpha,\beta}$ is the bounded self-adjoint operator in $l^2(\mathbb{Z})$. The spectrum is independent of $\theta$ if $(\alpha, 1)$ is rational independent. For any $f \in l^2(\mathbb{Z})$, the spectral measure $\mu_{V,\alpha,\beta}$ of $H_{V,\alpha,\beta}$ can be defined as

$$\langle (H_{V,\alpha,\beta} - E)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu_{V,\alpha,\beta}(E'), \quad \forall E \in \mathbb{C}\setminus \Sigma_{V,\alpha}. \quad (1.1)$$

Let $\mu_{V,\alpha,\beta} = \mu_{V,\alpha,\beta}^{e^{-1}} + \mu_{V,\alpha,\beta}^{f_{0}}$, where $\{\epsilon_i\}_{i \in \mathbb{Z}}$ is the canonical basis of $l^2(\mathbb{Z})$. Let $N_{V,\alpha}$ be the IDS of $H_{V,\alpha,\beta}$, it is well known that IDS is the average of the spectral measure $\mu_{V,\alpha,\beta}$ with respect to $\theta$, i.e.,

$$N_{V,\alpha}(E) = \int_{T^d} \mu_{V,\alpha,\beta}(-\infty, E]d\theta.$$ 

Hence the regularity of IDS is closely related to that of the spectral measure.

Recall that $\alpha \in \mathbb{T}^d$ is Diophantine if there exist $\gamma > 0$ and $\tau > d - 1$ such that $\alpha \in DC_{\delta}(\gamma, \tau)$, where

$$DC_{\delta}(\gamma, \tau) = \left\{ \alpha : \inf_{j \in \mathbb{Z}} |(n, \alpha) - j| > \frac{\gamma}{|n|^{\tau}}, \forall n \in \mathbb{Z}^d \setminus \{0\} \right\}.$$ 

Let $DC_{\delta} = \cup_{\gamma > 0, \tau > d - 1} DC_{\delta}(\gamma, \tau)$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $p_n/q_n$ be the continued fraction approximants to $\alpha$, then one can define

$$\beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}.$$ 

Given the operator $H_{V,\alpha,\beta}$, one can define the Lyapunov exponent $L(\alpha, S^V_E)$ (see Section 2.1) of the corresponding Schrödinger cocycle $(\alpha, S^V_E \theta)$, where $E \in \mathbb{R}$ and

$$S^V_E \theta = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Hadj Amor [16] proved the $\frac{1}{2}$-Hölder continuity of IDS if $\alpha \in DC_{\delta}$ and $V \in C^\infty(\mathbb{T}^d, \mathbb{R})$ is small, and her approach is based on the almost reducibility scheme developed by Eliasson [13]. Recall that the cocycle $(\alpha, A)$ is reducible if $(\alpha, A)$ can be conjugated to some constant cocycles and the cocycle $(\alpha, A)$ is almost reducible if the closure of its conjugates contains a constant. Avila and Jitomirskaya [4] proved the $\frac{1}{2}$-Hölder continuity of IDS for $\alpha \in DC_1$ and the small analytic potential. Their result was non-perturbative, which means that the smallness is independent of $\alpha$. After that, Avila [2,3] generalized the result for the small analytic potential with $\beta(\alpha) = 0$ if there is $\delta > 0$ such that $L(\alpha, S^V_{E+\epsilon}) = 0$ for $|\epsilon| < \delta$. Note that Leguil-You-Zhao-Zhou [20] showed the same result as well by the global theory of the one-frequency Schrödinger operators [1].
The Hölder continuity of $N_{V,\alpha}(E)$ is equivalent to the Hölder continuity of $L(\alpha, S^V_E)$ according to the famous Thouless formula [6], i.e.,

$$L(\alpha, S^V_E) = \int \ln |E - E'| dN_{V,\alpha}(E').$$

Suppose that $L(\alpha, S^V_E) > 0$. Goldstein and Schlag [14] proved the Hölder continuity of $L(\alpha, S^V_E)$ if $V \in C^\omega(\mathbb{T}, \mathbb{R})$, and $\alpha \in \text{SDC}^1$ by the avalanche principle and sharp large deviation theorem. Later, You and Zhang [22] proved that for analytic potential, the $L(\alpha, S^V_E)$ is Hölder continuous if $\alpha \in \text{DC}_1$ or some weaker Liouvillean $\alpha$ by a refined large deviation theorem.

For the special kinds of potentials, there are some other interesting results. If the potential is a small perturbation of a trigonometric polynomial of degree $d$ and $\alpha \in \text{DC}_1$, Goldstein-Schlag [15] proved that IDS is $\frac{1}{2}\sigma$-Hölder continuous for any $\epsilon > 0$ by assuming $L(\alpha, S^V_E) > 0$. In particular, if the potential $V(x) = 2\lambda \cos(x)$, we get the so called almost Mathieu operators (AMO), i.e.,

$$(H_{\lambda,\alpha}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(\theta + n\alpha)u_n,$$

where $\lambda \in \mathbb{R}$ is the coupling constant. For the large coupling, Bourgain [7] proved that Lyapunov exponent of AMO is $\frac{1}{2} - \epsilon$-Hölder continuous for any $\epsilon > 0$. Avila [2] obtained the exact $\frac{1}{2}$-Hölder continuity of IDS for AMO with $|\lambda| < 1$.

### 1.2 Hölder continuity of the spectral measure

In general, the spectral measure is less regular than IDS. In the $\beta(\alpha) = 0$ regime, Avila and Jitomirskaya [5] proved that if $V \in C^\omega(\mathbb{T}, \mathbb{R})$ is small enough and $\alpha \in \text{DC}_1$, then the spectral measure $\mu_{V,\alpha,\theta}^1$ of one-frequency Schrödinger operators is $\frac{1}{2}$-Hölder continuous for any $f \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$. They also showed $\frac{1}{2}$-Hölder continuity of absolutely continuous spectral measures for the one-frequency Schrödinger operators with $V \in C^\omega(\mathbb{T}, \mathbb{R})$. In the $\beta(\alpha) > 0$ regime, Liu and Yuan [21] extended the results in [5] to that for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ with $\beta(\alpha) < \infty$, whenever the analytic radius of $V$ is sufficiently large.

What can we say about the regularity of the spectral measure of Schrödinger operators with finitely differentiable potential? Recall that Cai-Chavaudret-You-Zhou [10] have shown that if $\alpha \in \text{DC}_d$ and $V \in C^k(\mathbb{T}^d, \mathbb{R})$ is small, then IDS of the Schrödinger operator is $\frac{1}{2}$-Hölder continuous. Recently, Zhao [24] also generalized [5] and [21] to the multi-frequency Schrödinger operators with $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$. This paper is motivated by [10] and [24], as a supplemental answer, we obtain the $\frac{1}{2}$-Hölder continuity of the spectral measure $\mu_{V,\alpha,\theta}^1$ for $V \in C^k(\mathbb{T}^d, \mathbb{R})$.

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\[\text{SDC}(\gamma, \tau) = \left\{ \alpha \in \mathbb{R}^d : \inf_{j \in \mathbb{Z}} |(n, \alpha) - j| > \frac{\gamma}{|n|(\ln((1 + |n|))^{\tau})} \right\},\]

and $\text{SDC} = \cup_{\gamma > 0, \tau > 1} \text{SDC}(\gamma, \tau)$.
Theorem 1.1. Let \( \alpha \in DC_d(\gamma, \tau) \), \( V \in C^k(T^d, \mathbb{R}) \) with \( k \geq 5D \tau \) and \( D \) is a numerical constant. There exists \( \bar{\varepsilon} = \bar{\varepsilon}(\gamma, \tau, k) \) such that if \( \|V\|_k \leq \bar{\varepsilon} \), then for any \( f \in L^2(\mathbb{Z}) \cap \ell^1(\mathbb{Z}) \),

\[
\mu_{V,\alpha,\theta}^f(f) \leq D_0 |J|^\frac{1}{2} \|f\|_{\ell^1}^2,
\]

for all intervals \( J \) and all \( \theta \), where \( D_0 = D_0(V, \alpha) > 0 \).

Remark 1.1. Note that Theorem 1.1 is perturbative, i.e., the smallness \( \bar{\varepsilon} \) depends not only on the potential \( V \), but also on the frequency \( \alpha \). From a counterexample of Bourgain [8], one cannot expect non-perturbative results in multi-frequency case.

2 Preliminaries

For a bounded analytic function \( F \) defined on \( S_r = \{ \theta : \theta = (\theta_1, \cdots, \theta_d) \in C^d, |3\theta_i| < r, \forall i = 1, \cdots, d \} \), let \( \|F\|_r = \sup_{\theta \in S_r} \|F(\theta)\| \) and denote by \( C^\omega(T^d, *) \) the set of these *-value functions ( * will usually denote \( \mathbb{R}, sl(2, \mathbb{R}) \) or \( SL(2, \mathbb{R}) \)). We also denote the set \( C^k(T^d, *) \) to be the space of \( k \) times differentiable with continuous \( k \)-th derivatives functions, endowed with the norm

\[
\|F\|_k := \sup_{k \leq k, \theta \in T^d} \|\partial^k F(\theta)\|.
\]

In particular,

\[
\|F\|_0 := \|F\|_{T^d} = \sup_{\theta \in T^d} \|F(\theta)\|.
\]

For \( \theta \in \mathbb{R} \), we set \( \|\theta\|_T = \inf_{j \in \mathbb{Z}} |\theta - j| \).

2.1 Uniform hyperbolicity

Given \( A \in C^\omega(T^d, SL(2, \mathbb{C})) \) and \( \alpha \in \mathbb{R}^d \) rationally independent, one can define the quasi-periodic (Q-P) cocycle \((\alpha, A)\):

\[
(\alpha, A) : T^d \times \mathbb{C}^2 \to T^d \times \mathbb{C}^2;
(\theta, v) \mapsto (\theta + \alpha, A(\theta) \cdot v).
\]

The iterations of \((\alpha, A)\) are of form \((\alpha, A)^n = (na, A_n)\), where

\[
A_n(\theta) := \begin{cases} 
A(\theta + (n-1)\alpha) \cdots A(\theta + \alpha) A(\theta), & n \geq 0, \\
A^{-1}(\theta + na) A^{-1}(\theta + (n+1)\alpha) \cdots A^{-1}(\theta - \alpha), & n < 0.
\end{cases}
\]

The Lyapunov exponent of the cocycle \((\alpha, A)\) is defined as

\[
L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int_{T^d} \|A_n(\theta)\|d\theta.
\]
We say the cocycle $(\alpha, A)$ is uniformly hyperbolic if for every $\theta \in \mathbb{T}^d$, there exists a continuous splitting $C^2 = E^s(\theta) \oplus E^u(\theta)$ such that for every $n \geq 0$,
\[ \|A_n(\theta)v\| \leq Ce^{-cn}\|v\|, \quad v \in E^s(\theta), \]
\[ \|A_n(\theta)^{-1}v\| \leq Ce^{-cn}\|v\|, \quad v \in E^u(\theta + na), \]
for some constants $C, c > 0$. And the splitting is invariant by the dynamics:
\[ A(\theta)E^s(\theta) = E^s(\theta + \alpha), \quad \forall \theta \in \mathbb{T}^d, \]
\[ A(\theta)E^u(\theta) = E^u(\theta + \alpha), \quad \forall \theta \in \mathbb{T}^d. \]

2.2 Spectral measure and Wely’s $m$ function

Typical examples of $SL(2, \mathbb{R})$ cocycles are the Schrödinger cocycles $(\alpha, S^V_E)$:
\[ A(\theta) = S^V_E(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \quad E \in \mathbb{R}. \]

Those cocycles come from the eigenvalue equation of one dimensional quasi-periodic Schrödinger operators on $\ell^2(\mathbb{Z})$:
\[ (H_{V,\alpha,\beta}u)_n = u_{n+1} + u_{n-1} + V(\theta + na)u_n = Eu_n, \]
and any formal solution $u = (u_n)_{n \in \mathbb{Z}}$ of $H_{V,\alpha,\beta}u = Eu$ satisfies
\[ \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S^V_E(\theta + na) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \forall n \in \mathbb{Z}. \]

For any $f \in \ell^2(\mathbb{Z})$, one can define the spectral measure $\mu_{V,\alpha,\theta}^f$ corresponding to $f$ as in (1.1). Given $E + i\epsilon$ with $E \in \mathbb{R}$ and $\epsilon > 0$, there exists a non-zero solution $u^\pm$ of $H_{V,\alpha,\beta}u^\pm = (E + i\epsilon)u^\pm$ being square-summable at $+\infty$. The Wely’s $m$ function is given by $m^\pm = -\frac{u_n^+}{u_n^-}$.

Let
\[ M(E + i\epsilon) = \int_{\mathbb{R}} \frac{1}{E' - (E + i\epsilon)} d\mu_{V,\alpha,\beta}(E'). \]

From the definition of $M(\cdot)$, it deduces immediately that $M(\cdot)$ is a Herglotz function defined on $\mathbb{H} = \{z : \Im z > 0\}$, and
\[ \Im M(E + i\epsilon) \geq \frac{1}{2\epsilon} \mu_{V,\alpha,\beta}(E - \epsilon, E + \epsilon). \quad (2.1) \]

Recall that in [12], by the usual action of $SL(2, \mathbb{C})$, we denote
\[ m^\pm_\beta := R_{-\frac{\beta}{\pi}} \cdot m^\pm = \frac{m^\pm \cos \beta - \sin \beta}{m^\pm \sin \beta + \cos \beta}, \]
then we have the following lemma:
Lemma 2.1 ([5]). Let $\psi(m^+) := \sup_{\beta} |m^+_{\beta}|$, then $\|M\|_0 \leq \psi(m^+)$. The spectral properties of $H_{V,a,\theta}$ and the dynamics of $(\alpha, S_E^V)$ are closely related by the fact: $E \in \Sigma_{V,a}$ if and only if $(\alpha, S_E^V)$ is not uniformly hyperbolic [19].

2.3 Analytic approximation
Assume $f \in C^k(T^d, sl(2, \mathbb{R}))$, according to [23], there exists a sequence $\{f_j\}_{j \geq 1}$ with $f_j \in C^{\omega}_0(T^d, sl(2, \mathbb{R}))$ and a universal constant $C' > 0$, such that
\[
\|f_j - f\|_k \to 0, \quad j \to +\infty, \quad (2.2a)
\]
\[
|f_j|_1 \leq C' \|f\|_k, \quad (2.2b)
\]
\[
|f_{j+1} - f_j|_1 \leq C'(j)^{-k} \|f\|_k. \quad (2.2c)
\]

2.4 Space decomposition
Given $A \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^d$ and $\eta > 0$, one can obtain the decomposition $\mathcal{B}_k = \mathcal{B}_k^{(nre)}(\eta) \oplus \mathcal{B}_k^{(re)}(\eta)$, where
\[
\mathcal{B}_k = \left\{ f \in C^k(T^d, sl(2, \mathbb{R})) : \|f\|_k < \infty \right\},
\]
and $\mathcal{B}_k^{(nre)}(\eta)$ is the subspace of $\mathcal{B}_k$ such that for any $Y \in \mathcal{B}_k^{(nre)}(\eta)$,
\[
A^{-1}Y(\theta + \alpha)A \in \mathcal{B}_k^{(nre)}(\eta), \quad \|A^{-1}Y(\theta + \alpha)A - Y(\theta)\|_k > \eta \|Y\|_k.
\]

Lemma 2.2 ([10, 17]). Assume that $A \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^d$, $\epsilon' \leq (4\|A\|)^{-4}$, and $\eta \geq 13\|A\|^2\epsilon'^{1/2}$. For any $g \in \mathcal{B}_k$ with $\|g\|_k \leq \epsilon'$, there exist $Y \in \mathcal{B}_k$ and $g^{re} \in \mathcal{B}_k^{(re)}(\eta)$ such that
\[
e^{-Y(\theta + \alpha)}Ae^{\theta}(e)^{Y(\theta)} = Ae^{g^{re}(\theta)},
\]
with $\|Y\|_k \leq \epsilon'^{1/2}$, and $\|g^{re}\|_k \leq 2\epsilon'$.

The continuous version ($C^\omega$ linear systems) and discrete version ($C^\omega$ cocyles) of Lemma 2.2 are shown in [17] and [10] respectively. The proof of Lemma 2.2 only depends on $\mathcal{B}_k$ being a Banach space, thus one can obtain the result by replacing analytic norm with $C^k$ norm in the Appendix of [10].

3 Dynamical estimates of $C^k$ quasi-periodic cocycles
To deal with the almost reducibility of the $C^k$ cocycles, the strategy used here is treating the analytic cocycles firstly and turning the estimates of analytic cocycles into those of finitely differentiable cocyles by analytic approximation.
Consider the following Q-P $SL(2, \mathbb{R})$ cocycle
\[
(\alpha, Ae^{f(\theta)}) : \mathbb{T}^d \times \mathbb{R}^2 \to \mathbb{T}^d \times \mathbb{R}^2;
\]
\[
(\theta, x) \mapsto (\theta + \alpha, Ae^{f(\theta)} \cdot x),
\]
where $\alpha \in DC_d(\gamma, \tau)$, $A \in SL(2, \mathbb{R})$ and $f(\theta) \in C^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ with $r > 0$. Assume $|f|_r \leq \varepsilon$, we are going to show that the perturbation will tend to zero by KAM scheme.

**Proposition 3.1 ([10, 20])**. Let $\alpha \in DC_d(\gamma, \tau)$, $\gamma, r > 0$, $\tau > d - 1$ and $\varepsilon = \frac{1}{10}$. Then for any $r_+ \in (0, r)$, there exist $c = c(\gamma, \tau, d)$ and a numerical constant $D$ such that if
\[
\varepsilon \leq \frac{c}{\| A \|^D (r - r_+)^Dr},
\]
then there exist $B(\theta) \in C^\omega_r(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_+ \in SL(2, \mathbb{R})$ and $f_+(\theta) \in C^\omega_r(T^d, sl(2, \mathbb{R}))$ such that $(\alpha, Ae^{f(\theta)})$ is conjugated to $(\alpha, A_+e^{f_+(\theta)})$ by $B(\theta)$, i.e.,
\[
B(\theta + \alpha)^{-1}Ae^{f(\theta)}B(\theta) = A_+e^{f_+(\theta)}.
\]

More precisely, let $N = \frac{2|\ln r|}{\pi r_+}$ and $\{e^{2\pi i p}, e^{-2\pi i p}\}$ be the two eigenvalues of $A$, we can distinguish between two cases:

**(A) (Non-resonant case).** Assume that
\[
\|2\rho - \langle n, \alpha \rangle\|_\mathbb{T} \geq \varepsilon', \quad \forall n \in \mathbb{Z}^d \quad \text{with} \quad 0 < |n| \leq N,
\]
then we have the estimates:
\[
|f_+(\theta)|_{r_+} \leq 4\varepsilon^{2-2\gamma}, \quad |B(\theta) - Id|_{r_+} \leq \varepsilon^{\frac{1}{2}}, \quad \|A_+ - A\| \leq 2\|A\|\varepsilon.
\]

**(B) (Resonant case).** If there exists $n^* \in \mathbb{Z}^d$ with $0 < |n^*| \leq N$ such that
\[
\|2\rho - \langle n^*, \alpha \rangle\|_\mathbb{T} < \varepsilon',
\]
then we have the estimates:
\[
|B(\theta)|_{r_+} \leq C_1|n^*|^{\frac{1}{2}}\varepsilon^{2|n^*|r_+}, \quad \|B(\theta)\|_0 \leq C_1|n^*|^{\frac{1}{2}}, \quad |f_+(\theta)|_{r_+} \ll \varepsilon^{100},
\]
where $C_1 = 4\|A\|^{\frac{1}{2}}\gamma^{-\frac{1}{2}}$. Moreover, let $A_+ := e^{A''}$ with $A'' \in sl(2, \mathbb{R})$, then $\|A''\| \leq 16\varepsilon^{\gamma'}$.

### 3.1 Real quantitative estimates for $C^k$ Q-P cocycles

We are going to deal with the almost reducibility of the $C^k$ cocycles by analytic approximation. Let $\{f_j\}_{j \geq 1}, f_j \in C^\omega_\frac{1}{2}(T^d, sl(2, \mathbb{R}))$ be the analytic sequence approximating
Let \( f \in C^k(T^d, sl(2, \mathbb{R})) \). We first recall some notations given in [10]. \( D \) is a numerical constant defined by Proposition 3.1, we denote
\[
\epsilon_0'(h, h') := \frac{c}{(2\|A\|)^D} (h - h')^{D\tau},
\]
and define
\[
\epsilon_m := \frac{c}{(2\|A\|)^{Dm^\frac{1}{2}}}.
\]
Then one can check that for any \( k \geq 5D\tau \) and any \( m \geq 10, m \in \mathbb{Z} \),
\[
\frac{c}{(2\|A\|)^{Dm^\frac{1}{2}}} \leq \epsilon_0'( \frac{1}{m'}, \frac{1}{m^\frac{1}{2}} ) .
\]
Denote \( l_j = M^{2^{j-1}}, \forall j \in \mathbb{Z}^+ \), where \( M > \max\{10, \frac{(2\|A\|)^D}{c}\} \) is an integer.

**Theorem 3.1.** Let \( \alpha \in DC_d(\gamma, \tau), A \in SL(2, \mathbb{R}), \sigma = \frac{1}{m}, f(\theta) \in C^k(T^d, sl(2, \mathbb{R})) \) with \( k \geq 5D\tau \). Let \( \{f_j\}_{j \geq 1} \) be the analytic sequence approximating \( f(\theta) \) defined in (2.2). There exists \( \bar{\varepsilon} = \bar{\varepsilon}(\gamma, \tau, \sigma, A) \) such that if \( \|f\| \leq \bar{\varepsilon} \), then the following holds:

**A** There exist \( B_j(\theta) \in C^\omega_{l_j+1}(2T^d, SL(2, \mathbb{R})), A_j \in SL(2, \mathbb{R}) \) and \( f_j(\theta) \in C^\omega_{l_j+1}(T^d, sl(2, \mathbb{R})) \) such that
\[
B_j(\theta + \alpha) A e^{f_j(\theta)} B_j(\theta) = A_j e^{f_j(\theta)},
\]
with following estimates
\[
\|B_j(\theta)\|_0 \leq (l_j |\ln l_j|)^{1+\sigma}, \quad |f_j(\theta)|_{l_j+1} \leq \frac{1}{2} \bar{\varepsilon}_{l_j}^\frac{1}{2},
\]
\[
|B_j(\theta)|_{l_j+1} \leq (l_j |\ln l_j|)^{1+\sigma} \epsilon_{l_j}^{-\frac{1}{2}},
\]
where \( \epsilon \in (\frac{2\pi}{M^{2^{j-1}}}, \frac{1}{M}) \) is a constant.

**B** Moreover, \( \|A_j\| \leq 2\|A\| \) and there exist some unitary matrices \( U_j \in SL(2, \mathbb{C}) \) such that \( A_j \) can be written as
\[
A_j = U_j \left( e^{2\pi i \rho} e^{\frac{c_j}{2\pi i \rho}} \right) U_j^{-1} \quad \text{with} \quad \rho_j \in i\mathbb{R} \cup \mathbb{R} \quad \text{and} \quad c_j \in \mathbb{C}.
\]
Then for any constant \( \kappa \in [1, \frac{M}{57}] \), there exists constant \( C = C(\|A\|) > 0 \) such that
\[
\|B_j(\theta)\|_0^\kappa \cdot |c_j| \leq C.
\]

**Proof.** We will prove Theorem 3.1 by the induction process as in [9] and [10].

**First Step.** Suppose that
\[
\|f\|_k \leq \frac{c}{C'(2\|A\|)^{Dm^\frac{1}{2}}}.
\]
where $C'$ is a universal constant defined in (2.2). Thus we have

$$|f_{l_1}|_{\frac{1}{\tau}} \leq \varepsilon_{l_1} \leq \varepsilon'_0 \left( \frac{1}{l_1}, \frac{1}{l_2} \right).$$

By Proposition 3.1, one can find $B_{l_1}(\theta) \in C^0_{\frac{T}{2}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_1} \in SL(2, \mathbb{R})$ and $f'_{l_1}(\theta) \in C^0_{\frac{T}{2}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$B_{l_1}(\theta + \alpha)^{-1} A e^{f'_{l_1}(\theta)} B_{l_1}(\theta) = A_{l_1} e^{f'_{l_1}(\theta)}.$$

More precisely, let $N_{l_1} = \frac{2|\ln \varepsilon_{l_1}|}{\tau} + \{e^{2\pi i \rho}, e^{-2\pi i \rho}\}$ be two eigenvalues of $A$, we can distinguish two cases:

(Non-resonant case). If the first step is obtained by non-resonant case:

$$\|2\rho - \langle n, \alpha \rangle\|_{\mathbb{T}} \geq \varepsilon_{l_1}^{\tau}, \quad \forall n \in \mathbb{Z}^d \quad \text{with} \quad 0 < |n| \leq N_{l_1},$$

then we have the estimates:

$$|f'_{l_1}(\theta)|_{\frac{1}{2}} \leq 4\varepsilon_{l_1}^{3-2\tau}, \quad |B_{l_1}(\theta)|_{\frac{1}{2}} \leq 1 + \varepsilon_{l_1}^{\frac{1}{2}}, \quad \|A_{l_1} - A\| \leq 2\|A\|\varepsilon_{l_1}.$$

(Resonant case). If the first step is obtained by resonant case: there exists $n_{l_1}^{*} \in \mathbb{Z}^d$ with $0 < |n_{l_1}^{*}| \leq N_{l_1}$ such that $\|2\rho - \langle n_{l_1}^{*}, \alpha \rangle\|_{\mathbb{T}} < \varepsilon_{l_1}^{\tau}$, then

$$\|B_{l_1}(\theta)\|_0 \leq C_2(\gamma, \tau, \|A\|)(l_1|\ln \varepsilon_{l_1}|)^{\frac{1}{2}} \leq (l_1|\ln \varepsilon_{l_1}|)^{(1+\rho)} e^{-\frac{l_{l_1}^{2\pi \rho}}{2}},$$

$$|B_{l_1}(\theta)|_{\frac{1}{2}} \leq (l_1|\ln \varepsilon_{l_1}|)^{(1+\rho)} e^{-\frac{l_{l_1}^{2\pi \rho}}{2}}, \quad |f'_{l_1}(\theta)|_{\frac{1}{2}} \ll \varepsilon_{l_1}^{100} < \frac{1}{2}\varepsilon_{l_1}^{\frac{5}{2}}.$$  

Moreover, let $A_{l_1} := e^{A_{l_1}''} \in sl(2, \mathbb{R})$ we have $\|A_{l_1}''\| \leq 16\varepsilon_{l_1}'.

**Induction Step:** Assume that in $(l_j)$-th step with $j \leq n$, we already have that

$$B_{l_j}(\theta + \alpha)^{-1} A e^{f'_{l_j}(\theta)} B_{l_j}(\theta) = A_{l_j} e^{f'_{l_j}(\theta)}$$

with the following estimates

$$|B_{l_j}(\theta)|_{\frac{1}{l_j^{\tau}}} \leq (l_j|\ln \varepsilon_{l_j}|)^{(\sigma+1)} e^{-\varepsilon_{l_j}'}, \quad \|A_{l_j}\| \leq 2\|A\|, \quad (3.3a)$$

$$\|B_{l_j}\|_0 \leq (l_j|\ln \varepsilon_{l_j}|)^{(\sigma+1)} e^{-\varepsilon_{l_j}'}, \quad |f'_{l_j}(\theta)|_{\frac{1}{l_j^{\tau}}} \leq \frac{1}{2}\varepsilon_{l_j}'. \quad (3.3b)$$

Moreover, if the $(l_j)$-th step is obtained by the resonant case, we have

$$A_{l_j} = e^{A_{l_j}'}, \quad \|A_{l_j}'\| \leq 8\varepsilon_{l_j}', \quad \|A_{l_j}\| \leq 1 + 16\varepsilon_{l_j}' \quad (3.4)$$
If the \((l_j)\)-th step is obtained by the non-resonant case, we have
\[
\|A_{l_j} - A_{l_{j-1}}\| \leq 2\|A_{l_{j-1}}\| \epsilon_{l_j}, \quad |B_{l_j}|_{\lambda + 1} \leq (1 + \epsilon_{l_j}^{1/2})|B_{l_{j-1}}|_{\lambda}.
\] (3.5)

Now let \(j = n + 1\) and focus on the cocycle \((\alpha, A e^{f_{l_{n+1}}(\theta)})\), it follows that
\[
B_{l_n}(\theta + \alpha)^{-1} A e^{f_{l_{n+1}}(\theta)} B_{l_n}(\theta) = A_{l_n} e^{f_{l_n}(\theta)} + B_{l_n}(\theta + \alpha)^{-1} (A e^{f_{l_{n+1}}} - A e^{f_{l_n}}) B_{l_n}(\theta).
\]
If we rewrite
\[
A_{l_n} e^{f_{l_n}} + B_{l_n}(\theta + \alpha)^{-1} (A e^{f_{l_{n+1}}} - A e^{f_{l_n}}) B_{l_n}(\theta) = A_{l_n} e^{f_{l_n}(\theta)},
\]
by (3.3) and \(\xi \in (\frac{2\pi}{M+1}, \frac{\pi}{2})\), we have
\[
|f_{l_n}(\theta)|_{l_{n+1}} \leq |f'_{l_n}(\theta)|_{l_{n+1}} + \|A_{l_n}^{-1}\| \cdot |B_{l_n}(\theta + \alpha)^{-1} (A e^{f_{l_{n+1}}} - A e^{f_{l_n}}) B_{l_n}|_{l_{n+1}} \leq \frac{1}{2} l_{n+1}^2 A \|\| l_n \epsilon_{l_n} + 2 \|A\| \times (l_n \ln \epsilon_{l_n}) 2^{\tau(\alpha + 1)} \epsilon_{l_n}^{-2\xi} \times \frac{c}{(2\|A\|)^D l_n^{2\xi}} \leq \epsilon_{l_{n+1}}.
\]

Apply Proposition 3.1 to the cocycle \((\alpha, A_{l_n} e^{f_{l_n}(\theta)})\), one can obtain \(\bar{B}_{l_n}(\theta) \in C_{l_{n+2}}^0(2T^d, SL(2, \mathbb{R}))\), \(A_{l_{n+1}} \in SL(2, \mathbb{R})\) and \(f'_{l_{n+1}}(\theta) \in C_{l_{n+2}}^0(2T^d, sl(2, \mathbb{R}))\) such that
\[
\bar{B}_{l_n}(\theta + \alpha)^{-1} A_{l_n} e^{f_{l_n}(\theta)} \bar{B}_{l_n}(\theta) = A_{l_{n+1}} e^{f'_{l_{n+1}}(\theta)}.
\]
Let \(B_{l_{n+1}} := \bar{B}_{l_n} B_{l_n} \in C_{l_{n+2}}^0(2T^d, SL(2, \mathbb{R}))\), and note that whether or not the \((l_{n+1})\)-th step is in the resonant case, the following estimate holds:
\[
|f'_{l_{n+1}}(\theta)|_{l_{n+2}} \leq \frac{1}{2} \epsilon_{l_{n+1}}^2.
\] (3.6)

We are going to analyze the structure of \(A_{l_{n+1}}\) and estimate the norm of the conjugation \(B_{l_{n+1}}\) in \((l_{n+1})\)-th step. Let \(A_{l_{n+1}} := e^{f_{l_{n+1}}'}\), and if \(\|A_{l_{n+1}}''\|\) is sufficiently small, one can always find some unitary matrices \(U \in SL(2, \mathbb{C})\) such that
\[
U^{-1} A_{l_{n+1}} U = \begin{pmatrix} e^{2\pi i \phi_{l_{n+1}}} & c_{l_{n+1}} \\ 0 & e^{-2\pi i \phi_{l_{n+1}}} \end{pmatrix}
\]
with the estimate \(|c_{l_{n+1}}| \leq 2 \|A_{l_{n+1}}''\|\). Let us focus on the cocycle \((\alpha, A_{l_n} e^{f_{l_n}(\theta)})\) and we need to distinguish between two cases.
We deduce that
\[ \|2\rho_i - \langle n_{i_{n+1}}, \alpha \rangle\|_T < \varepsilon_{i_{n+1}}^\sigma, \quad 0 < |n_{i_{n+1}}| \leq N_{i_{n+1}} := \frac{2\ln |\varepsilon_{i_{n+1}}|}{I_{n+1} - I_{n+2}}, \]
where \(\{e^{2\pi i \rho}, e^{-2\pi i \rho}\}\) are two eigenvalues of \(A_{i_n}\). Then by Proposition 3.1,
\[ |\bar{B}_{i_n}| \leq C_3(\gamma, \tau)(I_{n+1} |\ln \varepsilon_{i_{n+1}}|)^{1/2} \times \varepsilon_{i_{n+1}}^\frac{1}{2}. \]
We also have
\[ \|\bar{B}_{i_n}(\theta)\|_0 \leq C_3(I_{n+1} |\ln \varepsilon_{i_{n+1}}|)^{1/2}. \]
Hence,
\[ |B_{i_n+1}| \leq C_3(I_{n+1} |\ln \varepsilon_{i_{n+1}}|)^{1/2} \times \varepsilon_{i_{n+1}}^\frac{1}{2} \times (I_n |\ln \varepsilon_{i_n}|)^{\tau(\sigma+1)} \varepsilon_{i_n}^{-\xi}, \]
\[ \|B_{i_{n+1}}\|_0 \leq (I_{n+1} |\ln \varepsilon_{i_{n+1}}|)^{\tau(\sigma+1)}. \]
Moreover one can get that \(\|A''_{i_{n+1}}\| \leq 16\varepsilon_{i_{n+1}}^\sigma\), which gives \(|c_{n+1}| \leq 2\|A''_{i_{n+1}}\| \leq 32\varepsilon_{i_{n+1}}^\sigma\).
Combine with (3.8a), for any \(\kappa \in [1, \frac{c}{2\varepsilon}]\), we have
\[ \|B_{i_{n+1}}\|_0 \leq 32 |\ln \varepsilon_{i_{n+1}}|^{L_\sigma} \cdot \varepsilon_{i_{n+1}}^{\frac{\sigma(1-c)}{2\varepsilon}} < \infty. \]

(Non-resonant case). If the \((I_{m+1})\)-th step is obtained by non-resonant case, we track back to the nearest resonant step, says the \((I_m)\)-step. If such \(m\) does not exist, we deduce that each step is in non-resonant case, thus
\[ \|A_{i+1} - A\| \leq \|A_{i_1} - A\| + \sum_{j=1}^n \|A_{i_{j+1}} - A_{i_j}\| \leq 8\|A\|\|\varepsilon_{i_1}\|, \]
and \(|c_{n+1}| \leq \|A_{i_{n+1}}\| \leq 2\|A\|\). Since \(B_{i_{n+1}}\) is close to identity by construction, it follows that
\[ \|B_{i_{n+1}}\|_0 \leq |B_{i_{n+1}}|^{1/2} \leq 2. \]
We deduce that
\[ \|B_{i_{n+1}}\|_0 \leq 2^{\kappa+1} \|A\| < \infty. \]
If such \(m < n + 1\) exists, let \(A_{i_m} = e^{A''_{i_m}}\), then by (3.3) and (3.4), we have
\[ \|A''_{i_m}\| \leq 16\varepsilon_{i_m}^{\sigma}, \quad \|A_{i_m}\| \leq 1 + 32\varepsilon_{i_m}^{\sigma}, \quad |B_{i_m}|^{1/2} \leq (I_m |\ln \varepsilon_{i_m}|)^{\tau(\sigma+1)} \varepsilon_{i_m}^{-\xi}. \]
Since each step is in non-resonant case from \((l_m)\)-th step to \((l_{m+1})\)-th step, it deduces that
\[
\|A_{l_{m+1}} - A_{l_m}\| \leq 8\|A\|\varepsilon_{l_{m+1}}, \quad \|A_{l_{m+1}}\| \leq 2\|A\|.
\]
Moreover, (3.12) also implies
\[
\|c_{l+1}\| \leq 2\|A''_{l_{m+1}}\| \leq 64\varepsilon_{l_m}^2.
\]
Note that each conjugation is close to identity from \((l_m)\)-step to \((l_{m+1})\)-step by construction in (3.5), thus one can get that
\[
\|B_{l_{m+1}}\|_0 \leq |B_{l_{m+1}}| \frac{1}{\varepsilon_{l+1}} \leq 2\|B_l\| \frac{1}{\varepsilon_{l_{m+1}}} \leq (l_{m+1}) \ln \varepsilon_{l_{m+1}} \|\tau^{(s+1)} \| \varepsilon_{l_{m+1}}^{-\frac{s}{2}}.
\]
Combine (3.13) with (3.14), it follows that
\[
\|B_{l_{m+1}}\|^k \cdot |c_{l+1}| \leq 2^{k+6} \ln \varepsilon_l \frac{k}{2} \cdot \varepsilon_{l_m}^{-\frac{s}{2}} < \infty.
\]
This completes the proof of (A) by (3.6), (3.8a), (3.10) and (3.14). It also finishes the proof for (B) by (3.9), (3.11) and (3.15).

**Remark 3.1.** Theorem 3.1 has been proved in [10] essentially, however for the technical reason, we replace the estimate \(\|B_l\|_0 \leq \varepsilon_l^{-\frac{s}{2}}\) in Proposition 3.2 of [10] by \(\|B_l\|_0 \leq (l_{j}) \ln \varepsilon_{l_j} \|^\tau\) \(\|\tau^{(s+1)} \|\varepsilon_{l_{m+1}}^{-\frac{s}{2}}\), so that one can get \(\|B_{l_{m+1}}\|^k \cdot |c_j| < \infty\).

### 3.2 Complex almost triangularization for \(C^k\) Q-P cocycles

In Theorem 3.1, one can see that the conjugation \(B_l : 2T^d \rightarrow SL(2, \mathbb{R})\) is real, which results in \(A_j\) being real. In fact, we can choose the complex conjugation \(B_l : 2T^d \rightarrow SL(2, \mathbb{C})\) to make \(A_j\) complex almost triangularization provided that the cocycle is not uniformly hyperbolic.

**Theorem 3.2.** Suppose that all the conditions in Theorem 3.1 hold. Further assume that \((\alpha, Ae^{f(\theta)})\) is not uniformly hyperbolic. There exists \(\varepsilon_\ast = \varepsilon_\ast(\gamma, \tau, d, k, \|A\|)\) such that if \(\|f\|_k \leq \varepsilon_\ast\), then there exist \(A_{l_j} \in SL(2, \mathbb{C}), \tilde{F}_{l_j} \in C^{k_0}(2T^d, SL(2, \mathbb{C}))\) with
\[
k_0 = \left\lfloor \frac{k}{20} \right\rfloor \quad \text{and} \quad \Phi_{l_j} = C_{\gamma_{l_j}}^{k_0}(2T^d, SL(2, \mathbb{C}))\,
\]
such that
\[
\Phi_{l_j} (\theta + \alpha)^{-1} Ae^{f(\theta)} \Phi_{l_j} (\theta) = \tilde{A}_{l_j} + \tilde{F}_{l_j} (\theta),
\]
where
\[
\tilde{A}_{l_j} = \begin{pmatrix} e^{2\pi i \gamma_{l_j}} & c_j \\ 0 & e^{-2\pi i \gamma_{l_j}} \end{pmatrix}.
\]
with \( \rho_j \in \mathbb{R} \) and \( c_j \in \mathbb{C} \), also with estimates

\[
\| \tilde{F}_j(\theta) \| \leq 2\varepsilon_{ij}^2, \quad \| \Phi_{ij}(\theta) \| \leq (l_j |\ln \varepsilon_{ij}|)^\tau(\sigma+1), \quad \| \Phi_{ij}(\theta) \|_0^2 \cdot |c_j| \leq C, \tag{3.16}
\]

where \( C \) is a constant defined in Theorem 3.1 and \( \kappa \in [1, \frac{c_k}{\beta}] \).

**Proof.** Recall that in Theorem 3.1, we already have

\[
B_{ij}(\theta + \alpha)^{-1} A e^{f_j(\theta)} B_{ij}(\theta) = A_{ij} e^{f_j(\theta)}, \quad \forall j \in \mathbb{Z}.
\]

Denote

\[
A_{ij} + \tilde{F}_j = A_{ij} e^{f_j(\theta)} + B_{ij}(\theta + \alpha)^{-1}(A \epsilon - A e^{f_j(\theta)}) B_{ij}(\theta),
\]

then

\[
B_{ij}(\theta + \alpha)^{-1} A e^{f_j(\theta)} B_{ij}(\theta) = A_{ij} + \tilde{F}_j(\theta).
\]

By the estimates of (A) in Theorem 3.1, we have

\[
\| \tilde{F}_j \| \leq 2\| A_{ij} f_j(\theta) \|_0 + 2\| B_{ij}(\theta + \alpha)^{-1} A(\epsilon - f_j(\theta)) B_{ij}(\theta) \|_0 \\
\leq 2\| A \| \varepsilon_{ij}^2 + 2\| A \| (l_j |\ln \varepsilon_{ij}|)^{2\tau(\sigma+1)} \times \frac{c}{(2\| A \|)^{D} l_j^{k-1}} \\
\leq (2\| A \|)^{-1} \varepsilon_{ij}^2,
\]

where the second step uses the fact

\[
\sum_{i \geq j} \| f_{i+1} - f_i \| \leq \frac{c}{(2\| A \|)^{D} l_j^{k-1}}
\]

by (2.2).

Assume that \( \{ e^{2\pi i \phi_j}, e^{-2\pi i \phi_j} \} \) are two eigenvalues of \( A_{ij} \), then there exist unitary \( U \in SL(2, \mathbb{C}) \), such that

\[
U^{-1} A_{ij} U = \begin{pmatrix} e^{2\pi i \phi_j} & c_j \\ 0 & e^{-2\pi i \phi_j} \end{pmatrix}
\]

with \( |c_j| \leq \| A_{ij} \| \leq 2\| A \| \), where \( \rho_j \in i\mathbb{R} \cup \mathbb{R} \) and \( c_j \in \mathbb{C} \). In the following, we need to rule out that \( i\phi_j \in \mathbb{R} \setminus \{0\} \). Suppose that \( \lambda_j = i\phi_j \in \mathbb{R} \setminus \{0\} \). If \( 2\pi |\rho_j| > \varepsilon_{ij}^2 \), let \( P := \text{diag}\{ 2A^{\frac{1}{2}} \varepsilon_{ij}^{-\frac{1}{2}}, 2A \}^{-\frac{1}{2}} \varepsilon_{ij}^\frac{1}{2} \}, \) then we have

\[
P^{-1} A_{ij} + \tilde{F}_j(\theta) U P = \begin{pmatrix} e^{2\pi i \lambda_j} & 0 \\ 0 & e^{-2\pi i \lambda_j} \end{pmatrix} + F(\theta)
\]

with \( \| F \| \leq 2\varepsilon_{ij} \). We rewrite

\[
\begin{pmatrix} e^{2\pi i \lambda_j} & 0 \\ 0 & e^{-2\pi i \lambda_j} \end{pmatrix} + F(\theta) = \begin{pmatrix} e^{2\pi i \lambda_j} & 0 \\ 0 & e^{-2\pi i \lambda_j} \end{pmatrix} e^{f_j(\theta)} \tag{3.17}
\]
with \( \| \tilde{F}(\theta) \|_0 \leq 4 \| A \| \varepsilon_{l_i} \).

By Lemma 2.2 and Corollary 3.1 of [17], one can conjugate (3.17) to
\[
\begin{pmatrix}
e^{2\pi\lambda_j} & 0 \\
0 & e^{-2\pi\lambda_j}
\end{pmatrix}
\begin{pmatrix}
e^{\tilde{F}(\theta)} \\
e^{-\tilde{F}(\theta)}
\end{pmatrix}
= \begin{pmatrix}
e^{2\pi\lambda_j}e^{\tilde{F}(\theta)} & 0 \\
0 & e^{-2\pi\lambda_j}e^{-\tilde{F}(\theta)}
\end{pmatrix}
\]
with \( \| \tilde{F}(\theta) \|_0 \leq 8 \| A \| \varepsilon_{l_i} \), thus \( (a, Ae^{\theta}) \) is uniformly hyperbolic, which contradicts to the assumption. Hence we only need to consider
\[
2\pi |\rho_j| \leq \varepsilon_{l_i}^{\frac{1}{4}}.
\]
In this case, we put \( \rho_j \) into the perturbation so that the new perturbation satisfies
\[
\| \tilde{F}_j \|_0 \leq 2\varepsilon_{l_i}^{\frac{1}{4}} \quad \text{and} \quad A_{l_j} = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}.
\]

Denote
\[
\Phi_{l_j}(\theta) = B_{l_j}(\theta)U \in C_{\frac{1}{2}}(2\mathbb{T}^d, SL(2, \mathbb{C}))
\]
we always have
\[
\Phi_{l_j}(\theta + \alpha)^{-1}A e^{\theta} \Phi_{l_j}(\theta) = \begin{pmatrix} e^{2\pi ip_j} & c_j \\ 0 & e^{-2\pi ip_j} \end{pmatrix} \tilde{F}_j(\theta)
\]
with \( \rho_j \in \mathbb{R}, c_j \in \mathbb{C} \) and \( \| \tilde{F}_j \|_0 \leq 2\varepsilon_{l_i}^{\frac{1}{4}} \). Since \( U \) is unitary, by Theorem 3.1(A), (3.16) holds. \( \Box \)

## 4 Sharp Hölder continuity of the spectral measure

Consider the following discrete \( C^k \) Q-P Schrödinger operators:
\[
(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad \forall n \in \mathbb{Z}, \tag{4.1}
\]
where \( \alpha \in DC_d(\gamma, \tau) \) and \( V \in C^k(\mathbb{T}^d, \mathbb{R}) \). Let \( \mu_{V,\alpha,\theta} \) be the spectral measure of \( H_{V,\alpha,\theta} \). Based on the dynamical estimates of corresponding Schrödinger cocycles, we are able to show the \( \frac{1}{2} \)-Hölder continuity of \( \mu_{V,\alpha,\theta} \).

Let \( A(\theta) := S^V_\theta(\theta) \) and for any \( n \geq 1 \), we define
\[
P_n(\theta) = \sum_{s=1}^{n} A_{2s-1}(\theta + \alpha)A_{2s-1}(\theta + \alpha),
\]
where \( A_s(\theta) = A(\theta + (s-1)\alpha) \cdots A(\theta + \alpha)A(\theta) \). \( P_n \) is an increasing family of positive self-adjoint operators. \( \| P_n \| \) is unbounded since \( \text{tr} P_n \geq 2n \). Moreover, \( \text{det} P_n \) is also unbounded. For simplicity, we will use the notation \( a \approx b \) which denotes that there exist some constants \( C > 0 \) such that \( C^{-1}a \leq b \leq Ca \) and also use the notation \( a \lesssim b \) which denotes \( a \leq Cb \) for some constants \( C > 0 \).
Theorem 4.1. Let $\alpha \in DC_d(\gamma, \tau)$, $V \in C^k(T^d, \mathbb{R})$ with $k \geq 5D\tau$ and $D$ is a numerical constant. There exists $\varepsilon = \varepsilon(\gamma, \tau, k)$ such that if $\|V\|_k \leq \varepsilon$, then for any $f \in \ell^2(\mathbb{Z}) \cap \ell^1(\mathbb{Z})$, $$\mu_{V, \alpha, \theta}(J) \leq D_0\|f\|^2_{\ell^1} : \|J\|^2,$$

for all intervals $J$ and all $\theta$, where $D_0 = D_0(V, \alpha) > 0$.

Proof. Since the spectral measure $\mu_{V, \alpha, \theta}$ vanishes on $\mathbb{R} \setminus \Sigma_{V, \alpha}$, we only need to consider the case $E \in \Sigma_{V, \alpha}$.

Rewrite the Schrödinger cocycle $(\alpha, S^V_E)$ as $(\alpha, A_E e^{f(\theta)})$, where

$$A_E = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}, \quad f(\theta) = \ln(\text{Id} + A_E^{-1}F(\theta)), \quad F(\theta) = \begin{pmatrix} -V(\theta + n\alpha) & 0 \\ 0 & 0 \end{pmatrix}.$$ 

By the assumption on $\|V\|_k$ and Theorem 3.2, for $\kappa \in [1, \frac{e^k}{50}]$, there exist $\Phi_{ij} \in C^k(2T^d, SL(2, \mathbb{C}))$ with $\|\Phi_{ij}\|_0 \leq (I_1 \ln \varepsilon_{ij})^{(\tau+1)}$ and $\beta_i \in C^0(T^d, \mathbb{R})$, $i = 1, 2, 3, 4$ with $k_0 = [\frac{k}{50}]$ such that

$$\Phi_{ij}(\theta + \alpha)^{-1}S^V_E(\theta)\Phi_{ij}(\theta) = T_{ij} + \begin{pmatrix} \beta_1(\theta) & \beta_2(\theta) \\ \beta_3(\theta) & \beta_4(\theta) \end{pmatrix},$$

where

$$T_{ij} = \begin{pmatrix} e^{2\pi i\rho_j} & c_j e^{-2\pi i\rho_j} \\ 0 & e^{-2\pi i\rho_j} \end{pmatrix}$$

with $\rho_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, also with estimates

$$\|\Phi_{ij}\|_0 \cdot |c_j| \leq C, \quad \|\beta_i\|_0 \leq 2\varepsilon_{ij}^{\frac{1}{2}} \quad i = 1, 2, 3, 4.$$ (4.2)

For simplicity of notations, in the following $\Phi_{ij}, T_{ij}, \rho_j$ are written as $\Phi, T, \rho$ respectively. Let $\bar{T}(\theta) = \Phi(\theta + \alpha)^{-1}S^V_E(\theta)\Phi(\theta)$, then we have

$$\|\bar{T} - T\|_0 \leq 2\varepsilon_{ij}^{\frac{1}{2}}.$$ (4.3)

We need to compare the dynamics between $(\alpha, S^V_E)$ and $(\alpha, T)$. For this purpose, let $X = \sum_{j=1}^n T_{2j-1}^a T_{2j-1}^b$ and $\bar{X}(\theta) = \sum_{j=1}^n \bar{T}_{2j-1}(\theta)\bar{T}_{2j-1}(\theta)$. The following estimates on $\|X\|_0$ and $\|X^{-1}\|^{-1}_0$ are crucial.

Lemma 4.1 (Lemma 4.3 of [5]). Let

$$K(\theta) = \begin{pmatrix} e^{2\pi i\rho} & t(\theta) \\ 0 & e^{-2\pi i\rho} \end{pmatrix},$$

where $t(\theta) = \tilde{t}(r)e^{i\pi r\theta}$. Let $G = \sum_{j=1}^n K_{2j-1}^* K_{2j-1}$, then

$$\|G\|_0 \approx n(1 + |\tilde{t}(r)|^2 \min\{n^2, \|2\rho - \langle r, \alpha \rangle\|_T^2\}),$$

$$\|G^{-1}\|^{-1}_0 \approx n.$$
Lemma 4.2 (Lemma 4.4 of [5]). Let $G$ and $\tilde{t}(r)$ be as above. Define $\tilde{G} = \sum_{j=1}^{n} \tilde{K}_{2j-1} \tilde{K}_{2j-1}$. Then there exists $D_1 > 0$ such that if

$$\|\tilde{K} - K\|_0 \leq D_1 n^{-2} (1 + 2n|\tilde{t}(r)|)^{-2},$$

we have $\|\tilde{G} - G\|_0 \leq 1$.

Apply Lemma 4.1, one can get that

$$\|X\|_0 \approx n(1 + |c_j|^2 \min\{n^2, \|2\rho\|^{-2}_1\}), \quad \|X^{-1}\|^{-1}_0 \approx n. \quad (4.4)$$

Let $n^+$ be maximal such that $\|\tilde{X} - X\|_0 \leq 1$ for $1 \leq n < n^+$. So by Lemma 4.2 and (4.3), one can get that

$$2\epsilon_{n_j}^1 \geq D_1 (n^+)^{-2} (1 + 2n^+|c_j|)^{-2} \gtrsim (n^+)^{-4}. \quad (4.5)$$

It follows that

$$n^+ \gtrsim \epsilon_{n_j}^1. \quad (4.5)$$

Since $\|\tilde{X}\|_0 \leq \|X\| + 1$ and $\|\tilde{X}^{-1}\| \geq \|X^{-1}\|_0 - 1$ for $1 \leq n < n^+$, also we notice that

$$\|P\|_n \leq \|\Phi(\theta)\|_n^4 \cdot \|\tilde{X}(\theta + \alpha)\|_0,$$

$$\|P_n^{-1}\|_0 \geq \|\Phi(\theta)\|_0^{-4} \cdot \|\tilde{X}(\theta + \alpha)^{-1}\|_0^{-1}.$$

(4.4) implies

$$\|P_n\|_0 \leq D_2 n(1 + |c_j|^2 n^2) \|\Phi\|_0^4,$$

$$\|P_n^{-1}\|_0^{-1} \geq D_3 n \|\Phi\|_0^{-4}.$$ 

Hence by direct calculation,

$$\frac{\|P_n\|_0}{\|P_n^{-1}\|_0^{-1}} \leq D_4 |c_j|^2 \cdot \|\Phi\|_0^{16} + D_4 \frac{1}{n^2} \|\Phi\|_0^{16}.$$ 

From Theorem 3.2, we know that $\|\Phi\|_0^{8} \cdot |c_j| < \infty$ and $\|\Phi\|_0 \leq \epsilon_{n_j}^{-\frac{n}{2}}$, then

$$\|P_n\|_0 \leq \|P_n^{-1}\|_0^{-\frac{3}{2}}, \quad \forall n \in (n^-, n^+),$$

where

$$n^- := \epsilon_{n_j}^{-\frac{n}{2}}. \quad (4.6)$$

Denote the interval $I_j := [D_5 I_j^{k+}, D_6 I_j^{k+}], j \in \mathbb{Z}^+$, then by (4.5) and (4.6), for any $n \in I_j$, we have $n^- < n < n^+$. Since $I_j \cap I_{j+1} \neq \emptyset, \forall j \in \mathbb{Z}^+$, thus $\cup_{j \geq 1} I_j$ cover all the $n$ tending to
infinity. Hence \( \|P_n\|_0 \lesssim \|P_{n-1}\|_0^{-3} \) for any \( n \geq D_5 l_{150}^{1/2} \). On the other hand, the number of \( n \) satisfying \( n < D_5 l_{150}^{1/2} \) is finite, then
\[
\sup_{n < D_5 l_{150}^{1/2}} \frac{\|P_n\|_0}{\|P_{n-1}\|_0^{-3}} < \infty.
\]
Thus
\[
\|P_n\| \lesssim \|P_{n-1}\|^{-3}, \quad \forall n \in \mathbb{Z}^+.
\]  
\textbf{Lemma 4.3} (Lemma 4.2 of [5]). Let \( \epsilon_n = \frac{1}{2\sqrt{\det P_n}} \), then
\[
D_6^{-1} < \frac{\psi(m^+(E + i\epsilon_n))}{2\epsilon_n \|P_n\|_0} < D_6,
\]
where \( D_6 > 0 \) is a constant and \( \psi(m^+) \) is defined in Lemma 2.1.

For any bounded potential and any solution \( u \) satisfying \( (4.1) \), we have
\[
\|u\|_{L^{1+}} \lesssim \|u\|_L, \quad \text{where} \quad \|u\|_L = \left( \sum_{j=1}^{L} |u_j|^2 \right)^{1/2}.
\]
In particular, for the solution \( u^\beta \) with \( u_0^\beta \cos \beta + u_1^\beta \sin \beta = 0 \) and \( |u_0^\beta|^2 + |u_1^\beta|^2 = 1 \), we have
\[
\det P_n = \inf_{\beta} \|u^\beta\|_2^2 \|u^{\beta+\pi/2}\|_L^2.
\]  
\text{(4.8)}

By \( (4.7) \) and \( P_n : T^d \to gl(2, \mathbb{R}) \), we have
\[
\|P_n\|_0 = \det P_n \|P_{n-1}\|_0 \lesssim \epsilon_n^{-2} \|P_n\|^{-1/3},
\]
where \( \epsilon_n \) is defined in Lemma 4.3. Thus \( \|P_n\|_0 \lesssim \epsilon_n^{-2} \). According to Lemma 4.3, we deduce that
\[
\psi(m^+(E + i\epsilon_n)) \lesssim \epsilon_n \|P_n\|_0 \lesssim \epsilon_n^{-1/3}.
\]
Since \( \lim_{n \to \infty} \epsilon_n = 0 \), and we also have \( \epsilon_n \lesssim \epsilon_{n+1} \) by \( (4.8) \), we only need to consider the case of fixing \( \epsilon_n = \epsilon \). Combine \( (2.1) \) with Lemma 2.1, we have
\[
\mu_{V,\alpha,\beta}(E - \epsilon, E + \epsilon) \leq 2\epsilon \Im M(E + i\epsilon) \lesssim \epsilon^{1/2}.
\]
Since \( \mu_{V,\alpha,\beta} = 0 \) on \( \mathbb{R} \setminus \Sigma_{V,\alpha} \), then there exists \( D_0 = D_0(V, \alpha) > 0 \) such that
\[
\mu_{V,\alpha,\beta}(J) \leq D_0 |J|^{1/2}, \quad \forall J \subset \mathbb{R}.
\]
Let $\sigma : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the shift $f(i + 1) = \sigma f(i)$. Then $\sigma H_{V,\alpha,\theta}^{-1} = H_{V,\alpha,\theta+a}$. Thus $\mu^\sigma_{\theta+a} = \mu_{\theta}$ and $\mu^\sigma_{\theta+k\alpha} = \mu_{\theta+k\alpha}$. Let $\mathbb{E}(J)$ be the spectral projection of $H_{V,\alpha,\theta}$ on $J$, then

$$
\mu_f^{V,\alpha,\theta}(J) = \langle \mathbb{E}(J)f, f \rangle = \| \mathbb{E}(J) \sum_k f(k) e_k \|^2 \leq \left( \sum_k |f(k)| \right) \cdot \| \mathbb{E}(J)e_k \|^2 \\
= \left( \sum_k |f(k)| \mu^\sigma_{V,\alpha,\theta}(J)^{\frac{1}{2}} \right)^2 \leq \left( \sum_k |f(k)| \mu_{V,\alpha,\theta+k\alpha}(J)^{\frac{1}{2}} \right)^2 \\
\leq D_0 \|f\|_2^2 \cdot |J|^{\frac{1}{2}}.
$$

Thus, we complete the proof.

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**References**


