

Hölder Continuity of Spectral Measures for the Finitely Differentiable Quasi-Periodic Schrödinger Operators

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Abstract. In the present paper, we prove the $\frac{1}{2}$ -Hölder continuity of spectral measures for the C^k Schrödinger operators. This result is based on the quantitative almost reducibility and an estimate for the growth of the Schrödinger cocycles in [5].

Key Words: Schrödinger operator, quasi-periodic, almost reducibility, finitely differentiable.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

In this paper, we consider the Schrödinger operators defined on $\ell^2(\mathbb{Z})$

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n,$$

where $V : \mathbb{T}^d \rightarrow \mathbb{R}$ is the potential, $\theta \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the phase, and $\alpha \in \mathbb{T}^d$ is the frequency.

These operators have been extensively and thoroughly studied for the deep connection with quasi-crystal and quantum Hall effects [11, 18]. This paper concerns the regularity of the spectral measure of the quasi-periodic Schrödinger operators. For the analytic potential $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$, there is some significant progress [5, 21, 24]. However for the smooth potential $V \in C^k(\mathbb{T}^d, \mathbb{R})$, there is no similar result as far as we know, so we will give a supplementary answer to this situation.

Let us review some results on the Hölder continuity of the integrated density of states (IDS) and the individual spectral measures.

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1.1 Hölder continuity of IDS

Let $\Sigma_{V,\alpha,\theta}$ be the spectrum of $H_{V,\alpha,\theta}$, then $\Sigma_{V,\alpha,\theta} \subset \mathbb{R}$ since $H_{V,\alpha,\theta}$ is the bounded self-adjoint operator in $\ell^2(\mathbb{Z})$. The spectrum is independent of θ if $(\alpha, 1)$ is rational independent. For any $f \in \ell^2(\mathbb{Z})$, the spectral measure $\mu_{V,\alpha,\theta}^f$ of $H_{V,\alpha,\theta}$ can be defined as

$$\langle (H_{V,\alpha,\theta} - E)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu_{V,\alpha,\theta}^f(E'), \quad \forall E \in \mathbb{C} \setminus \Sigma_{V,\alpha}. \tag{1.1}$$

Let $\mu_{V,\alpha,\theta} = \mu_{V,\alpha,\theta}^{e_{-1}} + \mu_{V,\alpha,\theta}^{e_0}$, where $\{e_i\}_{i \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. Let $N_{V,\alpha}$ be the IDS of $H_{V,\alpha,\theta}$, it is well known that IDS is the average of the spectral measure $\mu_{V,\alpha,\theta}$ with respect to θ , i.e.,

$$N_{V,\alpha}(E) = \int_{\mathbb{T}^d} \mu_{V,\alpha,\theta}(-\infty, E] d\theta.$$

Hence the regularity of IDS is closely related to that of the spectral measure.

Recall that $\alpha \in \mathbb{T}^d$ is Diophantine if there exist $\gamma > 0$ and $\tau > d - 1$ such that $\alpha \in DC_d(\gamma, \tau)$, where

$$DC_d(\gamma, \tau) = \left\{ \alpha : \inf_{j \in \mathbb{Z}} |\langle n, \alpha \rangle - j| > \frac{\gamma}{|n|^\tau}, \forall n \in \mathbb{Z}^d \setminus \{0\} \right\}.$$

Let $DC_d = \cup_{\gamma > 0, \tau > d-1} DC_d(\gamma, \tau)$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, let $\frac{p_n}{q_n}$ be the continued fraction approximations to α , then one can define

$$\beta(\alpha) = \limsup_{n \rightarrow \infty} \frac{\ln q_{n+1}}{q_n}.$$

Given the operator $H_{V,\alpha,\theta}$, one can define the Lyapunov exponent $L(\alpha, S_E^V)$ (see Section 2.1) of the corresponding Schrödinger cocycle $(\alpha, S_E^V(\theta))$, where $E \in \mathbb{R}$ and

$$S_E^V(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Hadj Amor [16] proved the $\frac{1}{2}$ -Hölder continuity of IDS if $\alpha \in DC_d$ and $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$ is small, and her approach is based on the almost reducibility scheme developed by Eliasson [13]. Recall that the cocycle (α, A) is reducible if (α, A) can be conjugated to some constant cocycles and the cocycle (α, A) is almost reducible if the closure of its conjugates contains a constant. Avila and Jitomirskaya [4] proved the $\frac{1}{2}$ -Hölder continuity of IDS for $\alpha \in DC_1$ and the small analytic potential. Their result was non-perturbative, which means that the smallness is independent of α . After that, Avila [2,3] generalized the result for the small analytic potential with $\beta(\alpha) = 0$ if there is $\delta > 0$ such that $L(\alpha, S_{E+i\epsilon}^V) = 0$ for $|\epsilon| < \delta$. Note that Leguil-You-Zhao-Zhou [20] showed the same result as well by the global theory of the one-frequency Schrödinger operators [1].

The Hölder continuity of $N_{V,\alpha}(E)$ is equivalent to the Hölder continuity of $L(\alpha, S_E^V)$ according to the famous Thouless formula [6], i.e.,

$$L(\alpha, S_E^V) = \int \ln |E - E'| dN_{V,\alpha}(E').$$

Suppose that $L(\alpha, S_E^V) > 0$. Goldstein and Schlag [14] proved the Hölder continuity of $L(\alpha, S_E^V)$ if $V \in C^\omega(\mathbb{T}, \mathbb{R})$, and $\alpha \in SDC^\dagger$ by the avalanche principle and sharp large deviation theorem. Later, You and Zhang [22] proved that for analytic potential, the $L(\alpha, S_E^V)$ is Hölder continuous if $\alpha \in DC_1$ or some weaker Liovillean α by a refined large deviation theorem.

For the special kinds of potentials, there are some other interesting results. If the potential is a small perturbation of a trigonometric polynomial of degree d and $\alpha \in DC_1$, Goldetein-Schlag [15] proved that IDS is $(\frac{1}{2d} - \epsilon)$ -Hölder continuous for any $\epsilon > 0$ by assuming $L(\alpha, S_E^V) > 0$. In particular, if the potential $V(x) = 2\lambda \cos(x)$, we get the so called almost Mathieu operators (AMO), i.e.,

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos(\theta + n\alpha)u_n,$$

where $\lambda \in \mathbb{R}$ is the coupling constant. For the large coupling, Bourgain [7] proved that Lyapunov exponent of AMO is $(\frac{1}{2} - \epsilon)$ -Hölder continuous for any $\epsilon > 0$. Avila [2] obtained the exact $\frac{1}{2}$ -Hölder continuity of IDS for AMO with $|\lambda| < 1$.

1.2 Hölder continuity of the spectral measure

In general, the spectral measure is less regular than IDS. In the $\beta(\alpha) = 0$ regime, Avila and Jitomirskaya [5] proved that if $V \in C^\omega(\mathbb{T}, \mathbb{R})$ is small enough and $\alpha \in DC_1$, then the spectral measure $\mu_{V,\alpha,\theta}^f$ of one-frequency Schrödinger operators is $\frac{1}{2}$ -Hölder continuous for any $f \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$. They also showed $\frac{1}{2}$ -Hölder continuity of absolutely continuous spectral measures for the one-frequency Schrödinger operators with $V \in C^\omega(\mathbb{T}, \mathbb{R})$. In the $\beta(\alpha) > 0$ regime, Liu and Yuan [21] extended the results in [5] to that for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\beta(\alpha) < \infty$, whenever the analytic radius of V is sufficiently large.

What can we say about the regularity of the spectral measure of Schrödinger operators with finitely differentiable potential? Recall that Cai-Chavaudret-You-Zhou [10] have shown that if $\alpha \in DC_d$ and $V \in C^k(\mathbb{T}^d, \mathbb{R})$ is small, then IDS of the Schrödinger operator is $\frac{1}{2}$ -Hölder continuous. Recently, Zhao [24] also generalized [5] and [21] to the multi-frequency Schrödinger operators with $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$. This paper is motivated by [10] and [24], as a supplementray answer, we obtain the $\frac{1}{2}$ -Hölder continuity of the spectral measure $\mu_{V,\alpha,\theta}^f$ for $V \in C^k(\mathbb{T}^d, \mathbb{R})$.

[†]We say α satisfies strong Diophantine condition if there exist some $\gamma > 0, \tau > 1$ such that $\alpha \in SDC(\gamma, \tau)$, where

$$SDC(\gamma, \tau) = \left\{ \alpha \in \mathbb{R}^d : \inf_{j \in \mathbb{Z}} |\langle n, \alpha \rangle - j| > \frac{\gamma}{|n|(\ln(1 + |n|))^\tau} \right\},$$

and $SDC = \cup_{\gamma>0, \tau>1} SDC(\gamma, \tau)$.

Theorem 1.1. Let $\alpha \in \text{DC}_d(\gamma, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k \geq 5D\tau$ and D is a numerical constant. There exists $\tilde{\varepsilon} = \tilde{\varepsilon}(\gamma, \tau, k)$ such that if $\|V\|_k \leq \tilde{\varepsilon}$, then for any $f \in \ell^2(\mathbb{Z}) \cap \ell^1(\mathbb{Z})$,

$$\mu_{V, \alpha, \theta}^f(J) \leq D_0 |J|^{\frac{1}{2}} \|f\|_{\ell^1}^2,$$

for all intervals J and all θ , where $D_0 = D_0(V, \alpha) > 0$.

Remark 1.1. Note that Theorem 1.1 is perturbative, i.e., the smallness $\tilde{\varepsilon}$ depends not only on the potential V , but also on the frequency α . From a counterexample of Bourgain [8], one can not expect non-perturbative results in multi-frequency case.

2 Preliminaries

For a bounded analytic function F defined on $\mathcal{S}_r = \{\theta : \theta = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d, |\Im \theta_i| < r, \forall i = 1, \dots, d\}$, let $|F|_r = \sup_{\theta \in \mathcal{S}_r} \|F(\theta)\|$ and denote by $C_r^\omega(\mathbb{T}^d, *)$ the set of these $*$ -value functions ($*$ will usually denote $\mathbb{R}, \text{sl}(2, \mathbb{R})$ or $SL(2, \mathbb{R})$). We also denote the set $C^k(\mathbb{T}^d, *)$ to be the space of k times differentiable with continuous k -th derivatives functions, endowed with the norm

$$\|F\|_k := \sup_{k' \leq k, \theta \in \mathbb{T}^d} \|\partial^{k'} F(\theta)\|.$$

In particular,

$$\|F\|_0 := \|F\|_{\mathbb{T}^d} = \sup_{\theta \in \mathbb{T}^d} \|F(\theta)\|.$$

For $\theta \in \mathbb{R}$, we set $\|\theta\|_{\mathbb{T}} = \inf_{j \in \mathbb{Z}} |\theta - j|$.

2.1 Uniform hyperbolicity

Given $A \in C^\omega(\mathbb{T}^d, SL(2, \mathbb{C}))$ and $\alpha \in \mathbb{R}^d$ rationally independent, one can define the quasi-periodic (Q-P) cocycle (α, A) :

$$\begin{aligned} (\alpha, A) : \mathbb{T}^d \times \mathbb{C}^2 &\rightarrow \mathbb{T}^d \times \mathbb{C}^2; \\ (\theta, v) &\mapsto (\theta + \alpha, A(\theta) \cdot v). \end{aligned}$$

The iterations of (α, A) are of form $(\alpha, A)^n = (n\alpha, A_n)$, where

$$A_n(\theta) := \begin{cases} A(\theta + (n-1)\alpha) \cdots A(\theta + \alpha) A(\theta), & n \geq 0, \\ A^{-1}(\theta + n\alpha) A^{-1}(\theta + (n+1)\alpha) \cdots A^{-1}(\theta - \alpha), & n < 0. \end{cases}$$

The Lyapunov exponent of the cocycle (α, A) is defined as

$$L(\alpha, A) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathbb{T}^d} \|A_n(\theta)\| d\theta.$$

We say the cocycle (α, A) is uniformly hyperbolic if for every $\theta \in \mathbb{T}^d$, there exists a continuous splitting $\mathbb{C}^2 = E^s(\theta) \oplus E^u(\theta)$ such that for every $n \geq 0$,

$$\begin{aligned} \|A_n(\theta)v\| &\leq Ce^{-cn}\|v\|, & v \in E^s(\theta), \\ \|A_n(\theta)^{-1}v\| &\leq Ce^{-cn}\|v\|, & v \in E^u(\theta + n\alpha), \end{aligned}$$

for some constants $C, c > 0$. And the splitting is invariant by the dynamics:

$$\begin{aligned} A(\theta)E^s(\theta) &= E^s(\theta + \alpha), & \forall \theta \in \mathbb{T}^d, \\ A(\theta)E^u(\theta) &= E^u(\theta + \alpha), & \forall \theta \in \mathbb{T}^d. \end{aligned}$$

2.2 Spectral measure and Wely's m function

Typical examples of $SL(2, \mathbb{R})$ cocycles are the Schrödinger cocycles (α, S_E^V) :

$$A(\theta) = S_E^V(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \quad E \in \mathbb{R}.$$

Those cocycles come from the eigenvalue equation of one dimensional quasi-periodic Schrödinger operators on $\ell^2(\mathbb{Z})$:

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n = Eu_n,$$

and any formal solution $u = (u_n)_{n \in \mathbb{Z}}$ of $H_{V,\alpha,\theta}u = Eu$ satisfies

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = S_E^V(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \forall n \in \mathbb{Z}.$$

For any $f \in \ell^2(\mathbb{Z})$, one can define the spectral measure $\mu_{V,\alpha,\theta}^f$ corresponding to f as in (1.1). Given $E + i\epsilon$ with $E \in \mathbb{R}$ and $\epsilon > 0$, there exists a non-zero solution u^+ of $H_{V,\alpha,\theta}u^+ = (E + i\epsilon)u^+$ being square-summable at $+\infty$. The Wely's m function is given by $m^+ = -\frac{u_1^+}{u_0^+}$.

Let

$$M(E + i\epsilon) = \int_{\mathbb{R}} \frac{1}{E' - (E + i\epsilon)} d\mu_{V,\alpha,\theta}(E').$$

From the definition of $M(\cdot)$, it deduces immediately that $M(\cdot)$ is a Herglotz function defined on $\mathbb{H} = \{z : \Im z > 0\}$, and

$$\Im M(E + i\epsilon) \geq \frac{1}{2\epsilon} \mu_{V,\alpha,\theta}(E - \epsilon, E + \epsilon). \tag{2.1}$$

Recall that in [12], by the usual action of $SL(2, \mathbb{C})$, we denote

$$m_\beta^+ := R_{-\frac{\beta}{2\pi}} \cdot m^+ = \frac{m^+ \cos \beta - \sin \beta}{m^+ \sin \beta + \cos \beta'}$$

then we have the following lemma:

Lemma 2.1 ([5]). *Let $\psi(m^+) := \sup_{\beta} |m_{\beta}^+|$, then $\|M\|_0 \leq \psi(m^+)$.*

The spectral properties of $H_{V,\alpha,\theta}$ and the dynamics of (α, S_E^V) are closely related by the fact: $E \in \Sigma_{V,\alpha}$ if and only if (α, S_E^V) is not uniformly hyperbolic [19].

2.3 Analytic approximation

Assume $f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$, according to [23], there exists a sequence $\{f_j\}_{j \geq 1}$ with $f_j \in C^{\omega}_{\frac{1}{j}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ and a universal constant $C' > 0$, such that

$$\|f_j - f\|_k \rightarrow 0, \quad j \rightarrow +\infty, \tag{2.2a}$$

$$|f_j|_{\frac{1}{j}} \leq C' \|f\|_k, \tag{2.2b}$$

$$|f_{j+1} - f_j|_{\frac{1}{j+1}} \leq C'(j)^{-k} \|f\|_k. \tag{2.2c}$$

2.4 Space decomposition

Given $A \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^d$ and $\eta > 0$, one can obtain the decomposition $\mathcal{B}_k = \mathcal{B}_k^{(nre)}(\eta) \oplus \mathcal{B}_k^{(re)}(\eta)$, where

$$\mathcal{B}_k = \left\{ f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R})) : \|f\|_k < \infty \right\},$$

and $\mathcal{B}_k^{(nre)}(\eta)$ is the subspace of \mathcal{B}_k such that for any $Y \in \mathcal{B}_k^{(nre)}(\eta)$,

$$A^{-1}Y(\theta + \alpha)A \in \mathcal{B}_k^{(nre)}(\eta), \quad \|A^{-1}Y(\theta + \alpha)A - Y(\theta)\|_k > \eta \|Y\|_k.$$

Lemma 2.2 ([10, 17]). *Assume that $A \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^d$, $\varepsilon' \leq (4\|A\|)^{-4}$, and $\eta \geq 13\|A\|^2 \varepsilon'^{\frac{1}{2}}$. For any $g \in \mathcal{B}_k$ with $\|g\|_k \leq \varepsilon'$, there exist $Y \in \mathcal{B}_k$ and $g^{re} \in \mathcal{B}_k^{re}(\eta)$ such that*

$$e^{-Y(\theta+\alpha)} A e^{g(\theta)} e^{Y(\theta)} = A e^{g^{re}(\theta)},$$

with $\|Y\|_k \leq \varepsilon'^{\frac{1}{2}}$, and $\|g^{re}\|_k \leq 2\varepsilon'$.

The continuous version (C^ω linear systems) and discrete version (C^ω cocycles) of Lemma 2.2 are shown in [17] and [10] respectively. The proof of Lemma 2.2 only depends on \mathcal{B}_k being a Banach space, thus one can obtain the result by replacing analytic norm with C^k norm in the Appendix of [10].

3 Dynamical estimates of C^k quasi-periodic cocycles

To deal with the almost reducibility of the C^k cocycles, the strategy used here is treating the analytic cocycles firstly and turning the estimates of analytic cocycles into those of finitely differentiable cocycles by analytic approximation.

Consider the following Q-P $SL(2, \mathbb{R})$ cocycle

$$\begin{aligned} (\alpha, Ae^{f(\theta)}) : \mathbb{T}^d \times \mathbb{R}^2 &\rightarrow \mathbb{T}^d \times \mathbb{R}^2; \\ (\theta, x) &\mapsto (\theta + \alpha, Ae^{f(\theta)} \cdot x), \end{aligned}$$

where $\alpha \in DC_d(\gamma, \tau)$, $A \in SL(2, \mathbb{R})$ and $f(\theta) \in C_r^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ with $r > 0$. Assume $|f|_r \leq \varepsilon$, we are going to show that the perturbation will tend to zero by KAM scheme.

Proposition 3.1 ([10, 20]). *Let $\alpha \in DC_d(\gamma, \tau)$, $\gamma, r > 0, \tau > d - 1$ and $\sigma = \frac{1}{10}$. Then for any $r_+ \in (0, r)$, there exist $c = c(\gamma, \tau, d)$ and a numerical constant D such that if*

$$\varepsilon \leq \frac{c}{\|A\|^D} (r - r_+)^{D\tau}, \tag{3.1}$$

then there exist $B(\theta) \in C_{r_+}^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_+ \in SL(2, \mathbb{R})$ and $f_+(\theta) \in C_{r_+}^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that $(\alpha, Ae^{f(\theta)})$ is conjugated to $(\alpha, A_+e^{f_+(\theta)})$ by $B(\theta)$, i.e.,

$$B(\theta + \alpha)^{-1} Ae^{f(\theta)} B(\theta) = A_+ e^{f_+(\theta)}.$$

More precisely, let $N = \frac{2|\ln \varepsilon|}{r - r_+}$ and $\{e^{2\pi i \rho}, e^{-2\pi i \rho}\}$ be the two eigenvalues of A , we can distinguish between two cases:

(A) (Non-resonant case). Assume that

$$\|2\rho - \langle n, \alpha \rangle\|_{\mathbb{T}} \geq \varepsilon^\sigma, \quad \forall n \in \mathbb{Z}^d \quad \text{with } 0 < |n| \leq N,$$

then we have the estimates:

$$|f_+(\theta)|_{r_+} \leq 4\varepsilon^{3-2\sigma}, \quad |B(\theta) - Id|_{r_+} \leq \varepsilon^{\frac{1}{2}}, \quad \|A_+ - A\| \leq 2\|A\|\varepsilon.$$

(B) (Resonant case). If there exists $n^* \in \mathbb{Z}^d$ with $0 < |n^*| \leq N$ such that

$$\|2\rho - \langle n^*, \alpha \rangle\|_{\mathbb{T}} < \varepsilon^\sigma,$$

then we have the estimates:

$$|B(\theta)|_{r_+} \leq C_1 |n^*|^{\frac{\sigma}{2}} e^{\pi |n^*| r_+}, \quad \|B(\theta)\|_0 \leq C_1 |n^*|^{\frac{\sigma}{2}}, \quad |f_+(\theta)|_{r_+} \ll \varepsilon^{100}, \tag{3.2}$$

where $C_1 = 4\|A\|^{\frac{1}{2}} \gamma^{-\frac{1}{2}}$. Moreover, let $A_+ := e^{A''}$ with $A'' \in sl(2, \mathbb{R})$, then $\|A''\| \leq 16\varepsilon^\sigma$.

3.1 Real quantitative estimates for C^k Q-P cocycles

We are going to deal with the almost reducibility of the C^k cocycles by analytic approximation. Let $\{f_j\}_{j \geq 1}, f_j \in C_{\frac{1}{j}}^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ be the analytic sequence approximating

$f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$. We first recall some notations given in [10]. D is a numerical constant defined by Proposition 3.1, we denote

$$\varepsilon'_0(h, h') := \frac{c}{(2\|A\|)^D} (h - h')^{D\tau},$$

and define

$$\varepsilon_m := \frac{c}{(2\|A\|)^D m^{\frac{k}{4}}}.$$

Then one can check that for any $k \geq 5D\tau$ and any $m \geq 10, m \in \mathbb{Z}$,

$$\frac{c}{(2\|A\|)^D m^{\frac{k}{4}}} \leq \varepsilon'_0\left(\frac{1}{m}, \frac{1}{m^2}\right).$$

Denote $l_j = M^{2^{j-1}}, \forall j \in \mathbb{Z}^+$, where $M > \max\{10, \frac{(2\|A\|)^D}{c}\}$ is an integer.

Theorem 3.1. Let $\alpha \in DC_d(\gamma, \tau)$, $A \in SL(2, \mathbb{R})$, $\sigma = \frac{1}{10}$, $f(\theta) \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$ with $k \geq 5D\tau$. Let $\{f_j\}_{j \geq 1}$ be the analytic sequence approximating $f(\theta)$ defined in (2.2). There exists $\bar{\varepsilon} = \bar{\varepsilon}(\gamma, \tau, d, k, \|A\|)$ such that if $\|f\|_k \leq \bar{\varepsilon}$, then the following holds:

(A) There exist $B_{l_j}(\theta) \in C^{\omega_{\frac{1}{l_j+1}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_j} \in SL(2, \mathbb{R})$ and $f'_{l_j}(\theta) \in C^{\omega_{\frac{1}{l_j+1}}}(\mathbb{T}^d, sl(2, \mathbb{R}))$

such that

$$B_{l_j}(\theta + \alpha)^{-1} A e^{f_{l_j}(\theta)} B_{l_j}(\theta) = A_{l_j} e^{f'_{l_j}(\theta)},$$

with following estimates

$$\begin{aligned} \|B_{l_j}(\theta)\|_0 &\leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(1+\sigma)}, \quad |f'_{l_j}(\theta)|_{\frac{1}{l_j+1}} \leq \frac{1}{2} \varepsilon_{l_j}^{\frac{5}{2}}, \\ |B_{l_j}(\theta)|_{\frac{1}{l_j+1}} &\leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(1+\sigma)} \varepsilon_{l_j}^{-\xi}, \end{aligned}$$

where $\xi \in (\frac{2\pi}{M^2-1}, \frac{1}{2})$ is a constant.

(B) Moreover, $\|A_{l_j}\| \leq 2\|A\|$ and there exist some unitary matrices $U_j \in SL(2, \mathbb{C})$ such that A_{l_j} can be written as

$$A_{l_j} = U_j \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix} U_j^{-1} \quad \text{with } \rho_j \in i\mathbb{R} \cup \mathbb{R} \quad \text{and } c_j \in \mathbb{C}.$$

Then for any constant $\kappa \in [1, \frac{\sigma k}{8\tau})$, there exists constant $C = C(\|A\|) > 0$ such that

$$\|B_{l_j}(\theta)\|_0^\kappa \cdot |c_j| \leq C.$$

Proof. We will prove Theorem 3.1 by the induction process as in [9] and [10].

First Step. Suppose that

$$\|f\|_k \leq \frac{c}{C'(2\|A\|)^D M^{\frac{k}{4}}},$$

where C' is a universal constant defined in (2.2). Thus we have

$$|f_{l_1}|_{\frac{1}{l_1}} \leq \varepsilon_{l_1} \leq \varepsilon'_0 \left(\frac{1}{l_1}, \frac{1}{l_2} \right).$$

By Proposition 3.1, one can find $B_{l_1}(\theta) \in C^{\omega}_{\frac{1}{2}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_1} \in SL(2, \mathbb{R})$ and $f'_{l_1}(\theta) \in C^{\omega}_{\frac{1}{2}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$B_{l_1}(\theta + \alpha)^{-1} A e^{f_{l_1}(\theta)} B_{l_1}(\theta) = A_{l_1} e^{f'_{l_1}(\theta)}.$$

More precisely, let $N_{l_1} = \frac{2|\ln \varepsilon_{l_1}|}{\frac{1}{l_1} - \frac{1}{l_2}}$ and $\{e^{2\pi i \rho}, e^{-2\pi i \rho}\}$ be two eigenvalues of A , we can distinguish two cases:

(Non-resonant case). If the first step is obtained by non-resonant case:

$$\|2\rho - \langle n, \alpha \rangle\|_{\mathbb{T}} \geq \varepsilon_{l_1}^{\sigma}, \quad \forall n \in \mathbb{Z}^d \quad \text{with } 0 < |n| \leq N_{l_1},$$

then we have the estimates:

$$|f'_{l_1}(\theta)|_{\frac{1}{2}} \leq 4\varepsilon_{l_1}^{3-2\sigma}, \quad |B_{l_1}(\theta)|_{\frac{1}{2}} \leq 1 + \varepsilon_{l_1}^{\frac{1}{2}}, \quad \|A_{l_1} - A\| \leq 2\|A\|\varepsilon_{l_1}.$$

(Resonant case). If the first step is obtained by resonant case: there exists $n_{l_1}^* \in \mathbb{Z}^d$ with $0 < |n_{l_1}^*| \leq N_{l_1}$ such that $\|2\rho - \langle n_{l_1}^*, \alpha \rangle\|_{\mathbb{T}} < \varepsilon_{l_1}^{\sigma}$, then

$$\begin{aligned} \|B_{l_1}(\theta)\|_0 &\leq C_2(\gamma, \tau, \|A\|)(l_1 |\ln \varepsilon_{l_1}|)^{\frac{\tau}{2}} \leq (l_1 |\ln \varepsilon_{l_1}|)^{\tau(1+\sigma)}, \\ |B_{l_1}(\theta)|_{\frac{1}{2}} &\leq (l_1 |\ln \varepsilon_{l_1}|)^{\tau(1+\sigma)} \varepsilon_{l_1}^{\frac{-2\tau}{2-\tau}}, \quad |f'_{l_1}(\theta)|_{\frac{1}{2}} \ll \varepsilon_{l_1}^{100} < \frac{1}{2} \varepsilon_{l_1}^{\frac{5}{2}}. \end{aligned}$$

Moreover, let $A_{l_1} := e^{A''_{l_1}}$ with $A''_{l_1} \in sl(2, \mathbb{R})$ we have $\|A''_{l_1}\| \leq 16\varepsilon_{l_1}^{\sigma}$.

Induction Step: Assume that in (l_j) -th step with $j \leq n$, we already have that

$$B_{l_j}(\theta + \alpha)^{-1} A e^{f_{l_j}(\theta)} B_{l_j}(\theta) = A_{l_j} e^{f'_{l_j}(\theta)}$$

with the following estimates

$$|B_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(\sigma+1)} \varepsilon_{l_j}^{-\xi}, \quad \|A_{l_j}\| \leq 2\|A\|, \tag{3.3a}$$

$$\|B_{l_j}\|_0 \leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(\sigma+1)}, \quad |f'_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \leq \frac{1}{2} \varepsilon_{l_j}^{\frac{5}{2}}. \tag{3.3b}$$

Moreover, if the (l_j) -th step is obtained by the resonant case, we have

$$A_{l_j} = e^{A''_{l_j}}, \quad \|A''_{l_j}\| \leq 8\varepsilon_{l_j}^{\sigma}, \quad \|A_{l_j}\| \leq 1 + 16\varepsilon_{l_j}^{\sigma}. \tag{3.4}$$

If the (l_j) -th step is obtained by the non-resonant case, we have

$$\|A_{l_j} - A_{l_{j-1}}\| \leq 2\|A_{l_{j-1}}\|\varepsilon_{l_j}, \quad |B_{l_j}|_{\frac{1}{l_{j+1}}} \leq (1 + \varepsilon_{l_j}^{\frac{1}{2}})|B_{l_{j-1}}|_{\frac{1}{l_j}}. \tag{3.5}$$

Now let $j = n + 1$ and focus on the cocycle $(\alpha, Ae^{f_{n+1}}(\theta))$, it follows that

$$B_{l_n}(\theta + \alpha)^{-1} Ae^{f_{n+1}} B_{l_n}(\theta) = A_{l_n} e^{f_{l_n}} + B_{l_n}(\theta + \alpha)^{-1} (Ae^{f_{l_{n+1}}} - Ae^{f_{l_n}}) B_{l_n}(\theta).$$

If we rewrite

$$A_{l_n} e^{f_{l_n}} + B_{l_n}(\theta + \alpha)^{-1} (Ae^{f_{l_{n+1}}} - Ae^{f_{l_n}}) B_{l_n}(\theta) = A_{l_n} e^{\overline{f_{l_n}}(\theta)},$$

by (3.3) and $\zeta \in (\frac{2\pi}{M^2-1}, \frac{1}{2})$, we have

$$\begin{aligned} |\overline{f_{l_n}}(\theta)|_{\frac{1}{l_{n+1}}} &\leq |f'_{l_n}|_{\frac{1}{l_{n+1}}} + \|A_{l_n}^{-1}\| \cdot |B_{l_n}(\theta + \alpha)^{-1} (Ae^{f_{l_{n+1}}} - Ae^{f_{l_n}}) B_{l_n}|_{\frac{1}{l_{n+1}}} \\ &\leq \frac{1}{2}\varepsilon_{l_n}^{\frac{5}{2}} + 2\|A\|^2 \times (l_n |\ln \varepsilon_{l_n}|)^{2\tau(\sigma+1)} \varepsilon_{l_n}^{-2\zeta} \times \frac{c}{(2\|A\|)^{Dl_n^{k-1}}} \\ &\leq \frac{1}{2}\varepsilon_{l_{n+1}} + \frac{1}{2} \times \frac{c}{(2\|A\|)^{Dl_n^{\frac{k}{2}}}} \leq \varepsilon_{l_{n+1}}. \end{aligned}$$

Apply Proposition 3.1 to the cocycle $(\alpha, A_{l_n} e^{\overline{f_{l_n}}(\theta)})$, one can obtain $\tilde{B}_{l_n}(\theta) \in C^{\omega_{\frac{1}{l_{n+2}}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_{n+1}} \in SL(2, \mathbb{R})$ and $f'_{l_{n+1}}(\theta) \in C^{\omega_{\frac{1}{l_{n+2}}}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$\tilde{B}_{l_n}(\theta + \alpha)^{-1} A_{l_n} e^{\overline{f_{l_n}}(\theta)} \tilde{B}_{l_n}(\theta) = A_{l_{n+1}} e^{f'_{l_{n+1}}(\theta)}.$$

Let $B_{l_{n+1}} := \tilde{B}_{l_n} B_{l_n} \in C^{\omega_{\frac{1}{l_{n+2}}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, and note that whether or not the (l_{n+1}) -th step is in the resonant case, the following estimate holds:

$$|f'_{l_{n+1}}(\theta)|_{\frac{1}{l_{n+2}}} < \frac{1}{2}\varepsilon_{l_{n+1}}^{\frac{5}{2}}. \tag{3.6}$$

We are going to analyze the structure of $A_{l_{n+1}}$ and estimate the norm of the conjugation $B_{l_{n+1}}$ in (l_{n+1}) -th step. Let $A_{l_{n+1}} := e^{A''_{l_{n+1}}}$, and if $\|A''_{l_{n+1}}\|$ is sufficiently small, one can always find some unitary matrices $U \in SL(2, \mathbb{C})$ such that

$$U^{-1} A_{l_{n+1}} U = \begin{pmatrix} e^{2\pi i \rho_{n+1}} & c_{n+1} \\ 0 & e^{-2\pi i \rho_{n+1}} \end{pmatrix}$$

with the estimate $|c_{n+1}| \leq 2\|A''_{l_{n+1}}\|$. Let us focus on the cocycle $(\alpha, A_{l_n} e^{\overline{f_{l_n}}(\theta)})$ and we need to distinguish between two cases.

(Resonant case). If the (l_{n+1}) -th step is obtained by resonant case, i.e., there exists $n_{l_{n+1}}^* \in \mathbb{Z}^d$ such that

$$\|2\rho_{l_n} - \langle n_{l_{n+1}}^*, \alpha \rangle\|_{\mathbb{T}} < \varepsilon_{l_{n+1}}^\sigma, \quad 0 < |n_{l_{n+1}}^*| \leq N_{l_{n+1}} := \frac{2 \ln |\varepsilon_{l_{n+1}}|}{\frac{1}{l_{n+1}} - \frac{1}{l_{n+2}}},$$

where $\{e^{2\pi i \rho_n}, e^{-2\pi i \rho_n}\}$ are two eigenvalues of A_{l_n} . Then by Proposition 3.1,

$$|\tilde{B}_{l_n}|_{\frac{1}{l_{n+2}}} \leq C_3(\gamma, \tau)(l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\frac{\tau}{2}} \times \varepsilon_{l_{n+1}}^{-\frac{2\pi}{l_{n+1}-1}}. \tag{3.7}$$

We also have

$$\|\tilde{B}_{l_n}(\theta)\|_0 \leq C_3(l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\frac{\tau}{2}}.$$

Hence,

$$\begin{aligned} |B_{l_{n+1}}|_{\frac{1}{l_{n+2}}} &\leq C_3(l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\frac{\tau}{2}} \times \varepsilon_{l_{n+1}}^{-\frac{2\pi}{l_{n+1}-1}} \times (l_n |\ln \varepsilon_{l_n}|)^{\tau(\sigma+1)} \varepsilon_{l_n}^{-\xi} \\ &\leq (l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\tau(\sigma+1)} \varepsilon_{l_{n+1}}^{-\xi}, \end{aligned} \tag{3.8a}$$

$$\|B_{l_{n+1}}\|_0 \leq (l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\tau(\sigma+1)}. \tag{3.8b}$$

Moreover one can get that $\|A''_{l_{n+1}}\| \leq 16\varepsilon_{l_{n+1}}^\sigma$, which gives $|c_{n+1}| \leq 2\|A''_{l_{n+1}}\| \leq 32\varepsilon_{l_{n+1}}^\sigma$. Combine with (3.8a), for any $\kappa \in [1, \frac{\sigma k}{8\tau})$, we have

$$\|B_{l_{n+1}}\|_0^\kappa \cdot |c_{n+1}| \leq 32 |\ln \varepsilon_{l_{n+1}}|^{\frac{\kappa \sigma}{4}} \cdot \varepsilon_{l_{n+1}}^{\frac{\sigma(1-\sigma)}{2}} < \infty. \tag{3.9}$$

(Non-resonant case). If the (l_{n+1}) -th step is obtained by non-resonant case, we track back to the nearest resonant step, says the (l_m) -step. If such m does not exist, we deduce that each step is in non-resonant case, thus

$$\|A_{n+1} - A\| \leq \|A_{l_1} - A\| + \sum_{i=1}^n \|A_{l_{i+1}} - A_{l_i}\| \leq 8\|A\|\varepsilon_{l_1},$$

and $|c_{n+1}| \leq \|A_{l_{n+1}}\| \leq 2\|A\|$. Since $B_{l_{n+1}}$ is close to identity by construction, it follows that

$$\|B_{l_{n+1}}\|_0 \leq |B_{l_{n+1}}|_{\frac{1}{l_{n+1}}} \leq 2. \tag{3.10}$$

We deduce that

$$\|B_{l_{n+1}}\|_0^\kappa \cdot |c_{n+1}| \leq 2^{\kappa+1}\|A\| < \infty. \tag{3.11}$$

If such $m < n + 1$ exists, let $A_{l_m} = e^{A''_{l_m}}$, then by (3.3) and (3.4), we have

$$\|A''_{l_m}\| \leq 16\varepsilon_{l_m}^\sigma, \quad \|A_{l_m}\| \leq 1 + 32\varepsilon_{l_m}^\sigma, \quad |B_{l_m}|_{\frac{1}{l_{m+1}}} \leq (l_m |\ln \varepsilon_{l_m}|)^{\tau(\sigma+1)} \varepsilon_{l_m}^{-\xi}.$$

Since each step is in non-resonant case from (l_m) -th step to (l_{n+1}) -th step, it deduces that

$$\|A_{l_{n+1}} - A_{l_m}\| \leq 8\|A\|_{\varepsilon_{l_{m+1}}}, \quad \|A_{l_{n+1}}\| \leq 2\|A\|. \tag{3.12}$$

Moreover, (3.12) also implies

$$|c_{n+1}| \leq 2\|A''_{l_{n+1}}\| \leq 64\varepsilon_{l_m}^\sigma. \tag{3.13}$$

Note that each conjugation is close to identity from (l_m) -step to (l_{n+1}) -step by construction in (3.5), thus one can get that

$$\|B_{l_{n+1}}\|_0 \leq |B_{l_{n+1}}|_{\frac{1}{l_{n+1}}} \leq 2\|B_{l_m}\|_{\frac{1}{l_{m+1}}} \leq (l_{n+1} |\ln \varepsilon_{l_{n+1}}|)^{\tau(\sigma+1)} \varepsilon_{l_{n+1}}^{-\xi}. \tag{3.14}$$

Combine (3.13) with (3.14), it follows that

$$\|B_{l_{n+1}}\|_0^k \cdot |c_{n+1}| \leq 2^{k+6} |\ln \varepsilon_{l_m}|^{\frac{k\sigma}{4}} \cdot \varepsilon_{l_m}^{\frac{\sigma(1-\sigma)}{2}} < \infty. \tag{3.15}$$

This completes the proof of **(A)** by (3.6), (3.8a), (3.10) and (3.14). It also finishes the proof for **(B)** by (3.9), (3.11) and (3.15). \square

Remark 3.1. Theorem 3.1 has been proved in [10] essentially, however for the technical reason, we replace the estimate $\|B_{l_j}\|_0 \leq \varepsilon_{l_j}^{-\frac{\sigma}{4}}$ in Proposition 3.2 of [10] by $\|B_{l_j}\|_0 \leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(\sigma+1)}$, so that one can get $\|B_{l_j}\|_0^k \cdot |c_j| < \infty$.

3.2 Complex almost triangularization for C^k Q-P cocycles

In Theorem 3.1, one can see that the conjugation $B_{l_j} : 2\mathbb{T}^d \rightarrow SL(2, \mathbb{R})$ is real, which results in A_{l_j} being real. In fact, we can choose the complex conjugation $B_{l_j} : 2\mathbb{T}^d \rightarrow SL(2, \mathbb{C})$ to make A_{l_j} complex almost triangularization provided that the cocycle is not uniformly hyperbolic.

Theorem 3.2. *Suppose that all the conditions in Theorem 3.1 hold. Further assume that $(\alpha, Ae^{f(\theta)})$ is not uniformly hyperbolic. There exists $\varepsilon_* = \varepsilon_*(\gamma, \tau, d, k, \|A\|)$ such that if $\|f\|_k \leq \varepsilon_*$, then there exist $\tilde{A}_{l_j} \in SL(2, \mathbb{C}), \tilde{F}_{l_j} \in C^{k_0}(\mathbb{T}^d, SL(2, \mathbb{C}))$ with*

$$k_0 = \left\lceil \frac{k}{20} \right\rceil \quad \text{and} \quad \Phi_{l_j}(\theta) \in C^{\omega_{\frac{1}{l_{j+1}}}}(2\mathbb{T}^d, SL(2, \mathbb{C})),$$

such that

$$\Phi_{l_j}(\theta + \alpha)^{-1} Ae^{f(\theta)} \Phi_{l_j}(\theta) = \tilde{A}_{l_j} + \tilde{F}_{l_j}(\theta),$$

where

$$\tilde{A}_{l_j} = \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}$$

with $\rho_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, also with estimates

$$\|\tilde{F}_{l_j}(\theta)\|_0 \leq 2\varepsilon_{l_j}^{\frac{1}{4}}, \quad \|\Phi_{l_j}(\theta)\|_0 \leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(\sigma+1)}, \quad \|\Phi_{l_j}(\theta)\|_0^\kappa \cdot |c_j| \leq C, \quad (3.16)$$

where C is a constant defined in Theorem 3.1 and $\kappa \in [1, \frac{\sigma k}{8\tau})$.

Proof. Recall that in Theorem 3.1, we already have

$$B_{l_j}(\theta + \alpha)^{-1} A e^{f_{l_j}(\theta)} B_{l_j}(\theta) = A_{l_j} e^{f'_{l_j}(\theta)}, \quad \forall j \in \mathbb{Z}.$$

Denote

$$A_{l_j} + \tilde{F}_{l_j} = A_{l_j} e^{f'_{l_j}(\theta)} + B_{l_j}(\theta + \alpha)^{-1} (A e^f - A e^{f_{l_j}(\theta)}) B_{l_j}(\theta),$$

then

$$B_{l_j}(\theta + \alpha)^{-1} A e^{f_{l_j}(\theta)} B_{l_j}(\theta) = A_{l_j} + \tilde{F}_{l_j}(\theta).$$

By the estimates of **(A)** in Theorem 3.1, we have

$$\begin{aligned} \|\tilde{F}_{l_j}\|_0 &\leq 2\|A_{l_j} f'_{l_j}(\theta)\|_0 + 2\|B_{l_j}(\theta + \alpha)^{-1} A (f(\theta) - f_{l_j}(\theta)) B_{l_j}(\theta)\|_0 \\ &\leq 2\|A\| \varepsilon_{l_j}^{\frac{5}{2}} + 2\|A\| (l_j |\ln \varepsilon_{l_j}|)^{2\tau(\sigma+1)} \times \frac{c}{(2\|A\|)^{D l_j^{k-1}}} \\ &\leq (2\|A\|)^{-1} \varepsilon_{l_j}^2, \end{aligned}$$

where the second step uses the fact

$$\sum_{i \geq j} \|f_{l_{i+1}} - f_{l_i}\| \leq \frac{c}{(2\|A\|)^{D l_j^{k-1}}}$$

by (2.2).

Assume that $\{e^{2\pi i \rho_j}, e^{-2\pi i \rho_j}\}$ are two eigenvalues of A_{l_j} , then there exist unitary $U \in SL(2, \mathbb{C})$, such that

$$U^{-1} A_{l_j} U = \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}$$

with $|c_j| \leq \|A_{l_j}\| \leq 2\|A\|$, where $\rho_j \in i\mathbb{R} \cup \mathbb{R}$ and $c_j \in \mathbb{C}$. In the following, we need to rule out that $i\rho_j \in \mathbb{R} \setminus \{0\}$. Suppose that $\lambda_j = i\rho_j \in \mathbb{R} \setminus \{0\}$. If $2\pi|\rho_j| > \varepsilon_{l_j}^{\frac{1}{4}}$, let $P := \text{diag}\{\|2A\|^{\frac{1}{2}} \varepsilon_{l_j}^{-\frac{1}{2}}, \|2A\|^{-\frac{1}{2}} \varepsilon_{l_j}^{\frac{1}{2}}\}$, then we have

$$P^{-1} U^{-1} (A_{l_j} + \tilde{F}_{l_j}) U P = \begin{pmatrix} e^{2\pi \lambda_j} & 0 \\ 0 & e^{-2\pi \lambda_j} \end{pmatrix} + F(\theta)$$

with $\|F\|_0 \leq 2\varepsilon_{l_j}$. We rewrite

$$\begin{pmatrix} e^{2\pi \lambda_j} & 0 \\ 0 & e^{-2\pi \lambda_j} \end{pmatrix} + F(\theta) = \begin{pmatrix} e^{2\pi \lambda_j} & 0 \\ 0 & e^{-2\pi \lambda_j} \end{pmatrix} e^{\tilde{f}(\theta)} \quad (3.17)$$

with $\|\tilde{f}(\theta)\|_0 \leq 4\|A\|\varepsilon_{l_j}$.

By Lemma 2.2 and Corollary 3.1 of [17], one can conjugate (3.17) to

$$\begin{pmatrix} e^{2\pi\lambda_j} & 0 \\ 0 & e^{-2\pi\lambda_j} \end{pmatrix} \begin{pmatrix} e^{\tilde{f}^{re}(\theta)} & 0 \\ 0 & e^{-\tilde{f}^{re}(\theta)} \end{pmatrix} = \begin{pmatrix} e^{2\pi\lambda_j} e^{\tilde{f}^{re}(\theta)} & 0 \\ 0 & e^{-2\pi\lambda_j} e^{-\tilde{f}^{re}(\theta)} \end{pmatrix}$$

with $\|\tilde{f}^{re}(\theta)\|_0 \leq 8\|A\|\varepsilon_{l_j}$, thus $(\alpha, Ae^{f(\theta)})$ is uniformly hyperbolic, which contradicts to the assumption. Hence we only need to consider

$$2\pi|\rho_j| \leq \varepsilon_{l_j}^{\frac{1}{4}}.$$

In this case, we put ρ_j into the perturbation so that the new perturbation satisfies

$$\|\tilde{F}_{l_j}\|_0 \leq 2\varepsilon_{l_j}^{\frac{1}{4}} \quad \text{and} \quad A_{l_j} = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}.$$

Denote

$$\Phi_{l_j}(\theta) = B_{l_j}(\theta)U \in C^{\omega_{\frac{1}{l_j+1}}}(\mathbb{T}^d, SL(2, \mathbb{C})),$$

we always have

$$\Phi_{l_j}(\theta + \alpha)^{-1} Ae^{f(\theta)} \Phi_{l_j}(\theta) = \begin{pmatrix} e^{2\pi i \rho_j} & c_j \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix} + \tilde{F}_{l_j}(\theta)$$

with $\rho_j \in \mathbb{R}, c_j \in \mathbb{C}$ and $\|\tilde{F}_{l_j}\|_0 \leq 2\varepsilon_{l_j}^{\frac{1}{4}}$. Since U is unitary, by Theorem 3.1(A), (3.16) holds. □

4 Sharp Hölder continuity of the spectral measure

Consider the following discrete C^k Q-P Schrödinger operators:

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad \forall n \in \mathbb{Z}, \tag{4.1}$$

where $\alpha \in DC_d(\gamma, \tau)$ and $V \in C^k(\mathbb{T}^d, \mathbb{R})$. Let $\mu_{V,\alpha,\theta}$ be the spectral measure of $H_{V,\alpha,\theta}$. Based on the dynamical estimates of corresponding Schrödinger cocycles, we are able to show the $\frac{1}{2}$ -Hölder continuity of $\mu_{V,\alpha,\theta}$.

Let $A(\theta) := S_E^V(\theta)$ and for any $n \geq 1$, we define

$$P_n(\theta) = \sum_{s=1}^n A_{2s-1}^*(\theta + \alpha) A_{2s-1}(\theta + \alpha),$$

where $A_s(\theta) = A(\theta + (s-1)\alpha) \cdots A(\theta + \alpha)A(\theta)$. P_n is an increasing family of positive self-adjoint operators. $\|P_n\|$ is unbounded since $\text{tr}P_n \geq 2n$. Moreover, $\det P_n$ is also unbounded. For simplicity, we will use the notation $a \approx b$ which denotes that there exist some constants $C > 0$ such that $C^{-1}a \leq b \leq Ca$ and also use the notation $a \lesssim b$ which denotes $a \leq Cb$ for some constants $C > 0$.

Theorem 4.1. Let $\alpha \in \text{DC}_d(\gamma, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k \geq 5D\tau$ and D is a numerical constant. There exists $\tilde{\varepsilon} = \tilde{\varepsilon}(\gamma, \tau, k)$ such that if $\|V\|_k \leq \tilde{\varepsilon}$, then for any $f \in \ell^2(\mathbb{Z}) \cap \ell^1(\mathbb{Z})$,

$$\mu_{V,\alpha,\theta}^f(J) \leq D_0 \|f\|_{\ell^1}^2 \cdot |J|^{\frac{1}{2}},$$

for all intervals J and all θ , where $D_0 = D_0(V, \alpha) > 0$.

Proof. Since the spectral measure $\mu_{V,\alpha,\theta}$ vanishes on $\mathbb{R} \setminus \Sigma_{V,\alpha}$, we only need to consider the case $E \in \Sigma_{V,\alpha}$.

Rewrite the Schrödinger cocycle (α, S_E^V) as $(\alpha, A_E e^{f(\theta)})$, where

$$A_E = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}, \quad f(\theta) = \ln(\text{Id} + A_E^{-1}F(\theta)), \quad F(\theta) = \begin{pmatrix} -V(\theta + n\alpha) & 0 \\ 0 & 0 \end{pmatrix}.$$

By the assumption on $\|V\|_k$ and Theorem 3.2, for $\kappa \in [1, \frac{\sigma k}{8\tau})$, there exist $\Phi_{l_j} \in C^k(2\mathbb{T}^d, SL(2, \mathbb{C}))$ with $\|\Phi_{l_j}\|_0 \leq (l_j |\ln \varepsilon_{l_j}|)^{\tau(\sigma+1)}$ and $\beta_i \in C^{k_0}(\mathbb{T}^d, \mathbb{R})$, $i = 1, 2, 3, 4$ with $k_0 = \lfloor \frac{k}{20} \rfloor$ such that

$$\Phi_{l_j}(\theta + \alpha)^{-1} S_E^V(\theta) \Phi_{l_j}(\theta) = T_{l_j} + \begin{pmatrix} \beta_1(\theta) & \beta_2(\theta) \\ \beta_3(\theta) & \beta_4(\theta) \end{pmatrix},$$

where

$$T_{l_j} = \begin{pmatrix} e^{2\pi i \rho_j} & \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}$$

with $\rho_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$, also with estimates

$$\|\Phi_{l_j}\|_0^k \cdot |c_j| \leq C, \quad \|\beta_i\|_0 \leq 2\varepsilon_{l_j}^{\frac{1}{4}}, \quad i = 1, 2, 3, 4. \tag{4.2}$$

For simplicity of notations, in the following $\Phi_{l_j}, T_{l_j}, \rho_j$ are written as Φ, T, ρ respectively. Let $\tilde{T}(\theta) = \Phi(\theta + \alpha)^{-1} S_E^V(\theta) \Phi(\theta)$, then we have

$$\|\tilde{T} - T\|_0 \leq 2\varepsilon_{l_j}^{\frac{1}{4}}. \tag{4.3}$$

We need to compare the dynamics between (α, S_E^V) and (α, T) . For this purpose, let $X = \sum_{j=1}^n T_{2j-1}^* T_{2j-1}$ and $\tilde{X}(\theta) = \sum_{j=1}^n \tilde{T}_{2j-1}^*(\theta) \tilde{T}_{2j-1}(\theta)$. The following estimates on $\|X\|_0$ and $\|X^{-1}\|_0^{-1}$ are crucial.

Lemma 4.1 (Lemma 4.3 of [5]). *Let*

$$K(\theta) = \begin{pmatrix} e^{2\pi i \rho} & t(\theta) \\ 0 & e^{-2\pi i \rho} \end{pmatrix},$$

where $t(\theta) = \hat{t}(r) e^{2\pi i r \theta}$. Let $G = \sum_{j=1}^n K_{2j-1}^* K_{2j-1}$, then

$$\begin{aligned} \|G\|_0 &\approx n(1 + |\hat{t}(r)|^2 \min\{n^2, \|2\rho - \langle r, \alpha \rangle\|_{\mathbb{T}}^{-2}\}), \\ \|G^{-1}\|_0^{-1} &\approx n. \end{aligned}$$

Lemma 4.2 (Lemma 4.4 of [5]). *Let G and $\widehat{t}(r)$ be as above. Define $\widetilde{G} = \sum_{j=1}^n \widetilde{K}_{2j-1}^* \widetilde{K}_{2j-1}$. Then there exists $D_1 > 0$ such that if*

$$\|\widetilde{K} - K\|_0 \leq D_1 n^{-2} (1 + 2n|\widehat{t}(r)|)^{-2},$$

we have $\|\widetilde{G} - G\|_0 \leq 1$.

Apply Lemma 4.1, one can get that

$$\|X\|_0 \approx n(1 + |c_j|^2 \min\{n^2, \|2\rho\|_{\mathbb{T}}^{-2}\}), \quad \|X^{-1}\|_0^{-1} \approx n. \tag{4.4}$$

Let n^+ be maximal such that $\|\widetilde{X} - X\|_0 \leq 1$ for $1 \leq n < n^+$. So by Lemma 4.2 and (4.3), one can get that

$$2\varepsilon_{l_j}^{\frac{1}{4}} \geq D_1 (n^+)^{-2} (1 + 2n^+|c_j|)^{-2} \gtrsim (n^+)^{-4}.$$

It follows that

$$n^+ \gtrsim \varepsilon_{l_j}^{\frac{1}{16}}. \tag{4.5}$$

Since $\|\widetilde{X}\|_0 \leq \|X\|_0 + 1$ and $\|\widetilde{X}^{-1}\|_0 \geq \|X^{-1}\|_0 - 1$ for $1 \leq n < n^+$, also we notice that

$$\begin{aligned} \|P\|_n &\leq \|\Phi(\theta)\|_0^4 \cdot \|\widetilde{X}(\theta + \alpha)\|_0, \\ \|P_n^{-1}\|_0 &\geq \|\Phi(\theta)\|_0^{-4} \cdot \|\widetilde{X}(\theta + \alpha)^{-1}\|_0^{-1}. \end{aligned}$$

(4.4) implies

$$\begin{aligned} \|P_n\|_0 &\leq D_2 n(1 + |c_j|^2 n^2) \|\Phi\|_0^4, \\ \|P_n^{-1}\|_0^{-1} &\geq D_3 n \|\Phi\|_0^{-4}. \end{aligned}$$

Hence by direct calculation,

$$\frac{\|P_n\|_0}{\|P_n^{-1}\|_0^{-3}} \leq D_4 |c_j|^2 \cdot \|\Phi\|_0^{16} + D_4 \frac{1}{n^2} \|\Phi\|_0^{16}.$$

From Theorem 3.2, we know that $\|\Phi\|_0^8 \cdot |c_j| < \infty$ and $\|\Phi\|_0 \leq \varepsilon_{l_j}^{-\frac{\sigma}{30}}$, then

$$\|P_n\|_0 \lesssim \|P_n^{-1}\|_0^{-3}, \quad \forall n \in (n^-, n^+),$$

where

$$n^- := \varepsilon_{l_j}^{-\frac{8\sigma}{30}}. \tag{4.6}$$

Denote the interval $I_j := [D_5 l_j^{\frac{k}{150}}, D_6 l_j^{\frac{k}{64}}]$, $j \in \mathbb{Z}^+$, then by (4.5) and (4.6), for any $n \in I_j$, we have $n^- < n < n^+$. Since $I_j \cap I_{j+1} \neq \emptyset, \forall j \in \mathbb{Z}^+$, thus $\cup_{j \geq 1} I_j$ cover all the n tending to

infinity. Hence $\|P_n\|_0 \lesssim \|P_n^{-1}\|_0^{-3}$ for any $n \geq D_5 l_1^{\frac{k}{150}}$. On the other hand, the number of n satisfying $n < D_5 l_1^{\frac{k}{150}}$ is finite, then

$$\sup_{n < D_5 l_1^{\frac{k}{150}}} \frac{\|P_n\|_0}{\|P_n^{-1}\|_0^{-3}} < \infty.$$

Thus

$$\|P_n\| \lesssim \|P_n^{-1}\|^{-3}, \quad \forall n \in \mathbb{Z}^+. \tag{4.7}$$

Lemma 4.3 (Lemma 4.2 of [5]). *Let $\epsilon_n = \frac{1}{2\sqrt{\det P_n}}$, then*

$$D_6^{-1} < \frac{\psi(m^+(E + i\epsilon_n))}{2\epsilon_n \|P_n\|_0} < D_6,$$

where $D_6 > 0$ is a constant and $\psi(m^+)$ is defined in Lemma 2.1.

For any bounded potential and any solution u satisfying (4.1), we have

$$\|u\|_{L+1} \lesssim \|u\|_L, \quad \text{where } \|u\|_L = \left(\sum_{j=1}^L |u_j|^2 \right)^{\frac{1}{2}}.$$

In particular, for the solution u^β with $u_0^\beta \cos \beta + u_1^\beta \sin \beta = 0$ and $|u_0^\beta|^2 + |u_1^\beta|^2 = 1$, we have

$$\det P_n = \inf_{\beta} \|u^\beta\|_L^2 \|u^{\beta+\pi/2}\|_L^2. \tag{4.8}$$

By (4.7) and $P_n : \mathbb{T}^d \rightarrow \mathfrak{gl}(2, \mathbb{R})$, we have

$$\|P_n\|_0 = \det P_n \|P_n^{-1}\|_0 \lesssim \epsilon_n^{-2} \|P_n\|^{-\frac{1}{3}},$$

where ϵ_n is defined in Lemma 4.3. Thus $\|P_n\|_0 \lesssim \epsilon_n^{-\frac{3}{2}}$. According to Lemma 4.3, we deduce that

$$\psi(m^+(E + i\epsilon_n)) \lesssim \epsilon_n \|P_n\|_0 \lesssim \epsilon_n^{-\frac{1}{2}}.$$

Since $\lim_{n \rightarrow \infty} \epsilon_n = 0$, and we also have $\epsilon_n \lesssim \epsilon_{n+1}$ by (4.8), we only need to consider the case of fixing $\epsilon_n = \epsilon$. Combine (2.1) with Lemma 2.1, we have

$$\mu_{V,\alpha,\theta}(E - \epsilon, E + \epsilon) \leq 2\epsilon \Im M(E + i\epsilon) \lesssim \epsilon^{\frac{1}{2}}.$$

Since $\mu_{V,\alpha,\theta} = 0$ on $\mathbb{R} \setminus \Sigma_{V,\alpha}$, then there exists $D_0 = D_0(V, \alpha) > 0$ such that

$$\mu_{V,\alpha,\theta}(J) \leq D_0 |J|^{\frac{1}{2}}, \quad \forall J \subset \mathbb{R}.$$

Let $\sigma : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ be the shift $f(i+1) = \sigma f(i)$. Then $\sigma H_{V,\alpha,\theta} \sigma^{-1} = H_{V,\alpha,\theta+\alpha}$. Thus $\mu_{\theta+\alpha}^{\sigma f} = \mu_{\theta}^f$ and $\mu_{\theta}^{e_k} = \mu_{\theta+k\alpha}^{e_0} \leq \mu_{\theta+k\alpha}$. Let $\mathbb{E}(J)$ be the spectral projection of $H_{V,\alpha,\theta}$ on J , then

$$\begin{aligned} \mu_{V,\alpha,\theta}^f(J) &= \langle \mathbb{E}(J)f, f \rangle = \|\mathbb{E}(J) \sum_k f(k)e_k\|^2 \leq \left(\sum_k |f(k)| \cdot \|\mathbb{E}(J)e_k\| \right)^2 \\ &= \left(\sum_k |f(k)| \mu_{V,\alpha,\theta}^{e_k}(J)^{\frac{1}{2}} \right)^2 \leq \left(\sum_k |f(k)| \mu_{V,\alpha,\theta+k\alpha}(J)^{\frac{1}{2}} \right)^2 \\ &\leq D_0 \|f\|_{\ell^1}^2 \cdot |J|^{\frac{1}{2}}. \end{aligned}$$

Thus, we complete the proof. \square

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