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Existence and Uniqueness of Solution for a Class of Nonlinear Degenerate Elliptic Equations

Albo Carlos Cavalheiro*

Department of Mathematics, State University of Londrina, Londrina, Brazil

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Abstract. In this work we are interested in the existence and uniqueness of solutions for the Navier problem associated to the degenerate nonlinear elliptic equations

$$\begin{aligned}&\Delta\Big[\omega_1(x)|\Delta u|^{p-2}\Delta u+v_1(x)|\Delta u|^{q-2}\Delta u\Big]\\&\quad-\operatorname{div}\Big[\omega_2(x)|\nabla u|^{r-2}\nabla u+v_2(x)|\nabla u|^{s-2}\nabla u\Big]\\=&f(x)-\operatorname{div}(G(x))\quad\text{in }\ \Omega,\end{aligned}$$

in the setting of the weighted Sobolev spaces.

Key Words: Degenerate nonlinear elliptic equation, Weighted Sobolev spaces.
AMS Subject Classifications: 35J60, 35J70

1 Introduction

In this work we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $X=W^{2,p}(\Omega,\omega_1)\cap W_0^{1,r}(\Omega,\omega_2)$ (see Definition 2.4 and Definition 2.5 for the Navier problem

$$(P)\qquad\begin{cases}Lu(x)=f(x)-\operatorname{div}(G(x)) & \text{in }\ \Omega,\\ u(x)=\Delta u(x)=0 & \text{on }\ \partial\Omega,\end{cases}$$

where L is the partial differential operator

$$\begin{aligned}Lu(x)=&\Delta\Big[\omega_1(x)|\Delta u|^{p-2}\Delta u+v_1(x)|\Delta u|^{q-2}\Delta u\Big]\\&\quad-\operatorname{div}\Big[\omega_2(x)|\nabla u|^{r-2}\nabla u+v_2(x)|\nabla u|^{s-2}\nabla u\Big],\end{aligned}$$

*Corresponding author. *Email address:* accava@gmail.com (A. C. Cavalheiro)

where Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2, v_1 and v_2 are four weight functions, Δ is the Laplacian operator and $2 \le q, s < r < p < \infty$.

Let Ω be an open set in $\mathbb{R}^n$. We denote by $\mathcal{W}(\Omega)$ the set of all measurable, a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called weight functions. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ_ω. Thus,

$$\mu_\omega(E) = \int_E \omega(x)dx$$

for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1–3, 5, 10] and [15]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [12]). These classes have found many useful applications in harmonic analysis (see [14]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [11]). There are, in fact, many interesting examples of weights (see [10] for p-admissible weights).

In the non-degenerate case (i.e., with $\omega(x) \equiv 1$), for all $f \in L^p(\Omega)$ the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [9]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [3]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [13] and the references therein), and the degenerated p-Laplacian has been studied in [5]. The problem with degenerated p-Laplacian and p-Biharmonic operators in the case $\omega_1 = \omega_2 = v_1 = v_2$ and $p = q = r = s$

$$\begin{cases} \Delta(\omega(x)|\Delta u|^{p-2}\Delta u) - \operatorname{div}[\omega(x)|\nabla u|^{p-2}\nabla u] = f(x) - \operatorname{div}(G(x)) & \text{in } \Omega, \\ u(x) = \Delta u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied by the author in [2].

The following theorem will be proved in Section 3.

Theorem 1.1. *Let $2 \leq q, s < r < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set. Assume*

(H1) $\omega_1 \in A_p$, $\omega_2 \in A_r$ *and* $v_1, v_2 \in \mathcal{W}(\Omega)$;

(H2) $\frac{v_1}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1)$ *and* $\frac{v_2}{\omega_2} \in L^{r/(r-s)}(\Omega, \omega_2)$;

(H3) $\frac{f}{\omega_2} \in L^{r'}(\Omega, \omega_2)$ *and* $\frac{|G|}{\omega_2} \in L^{r'}(\Omega, \omega_2)$, *where* $G = (g_1, \cdots, g_n)$.

Then the problem (P) has a unique solution $u \in X = W^{2,p}(\Omega, \omega_1) \cap W_0^{1,r}(\Omega, \omega_2)$. Moreover, since $0 < \frac{1}{p} + \frac{1}{r} < 1$, then

$$\|u\|_X \leq C_{p,r}\Big(\frac{M^{p'-1}}{p'} + \frac{M^{r'-1}}{r'}\Big),$$

where

$$M = C_\Omega \|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)},$$

C_Ω is the constant in Theorem 2.2 and $C_{p,r} = pr/(pr - p - r)$.

2 Definitions and basic results

Definition 2.1. Let $1 \leq p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant $C = C(p, \omega)$ such that, for every ball $B \subset \mathbb{R}^n$

$$\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)dx\Big)^{p-1} \leq C, \qquad \text{if } p > 1,$$

$$\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\operatorname*{ess\,sup}_{x\in B} \frac{1}{\omega(x)}\Big) \leq C, \qquad \text{if } p = 1,$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

If $1 < q \leq p$, then $A_q \subset A_p$ (see [8,10] or [14] for more information about A_p-weights). We say the weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x;2r)) \leq C\mu(B(x;r))$ for every ball $B = B(x;r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [10]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [14]).

Definition 2.2. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \Big(\int_\Omega |f(x)|^p \omega(x)dx\Big)^{1/p} < \infty.$$

We denote $[L^p(\Omega,\omega)]^n = L^p(\Omega,\omega)\times\cdots\times L^p(\Omega,\omega)$.

If $\omega\in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\rm loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [15]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.3. Let $\Omega\subset\mathbb{R}^n$ be open, k be a nonnegative integer and $\omega\in A_p$ $(1 < p < \infty)$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u\in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u\in L^p(\Omega,\omega)$ for $1 \le |\alpha| \le k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\omega(x)dx + \sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p\omega(x)dx\right)^{1/p}. \tag{2.1}$$

We also define $W^{k,p}_0(\Omega,\omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega,\omega)}$.

If $\omega\in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Theorem 2.1.4 in [15]). The spaces $W^{k,p}(\Omega,\omega)$ and $W^{k,p}_0(\Omega,\omega)$ are Banach spaces.

It is evident that the weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x\in\Omega$ (c_1 and c_2 positive constants), gives nothing new (the space $W^{k,p}_0(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}_0(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish somewhere in $\Omega\cup\partial\Omega$ or increase to infinity (or both).

The space $W^{1,p}_0(\Omega,\omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). The dual space of $W^{1,p}_0(\Omega,\omega)$ is the space

$$\begin{aligned}&[W^{1,p}_0(\Omega,\omega]^* = W^{-1,p'}(\Omega,\omega)\\ =&\left\{T = f - \mathrm{div}G : G = (g_1,\cdots,g_n), \frac{f}{\omega}, \frac{g_j}{\omega}\in L^{p'}(\Omega,\omega),\ j = 1,\cdots,n\right\}.\end{aligned}$$

In this article we use the following results.

Lemma 2.1. *Let* $1 < p < \infty$.

(a) There exists a constant $\alpha_p > 0$ *such that*

$$\big||x|^{p-2}x - |y|^{p-2}y\big| \le \alpha_p|x-y|(|x|+|y|)^{p-2}, \quad \forall x,y\in\mathbb{R}^n;$$

(b) There exist two positive constants β_p, γ_p *such that for every* $x,y\in\mathbb{R}^n$

$$\beta_p(|x|+|y|)^{p-2}|x-y|^2 \le (|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) \le \gamma_p(|x|+|y|)^{p-2}|x-y|^2.$$

Proof. See [4], Proposition 17.2 and Proposition 17.3. □

Lemma 2.2. *If $\omega\in A_p$, then*

$$\left(\frac{|E|}{|B|}\right)^p \le C_{p,\omega}\frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (where $\mu(E)=\int_E \omega(x)dx$).

Proof. See Theorem 15.5 Strong doubling of A_p-weights in [10]. □

By Lemma 2.2, if $\omega\in A_p$, then $\mu(E)=0$ if and only if $|E|=0$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Theorem 2.1. *Let $\omega\in A_p$, $1<p<\infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m\to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi\in L^p(\Omega,\omega)$ such that*

(i) $u_{m_k}(x)\to u(x)$, $m_k\to\infty$, *a.e. on* Ω;

(ii) $|u_{m_k}(x)|\le\Phi(x)$, *a.e. on* Ω.

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [7]. □

Theorem 2.2 (The weighted Sobolev inequality). *Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega\in A_p$, $1<p<\infty$. There exist positive constants C_Ω and δ such that for all $u\in W^{1,p}_0(\Omega,\omega)$ and all θ satisfying $1\le\theta\le n/(n-1)+\delta$,*

$$\|u\|_{L^{p\theta}(\Omega,\omega)}\le C_\Omega\|\nabla u\|_{L^p(\Omega,\omega)}, \tag{2.2}$$

where C_Ω depends only on n, p, the A_p-constant $C(p,\omega)$ of ω and the diameter of Ω.

Proof. Its suffices to prove the inequality for functions $u\in C^\infty_0(\Omega)$ (see Theorem 1.3 in [6]). To extend the estimates (2.2) to arbitrary $u\in W^{1,p}_0(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C^\infty_0(\Omega)$ functions tending to u in $W^{1,p}_0(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1}-u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^p(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

Definition 2.4. We denote by $X=W^{2,p}(\Omega,\omega_1)\cap W^{1,r}_0(\Omega,\omega_2)$ with the norm

$$\|u\|_X=\||\nabla u|\|_{L^r(\Omega,\omega_2)}+\|\Delta u\|_{L^p(\Omega,\omega_1)}.$$

Definition 2.5. We say that an element $u\in X$ is a (weak) solution of problem (P) if, for all $\varphi\in X$,

$$\begin{aligned}&\int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi\omega_1 dx+\int_\Omega |\Delta u|^{q-2}\Delta u\Delta\varphi v_1 dx\\ &\quad+\int_\Omega|\nabla u|^{r-2}\langle\nabla u,\nabla\varphi\rangle\omega_2 dx+\int_\Omega|\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle v_2 dx\\ =&\int_\Omega f(x)\varphi(x)dx+\int_\Omega\langle G,\nabla\varphi(x)\rangle dx.\end{aligned}$$

Remark 2.1. Since $2 \le q, s < r < p < \infty$, if

$$\frac{v_1}{\omega_1} \in L^{p/(p-q)}(\Omega, \omega_1) \quad \text{and} \quad \frac{v_2}{\omega_2} \in L^{r/(r-s)}(\Omega, \omega_2),$$

there exist two constants $M_1 > 0$, $M_2 > 0$ such that

$$\|u\|_{L^q(\Omega, v_1)} \le M_1 \|u\|_{L^p(\Omega, \omega_1)} \quad \text{and} \quad \|u\|_{L^s(\Omega, v_2)} \le M_2 \|u\|_{L^r(\Omega, \omega_2)},$$

where

$$M_1 = \left[\int_\Omega \left(\frac{v_1}{\omega_1} \right)^{p/(p-q)} \omega_1 dx \right]^{(p-q)/pq} = \|v_1/\omega_1\|^{1/q}_{L^{p/(p-q)}(\Omega, \omega_1)},$$

$$M_2 = \left[\int_\Omega \left(\frac{v_2}{\omega_2} \right)^{r/(r-s)} \omega_2 dx \right]^{(r-s)/rs} = \|v_2/\omega_2\|^{1/s}_{L^{r/(r-s)}(\Omega, \omega_2)}.$$

In fact, since $2 \le q, s < r < p < \infty$, we have $\theta = p/q > 1$ and $\theta' = p/(p-q)$,

$$\begin{aligned} \|u\|^q_{L^q(\Omega, v_1)} &= \int_\Omega |u|^q v_1 dx = \int_\Omega |u|^q \frac{v_1}{\omega_1} \omega_1 dx \\ &\le \left(\int_\Omega |u|^{q\theta} \omega_1 dx \right)^{1/\theta} \left(\int_\Omega \left(\frac{v_1}{\omega_1} \right)^{\theta'} \omega_1 dx \right)^{1/\theta'} \\ &= \left(\int_\Omega |u|^p \omega_1 dx \right)^{q/p} \left(\int_\Omega \left(\frac{v_1}{\omega_1} \right)^{p/(p-q)} \omega_1 dx \right)^{(p-q)/p}. \end{aligned}$$

Hence, $\|u\|_{L^q(\Omega, v_1)} \le M_1 \|u\|_{L^p(\Omega, \omega_1)}$.

Analogously, we obtain

$$\|u\|_{L^s(\Omega, v_2)} \le M_2 \|u\|_{L^r(\Omega, \omega_2)}.$$

Remark 2.2. In this paper, we will use many times the following Convergence Principle in Banach spaces: Let X be a Banach space, $x \in X$ and a sequence $\{x_n\}$ in X. If every subsequence of $\{x_n\}$ has, in turn, a subsequence which converges strongly to x, then the original sequence converges strongly to x, i.e., $x_n \to x$ as $n \to \infty$ (see [16], Proposition 10.13).

3 Proof of Theorem 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then for each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$.*

Proof. See Theorem 26.A in [17]. □

To prove the existence of solutions, we define $B, B_1, B_2, B_3, B_4 : X\times X\to\mathbb{R}$ and $T : X\to\mathbb{R}$ by

$$\begin{aligned} B(u,\varphi) &= B_1(u,\varphi)+B_2(u,\varphi)+B_3(u,\varphi)+B_4(u,\varphi),\\ B_1(u,\varphi) &= \int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi\omega_1 dx,\\ B_2(u,\varphi) &= \int_\Omega |\Delta u|^{q-2}\Delta u\Delta\varphi v_1 dx,\\ B_3(u,\varphi) &= \int_\Omega |\nabla u|^{r-2}\langle\nabla u,\nabla\varphi\rangle\omega_2 dx,\\ B_4(u,\varphi) &= \int_\Omega |\nabla u|^{s-2}\langle\nabla u,\nabla\varphi\rangle v_2 dx,\\ T(\varphi) &= \int_\Omega f\varphi dx+\int_\Omega \langle G,\nabla\varphi\rangle dx. \end{aligned}$$

Then $u\in X$ is a (weak) solution to problem (P) if for all $\varphi\in X$

$$B(u,\varphi)=B_1(u,\varphi)+B_2(u,\varphi)+B_3(u,\varphi)+B_4(u,\varphi)=T(\varphi).$$

Step 1. We define the operator $F_1 : X\to L^{p'}(\Omega,\omega_1)$ by

$$(F_1u)(x)=|\Delta u(x)|^{p-2}\Delta u(x).$$

We now show that operator F_1 is bounded and continuous.
(i) We have

$$\begin{aligned} \|F_1u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_1u(x)|^{p'}\omega_1 dx=\int_\Omega \big||\Delta u|^{p-2}\Delta u\big|^{p'}\omega_1 dx\\ &= \int_\Omega |\Delta u|^p\omega_1 dx=\|\Delta u\|^p_{L^p(\Omega,\omega_1)}\le \|u\|^p_X. \end{aligned} \tag{3.1}$$

Therefore, by (3.1) we obtain

$$\|F_1u\|_{L^{p'}(\Omega,\omega_1)}\le \|u\|_X^{p-1}, \tag{3.2}$$

and hence the boundedness.
(ii) Let $u_m\to u$ in X as $m\to\infty$. We need to show that $F_1u_m\to F_1u$ in $L^{p'}(\Omega,\omega_1)$. If $u_m\to u$ in X, then $|\Delta u_m|\to|\Delta u|$ in $L^p(\Omega,\omega_1)$ and $|\nabla u_m|\to|\nabla u|$ in $L^r(\Omega,\omega_2)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and two functions $\Phi_1\in L^p(\Omega,\omega_1)$ and $\Phi_2\in L^r(\Omega,\omega_2)$ such that

$$|\nabla u_{m_k}(x)|\to|\nabla u(x)| \quad \text{a.e. in } \Omega, \tag{3.3a}$$
$$|\nabla u_{m_k}(x)|\le\Phi_2(x) \quad \text{a.e. in } \Omega, \tag{3.3b}$$
$$|\Delta u_{m_k}(x)|\to|\Delta u(x)| \quad \text{a.e. in } \Omega, \tag{3.3c}$$
$$|\Delta u_{m_k}(x)|\le\Phi_1(x) \quad \text{a.e. in } \Omega. \tag{3.3d}$$

Now, since $p>2$, using (3.3c), (3.3d), $a=p/p'=p-1$ and $a'=(p-1)/(p-2)$, there exists a constant $\alpha_p>0$ (by Lemma 2.1(a)) such that

$$\begin{aligned}
&\|F_1u_{m_k}-F_1u\|^{p'}_{L^{p'}(\Omega,\omega_1)}=\int_\Omega |F_1u_{m_k}-F_1u|^{p'}\omega_1dx\\
=&\int_\Omega \Big||\Delta u_{m_k}|^{p-2}\Delta u_{m_k}-|\Delta u|^{p-2}\Delta u\Big|^{p'}\omega_1dx\\
\le&\int_\Omega \Big[\alpha_p|\Delta u_{m_k}-\Delta u|(|\Delta u_{m_k}|+|\Delta u|)^{p-2}\Big]^{p'}\omega_1dx\\
\le&\alpha_p^{p'}\int_\Omega |\Delta u_{m_k}-\Delta u|^{p'}(2\Phi_1)^{(p-2)p'}\omega_1dx\\
=&2^{(p-2)p'}\alpha_p^{p'}\int_\Omega |\Delta u_{m_k}-\Delta u|^{p'}\Phi_1^{(p-2)p'}\omega_1dx\\
\le&2^{(p-2)p'}\alpha_p^{p'}\Big(\int_\Omega |\Delta u_{m_k}-\Delta u|^{p'a}\omega_1dx\Big)^{1/a}\Big(\int_\Omega \Phi_1^{(p-2)p'a'}\omega_1dx\Big)^{1/a'}\\
=&2^{(p-2)p'}\alpha_p^{p'}\Big(\int_\Omega |\Delta u_{m_k}-\Delta u|^{p}\omega_1dx\Big)^{p'/p}\Big(\int_\Omega \Phi_1^{p}\omega_1dx\Big)^{(p-2)/(p-1)}\\
=&2^{(p-2)p'}\alpha_p^{p'}\|\Delta u_{m_k}-\Delta u\|^{p'}_{L^p(\Omega,\omega_1)}\|\Phi_1\|^{p'(p-2)}_{L^p(\Omega,\omega_1)}\\
\le&2^{(p-2)p'}\alpha_p^{p'}\|u_{m_k}-u\|^{p'}_X\|\Phi_1\|^{p'(p-2)}_{L^p(\Omega,\omega_1)}.
\end{aligned}$$

Hence,

$$\|F_1u_{m_k}-F_1u\|_{L^{p'}(\Omega,\omega)}\le 2^{p-2}\alpha_p\|u_{m_k}-u\|_X\|\Phi_1\|^{p-2}_{L^p(\Omega,\omega_1)}.$$

Therefore (since $2<p<\infty$), we obtain $\|F_1u_{m_k}-F_1u\|_{L^{p'}(\Omega,\omega_1)}\to 0$, that is,

$$F_1u_{m_k}\to F_1u \quad \text{in } L^{p'}(\Omega,\omega_1).$$

By the Convergence Principle in Banach spaces (see Remark 2.2), we have

$$F_1u_m\to F_1u \quad \text{in } L^{p'}(\Omega,\omega_1). \tag{3.4}$$

Step 2. Define the operator $F_2: X\to L^{q'}(\Omega,v_1)$, $(F_2u)(x)=|\Delta u(x)|^{q-2}\Delta u(x)$. We also have that the operator F_2 is continuous and bounded. In fact:

(i) If $q>2$, we have by Remark 2.1

$$\begin{aligned}
\|F_2u\|^{q'}_{L^{q'}(\Omega,v_1)}=&\int_\Omega \big||\Delta u|^{q-2}\Delta u\big|^{q'}v_1dx=\int_\Omega |\Delta u|^q v_1dx\\
=&\|\Delta u\|^q_{L^q(\Omega,v_1)}\le M_1^q\|\Delta u\|^q_{L^p(\Omega,\omega_1)}\le M_1^q\|u\|^q_X.
\end{aligned}$$

Hence,

$$\|F_2u\|_{L^{q'}(\Omega,v_1)}\le M_1^{q-1}\|u\|^{q-1}_X.$$

(ii) Now using (3.3c), (3.3d), Remark 2.1, $b=q/q'=q-1$ and $b'=(q-1)/(q-2)$ (if $q>2$), there exists a constant $\alpha_q>0$ (by Lemma 2.1(a)) such that

$$\begin{aligned}
&\|F_2u_{m_k}-F_2u\|^{q'}_{L^{q'}(\Omega,v_1)}=\int_\Omega |F_2u_{m_k}-F_2u|^{q'}v_1dx\\
=&\int_\Omega \Big||\Delta u_{m_k}|^{q-2}\Delta u_{m_k}-|\Delta u|^{q-2}\Delta u\Big|^{q'}v_1dx\\
\le&\int_\Omega \Big[\alpha_q|\Delta u_{m_k}-\Delta u|(|\Delta u_{m_k}|+|\Delta u|)^{(q-2)}\Big]^{q'}v_1dx\\
\le&\alpha_q^{q'}\int_\Omega |\Delta u_{m_k}-\Delta u|^{q'}(2\Phi_1)^{(q-2)q'}v_1dx\\
\le&2^{(q-2)q'}\alpha_q^{q'}\Big(\int_\Omega |\Delta u_{m_k}-\Delta u|^{q'b}v_1dx\Big)^{1/b}\Big(\int_\Omega \Phi_1^{(q-2)q'b'}v_1dx\Big)^{1/b'}\\
=&\alpha_q^{q'}2^{(q-2)q'}\Big(\int_\Omega |\Delta u_{m_k}-\Delta u|^{q}v_1dx\Big)^{q'/q}\Big(\int_\Omega \Phi_1^{q}vdx\Big)^{(q-2)/(q-1)}\\
=&\alpha_q^{q'}2^{(q-2)q'}\|\Delta u_{m_k}-\Delta u\|^{q'}_{L^q(\Omega,v_1)}\|\Phi_1\|^{q'(q-2)}_{L^q(\Omega,v_1)}\\
\le&\alpha_q^{q'}2^{(q-2)q'}M_1^{q'}\|\Delta u_{m_k}-\Delta u\|^{q'}_{L^p(\Omega,\omega_1)}M_1^{q'(q-2)}\|\Phi_1\|^{q'(q-2)}_{L^p(\Omega,\omega_1)}\\
\le&\alpha_q^{q'}2^{(q-2)q'}M_1^{q}\|u_{m_k}-u\|^{q'}_X\|\Phi_1\|^{q'(q-2)}_{L^p(\Omega,\omega_1)}.
\end{aligned}$$

Hence,

$$\|F_2u_{m_k}-F_2u\|_{L^{q'}(\Omega,v_1)}\le 2^{q-2}\alpha_qM_1^{q-1}\|\Phi_1\|^{q-2}_{L^p(\Omega,\omega_1)}\|u_{m_k}-u\|_X.$$

In the case $q=2$, we have $(F_2u)(x)=\Delta u(x)$. Hence,

$$\begin{aligned}
&\|F_2u\|_{L^2(\Omega,v_1)}=\|\Delta u\|_{L^2(\Omega,v_1)}\le M_1\|\Delta u\|_{L^p(\Omega,\omega_1)}\le M_1\|u\|_X,\\
&\|F_2u_{m_k}-F_2u\|_{L^2(\Omega,v_1)}\le M_1\|\Delta u_{m_k}-\Delta u\|_{L^p(\Omega,\omega_1)}\le M_1\|u_{m_k}-u\|_X.
\end{aligned}$$

Therefore, for $2\le q<\infty$, we obtain $\|F_2u_{m_k}-F_2u\|_{L^{q'}(\Omega,v_1)}\to 0$, that is,

$$F_2u_{m_k}\to F_2u \quad \text{in } L^{q'}(\Omega,v_1).$$

By the Convergence Principle in Banach spaces (see Remark 2.2), we have

$$F_2u_m\to F_2u \quad \text{in } L^{q'}(\Omega,v_1). \tag{3.5}$$

Step 3. We define $F_3: X\to[L^{r'}(\Omega,\omega_2)]^n$ by $(F_3u)(x)=|\nabla u(x)|^{r-2}\nabla u(x)$. We also have that the operator F_3 is continuous and bounded. In fact, we have

(i) if $u\in X$, since $r>2$, we have

$$\|F_3u\|^{r'}_{L^{r'}(\Omega,\omega_2)}=\int_\Omega |F_3u|^{r'}\omega_2dx=\int_\Omega \Big||\nabla u|^{r-2}\nabla u\Big|^{r'}\omega_2dx$$
$$=\int_\Omega |\nabla u|^{(r-1)r'}\omega_2dx=\int_\Omega |\nabla u|^{r}\omega_2dx=\||\nabla u|\|^{r}_{L^{r}(\Omega,\omega_2)}\le \|u\|^r_X.$$

Hence,

$$\|F_3u\|_{L^{r'}(\Omega,\omega_2)}\le \|u\|^{r-1}_X.$$

(ii) If $u_m\to u$ in X, then $|\nabla u_m|\to|\nabla u|$ in $L^r(\Omega,\omega_2)$. Using Theorem 2.1, there exists a subsequence $\{u_{m_k}\}$ and $\Phi_2\in L^r(\Omega,\omega_2)$ such that

$$|\nabla u_{m_k}|\to|\nabla u| \qquad \text{a.e in } \Omega, \tag{3.6a}$$
$$|\nabla u_{m_k}|\le \Phi_2 \qquad \text{a.e in } \Omega. \tag{3.6b}$$

Hence, using (3.6a), (3.10), $\gamma=r/r'=r-1$ and $\gamma'=(r-1)/(r-2)$, there exists a constant $\alpha_r>0$ (by Lemma 2.1(a)) such that

$$\begin{aligned}
&\|F_3u_{m_k}-F_3u\|^{r'}_{L^{r'}(\Omega,\omega_2)}=\int_\Omega |F_3u_{m_k}-F_3u|^{r'}\omega_2dx\\
=&\int_\Omega \Big||\nabla u_{m_k}|^{r-2}\nabla u_{m_k}-|\nabla u|^{r-2}\nabla u\Big|^{r'}\omega_2dx\\
\le&\int_\Omega \Big[\alpha_r|\nabla u_{m_k}-\nabla u|(|\nabla u_{m_k}|+|\nabla u|)^{r-2}\Big]^{r'}\omega_2dx\\
\le&\alpha_r^{r'}\int_\Omega |\nabla u_{m_k}-\nabla u|^{r'}(2\Phi_2)^{(r-2)r'}\omega_2dx\\
\le&2^{(r-2)r'}\alpha_r^{r'}\Big(\int_\Omega |\nabla u_{m_k}-\nabla u|^{r'\gamma}\omega_2dx\Big)^{1/\gamma}\Big(\int_\Omega \Phi_2^{(r-2)r'\gamma'}\omega_2dx\Big)^{1/\gamma'}\\
=&2^{(r-2)r'}\alpha_r^{r'}\|\nabla u_{m_k}-\nabla u\|^{r'}_{L^r(\Omega,\omega_2)}\|\Phi_2\|^{(r-2)r'}_{L^r(\Omega,\omega_2)}\\
\le&2^{(r-2)r'}\alpha_r^{r'}\|u_{m_k}-u\|^{r'}_X\|\Phi_2\|^{(r-2)r'}_{L^r(\Omega,\omega_2)}.
\end{aligned}$$

Hence,

$$\|F_3u_{m_k}-F_3u\|_{L^{r'}(\Omega,\omega_2)}\le 2^{r-2}\alpha_r\|\Phi_2\|^{r-2}_{L^r(\Omega,\omega_2)}\|u_{m_k}-u\|_X.$$

Therefore (since $2<r<\infty$), we obtain $\|F_3u_{m_k}-F_3u\|_{L^{r'}(\Omega,\omega_2)}\to0$ as $m_k\to\infty$, that is, $F_3u_{m_k}\to F_u$ in $L^{r'}(\Omega,\omega_2)$. By the Convergence Principle in Banach spaces (see Remark 2.2), we obtain

$$F_3u_m\to F_3u \quad \text{in } L^{r'}(\Omega,\omega_2). \tag{3.7}$$

Step 4. We define $F_4: X\to[L^{s'}(\Omega,v_2)]^n$ by $(F_4u)(x)=|\nabla u(x)|^{s-2}\nabla u(x)$. We also have that the operator F_4 is continuous and bounded. In fact, we have

(i) If $s>2$ and $u\in X$, by Remark 2.1, we obtain

$$\begin{aligned}\|F_4u\|^{s'}_{L^{s'}(\Omega,v_2)} &= \int_\Omega |F_4u|^{s'}v_2dx = \int_\Omega \Big||\nabla u|^{s-2}\nabla u\Big|^{s'} v_2dx\\ &= \int_\Omega |\nabla u|^{(s-1)s'} v_2dx = \int_\Omega |\nabla u|^{s} v_2dx = \||\nabla u|\|^s_{L^s(\Omega,v_2)}\\ &\le M_2^s\||\nabla u|\|^s_{L^r(\Omega,\omega_2)} \le M_2^s\|u\|^s_X.\end{aligned}$$

Hence,

$$\|F_4u\|_{L^{s'}(\Omega,v_2)} \le M_2^{s-1}\|u\|^{s-1}_X.$$

(ii) If $u_m\to u$ in X, then $|\nabla u_m|\to|\nabla u|$ in $L^r(\Omega,\omega_2)$. Using Theorem 2.1, there exists a subsequence $\{u_{m_k}\}$ and $\Phi_2\in L^r(\Omega,\omega_2)$ such that

$$|\nabla u_{m_k}|\to|\nabla u| \qquad \text{a.e in } \Omega, \tag{3.8a}$$

$$|\nabla u_{m_k}| \le \Phi_2 \qquad \text{a.e in } \Omega. \tag{3.8b}$$

Hence, using (3.8a), (3.8b), $\eta=s/s'=s-1$ and $\eta'=(s-1)/(s-2)$, there exists a constant $\alpha_s>0$ (by Lemma 2.1(a)) such that

$$\begin{aligned}\|F_4u_{m_k}-F_4u\|^{s'}_{Ls'(\Omega,v_2)} &= \int_\Omega |F_4u_{m_k}-F_4u|^{s'}v_2dx\\ &= \int_\Omega \Big||\nabla u_{m_k}|^{s-2}\nabla u_{m_k}-|\nabla u|^{s-2}\nabla u\Big|^{s'}v_2dx\\ &\le \int_\Omega \Big[\alpha_s|\nabla u_{m_k}-\nabla u|(|\nabla u_{m_k}|+|\nabla u|)^{s-2}\Big]^{s'}v_2dx\\ &\le \alpha_s^{s'}\int_\Omega |\nabla u_{m_k}-\nabla u|^{s'}(2\Phi_2)^{(s-2)s'}v_2dx\\ &\le 2^{(s-2)s'}\alpha_s^{s'}\left(\int_\Omega |\nabla u_{m_k}-\nabla u|^{s'\eta}v_2dx\right)^{1/\eta}\left(\int_\Omega \Phi_2^{(s-2)s'\eta'}v_2dx\right)^{1/\eta'}\\ &= 2^{(s-2)s'}\alpha_s^{s'}\||\nabla u_{m_k}-\nabla u|\|^{s'}_{L^s(\Omega,v_2)}\|\Phi_2\|^{(s-2)s'}_{L^s(\Omega,v_2)}\\ &\le 2^{(s-2)s'}\alpha_s^{s'}M_2^{s'}\||\nabla u_{m_k}-\nabla u|\|^{s'}_{L^r(\Omega,\omega_2)}M_2^{s'(s-2)}\|\Phi_2\|^{s'(s-2)}_{L^r(\Omega,\omega_2)}\\ &\le 2^{(s-2)s'}\alpha_s^{s'}M_2^{s'(s-1)}\|u_{m_k}-u\|^{s'}_X\|\Phi_2\|^{s'(s-2)}_{L^r(\Omega,\omega_2)}.\end{aligned}$$

Hence,

$$\|F_4u_{m_k}-F_4u\|_{L^{s'}(\Omega,v_2)} \le 2^{s-2}\alpha_sM_2^{s-1}\|\Phi_2\|^{s-2}_{L^r(\Omega,\omega_2)}\|u_{m_k}-u\|_X.$$

In case $s=2$, we have $(F_4(u))(x)=\nabla u$. Then

$$\begin{aligned}&\|F_4u\|_{L^2(\Omega,v_2)} = \||\nabla u|\|_{L^2(\Omega,v_2)} \le M_2\||\nabla u|\|_{L^2(\Omega,\omega_2)} \le M_2\|u\|_X,\\ &\|F_4u_{m_k}-F_4u\|_{L^2(\Omega,v_2)} \le M_2\|u_{m_k}-u\|_X.\end{aligned}$$

Therefore (for $2 \le s < \infty$), we obtain that $\|F_4u_{m_k} - F_4u\|_{L^{s'}(\Omega,v_2)} \to 0$ as $m_k \to \infty$, that is, $F_4u_{m_k} \to F_4u$ in $L^{s'}(\Omega, v_2)$. By the Convergence Principle in Banach spaces (see Remark 2.2) we obtain

$$F_4u_m \to F_4u \quad \text{in } L^{s'}(\Omega, v_2). \tag{3.9}$$

Step 5. We also have, by (H3) and Theorem 2.2,

$$\begin{aligned}
|T(\varphi)| \le &\int_\Omega |f||\varphi|dx + \int_\Omega |G||\nabla\varphi|dx \\
= &\int_\Omega \frac{|f|}{\omega_2}|\varphi|\omega_2 dx + \int_\Omega \frac{|G|}{\omega_2}|\nabla\varphi|\omega_2 dx \\
\le &\|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)}\|\varphi\|_{L^r(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)}\||\nabla\varphi|\|_{L^r(\Omega,\omega_2)} \\
\le &C_\Omega\|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)}\||\nabla\varphi|\|_{L^r(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)}\||\nabla\varphi|\|_{L^r(\Omega,\omega_2)} \\
\le &\Big(C_\Omega\|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)}\Big)\|\varphi\|_X,
\end{aligned}$$

and $T \in [W_0^{1,r}(\Omega,\omega_2)]^* \subset X^*$ (i.e., $T \in X^*$). Moreover, we also have for all $u, \varphi \in X$

$$|B(u,\varphi)| \le |B_1(u,\varphi)| + |B_2(u,\varphi)| + |B_3(u,\varphi)| + |B_4(u,\varphi)|. \tag{3.10}$$

Note that by Definition 2.4, we have

$$\begin{aligned}
|B_1(u,\varphi)| \le &\int_\Omega |\Delta u|^{p-1}|\Delta\varphi|\omega_1 dx \\
\le &\Big(\int_\Omega |\Delta u|^{(p-1)p'}\omega_1 dx\Big)^{1/p'}\Big(\int_\Omega |\Delta\varphi|^p\omega_1 dx\Big)^{1/p} \\
= &\|\Delta u\|^{p-1}_{L^p(\Omega,\omega_1)}\|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \le \|u\|_X^{p-1}\|\varphi\|_X,
\end{aligned}$$

by Remark 2.1

$$\begin{aligned}
|B_2(u,\varphi)| \le &\int_\Omega |\Delta u|^{q-1}|\Delta\varphi|v_1 dx \\
\le &\Big(\int_\Omega |\Delta u|^{(q-1)q'}v_1 dx\Big)^{1/q'}\Big(\int_\Omega |\Delta\varphi|^q v_1 dx\Big)^{1/q} \\
= &\|\Delta u\|^{q-1}_{L^q(\Omega,v_1)}\|\Delta\varphi\|_{L^q(\Omega,v_1)} \\
\le &M_1^{q-1}\|\Delta u\|^{q-1}_{L^p(\Omega,\omega_1)}M_1\|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \le M_1^q\|u\|_X^{q-1}\|\varphi\|_X,
\end{aligned}$$

and by Definition 2.4,

$$\begin{aligned}|B_3(u,\varphi)| \leq & \int_\Omega |\nabla u|^{r-1}|\nabla\varphi|\omega_2 dx\\ \leq & \Big(\int_\Omega |\nabla u|^{(r-1)r'}\omega_2 dx\Big)^{1/r'}\Big(\int_\Omega |\nabla\varphi|^{r}\omega_2 dx\Big)^{1/r}\\ = & \||\nabla u|\|^{r-1}_{L^r(\Omega,\omega_2)}\||\nabla\varphi|\|_{L^r(\Omega,\omega_2)} \leq \|u\|^{r-1}_X\|\varphi\|_X.\end{aligned}$$

Moreover, by Remark 2.1,

$$\begin{aligned}|B_4(u,\varphi) \leq & \int_\Omega |\nabla u|^{s-1}|\nabla\varphi|v_2 dx\\ \leq & \Big(\int_\Omega |\nabla u|^{(s-1)s'}v_2 dx\Big)^{1/s'}\Big(\int_\Omega |\nabla\varphi|^{s}v_2 dx\Big)^{1/s}\\ = & \||\nabla u|\|^{s-1}_{L^s(\Omega,v_2)}\||\nabla\varphi|\|_{L^s(\Omega,v_2)}\\ \leq & M_2^{s-1}\||\nabla u|\|^{s-1}_{L^r(\Omega,\omega_2)}M_2\||\nabla\varphi|\|_{L^r(\Omega,\omega_2)} \leq M_2^s\|u\|^{s-1}_x\|\varphi\|_X.\end{aligned}$$

Consequently, we obtain in (3.10) that

$$|B(u,\varphi)| \leq \Big(\|u\|^{p-1}_X + M_1^q\|u\|^{q-1}_X + \|u\|^{r-1}_X + M_2^s\|u\|^{s-1}_X\Big)\|\varphi\|_X.$$

Since $B(u,\cdot)$ is linear, for each $u\in X$, there exists a linear and continuous functional on X denoted by Au such that $\langle Au,\varphi\rangle = B(u,\varphi)$, for all $u,\varphi\in X$ (where $\langle f,x\rangle$ denotes the value of the linear functional f at the point x). Moreover,

$$\|Au\|_* \leq \|u\|^{p-1}_X + M_1^q\|u\|^{q-1}_X + \|u\|^{r-1}_X + M_2^s\|u\|^{s-1}_X, \tag{3.11}$$

where $\|Au\|_* = \sup\{|\langle Au,\varphi\rangle| = |B(u,\varphi)| : \varphi\in X,\ \|\varphi\|_X = 1\}$ is the norm of the operator Au.

Hence, we obtain the operator

$$\begin{aligned}A : X & \to X^*,\\ u & \mapsto Au.\end{aligned}$$

Consequently, problem (P) is equivalent to the operator equation

$$Au = T, \quad u\in X.$$

Step 6. If $u_1, u_2 \in X$ we have by Lemma 2.1(b) and $2 \leq q, s < r < p < \infty$,

$$\begin{aligned}
&\langle Au_1 - Au_2, u_1 - u_2\rangle = B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2)\\
=&\int_\Omega \Big(|\Delta u_1|^{p-1}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\Big)\Delta(u_1 - u_2)\omega_1 dx\\
&+\int_\Omega \Big(|\Delta u_1|^{q-1}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2\Big)\Delta(u_1 - u_2)v_1 dx\\
&+\int_\Omega \langle|\nabla u_1|^{r-1}\nabla u_1 - |\nabla u_2|^{r-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle\omega_2 dx\\
&+\int_\Omega \langle|\nabla u_1|^{s-1}\nabla u_1 - |\nabla u_2|^{s-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle v_2 dx\\
\geq&\beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2}|\Delta u_1 - \Delta u_2|^2\omega_1 dx\\
&+\beta_q \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{q-2}|\Delta u_1 - \Delta u_2|^2 v_1 dx\\
&+\beta_r \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{r-2}|\nabla u_1 - \nabla u_2|^2\omega_2 dx\\
&+\beta_s \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{s-2}|\nabla u_1 - \nabla u_2|^2 v_2 dx\\
\geq&\beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2|\Big)^{p-2}|\Delta u_1 - \Delta u_2|^2\omega_1 dx\\
&+\beta_r \int_\Omega \Big(|\nabla u_1| + |\nabla u_2|\Big)^{r-2}|\nabla u_1 - \nabla u_2|^2\omega_2 dx\\
\geq&\beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2}|\Delta u_1 - \Delta u_2|^2\omega_1 dx\\
&+\beta_r \int_\Omega |\nabla u_1 - \nabla u_2|^{r-2}|\nabla u_1 - \nabla u_2|^2\omega_2 dx\\
=&\beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^p\omega_1 dx + \beta_r \int_\Omega |\nabla u_1 - \nabla u_2|^r\omega dx \geq 0.
\end{aligned}$$

Therefore, the operator A is monotone. Moreover, we have

$$\begin{aligned}
&\langle Au, u\rangle = B(u,u)\\
=&B_1(u,u) + B_2(u,u) + B_3(u,u) + B_4(u,u)\\
=&\int_\Omega |\Delta u|^p\omega_1 dx + \int_\Omega |\Delta u|^q v_1 dx + \int_\Omega |\nabla u|^r\omega_2 dx + \int_\Omega |\nabla u|^s v_2 dx\\
\geq&\int_\Omega |\Delta u|^p\omega_1 dx + \int_\Omega |\nabla u|^r\omega_2 dx\\
=&\|\Delta u\|^p_{L^p(\Omega,\omega_2)} + \||\nabla u|\|^r_{L^r(\Omega,\omega_2)}.
\end{aligned}$$

Hence, since $2 < r < p < \infty$, we have

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to +\infty \quad \text{as} \quad \|u\|_X \to +\infty,$$

that is, A is coercive (using that $\lim\limits_{t+a\to\infty} \dfrac{t^p + a^r}{t+a} = \infty$, with $t > 0$ and $a > 0$).

Step 7. We need to show that the operator A is continuous. Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}
&|B_1(u_m, \varphi) - B_1(u, \varphi)| \\
\leq &\int_\Omega \Big| |\Delta u_m|^{p-2}\Delta u_m - |\Delta u_m|^{p-2}\Delta u \Big| |\Delta \varphi| \omega_1 dx \\
= &\int_\Omega |F_1 u_m - F_1 u| |\Delta\varphi| \omega_1 dx \\
\leq &\|F_1 u_m - F_1 u\|_{L^{p'}(\Omega,\omega_1)} \|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\
\leq &\|F_1 u_m - F_1 u\|_{L^{p'}(\Omega,\omega_1)} \|\varphi\|_X,
\end{aligned}$$

and by Remark 2.1,

$$\begin{aligned}
&|B_2(u_m, \varphi) - B_2(u, \varphi)| \\
\leq &\int_\Omega \Big| |\Delta u_m|^{q-2}\Delta u_m - |\Delta u|^{q-2}\Delta u \Big| |\Delta \varphi| v_1 dx \\
= &\int_\Omega |F_2 u_m - F_2 u| |\Delta\varphi| v_1 dx \\
\leq &\|F_2 u_m - F_2 u\|_{L^{q'}(\Omega,v_1)} \|\Delta\varphi\|_{L^q(\Omega,v_1)} \\
\leq &M_1 \|F_2 u_m - F_2 u\|_{L^{q'}(\Omega,v_1)} \|\Delta\varphi\|_{L^p(\Omega,\omega_1)} \\
\leq &M_1 \|F_2 u_m - F_2 u\|_{L^{q'}(\Omega,\omega_1)} \|\varphi\|_X,
\end{aligned}$$

and

$$\begin{aligned}
&|B_3(u_m, \varphi) - B_3(u, \varphi)| \\
\leq &\int_\Omega \Big| |\nabla u_m|^{r-2}\nabla u_m - |\nabla u|^{r-2}\nabla u \Big| |\nabla \varphi| \omega_2 dx \\
= &\int_\Omega |F_3 u_m - F_3 u| |\nabla\varphi| \omega_2 dx \\
\leq &\|F_3 u_m - F_3 u\|_{L^{r'}(\Omega,\omega_2)} \|\nabla\varphi\|_{L^r(\Omega,\omega_2)} \\
\leq &\|F_3 u_m - F_3 u\|_{L^{r'}(\Omega,\omega_2)} \|\varphi\|_X,
\end{aligned}$$

and by Remark 2.1

$$\begin{aligned}
&|B_4(u_m,\varphi)-B_4(u,\varphi)|\\
\leq&\int_\Omega \Big||\nabla u_m|^{s-2}\nabla u_m-|\nabla u|^{s-2}\nabla u\Big||\nabla\varphi|v_2dx\\
=&\int_\Omega |F_4u_m-F_4u||\nabla\varphi|v_2dx\\
\leq&\|F_4u_m-F_4u\|_{L^{s'}(\Omega,v_2)}\||\nabla\varphi|\|_{L^s(\Omega,v_2)}\\
\leq&M_2\|F_4u_m-F_4u\|_{L^{s'}(\Omega,v_2)}\||\nabla\varphi|\|_{L^r(\Omega,\omega_2)}\\
\leq&M_2\|F_4u_m-F_4u\|_{L^{s'}(\Omega,v_2)}\|\varphi\|_X.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|B(u_m,\varphi)-B(u,\varphi)|\\
\leq&|B_1(u_m,\varphi)-B_1(u,\varphi)|+|B_2(u_m,\varphi)-B_2(u,\varphi)|\\
&\quad+|B_3(u_m,\varphi)-B_3(u,\varphi)|+|B_4(u_m,\varphi)-B_4(u,\varphi)|\\
\leq&\Big(\|F_1u_m-F_1u\|_{L^{p'}(\Omega,\omega_1)}+M_1\|F_2u_m-F_2u\|_{L^{q'}(\Omega,v_1)}\\
&\quad+\|F_3u_m-F_3u\|_{L^{r'}(\Omega,\omega_2)}+M_2\|F_4u_m-F_4u\|_{L^{s'}(\Omega,v_2)}\Big)\|\varphi\|_X.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\|Au_m-Au\|_*\\
\leq&\|F_1u_m-F_1u\|_{L^{p'}(\Omega,\omega_1)}+M_1\|F_2u_m-F_2u\|_{L^{q'}(\Omega,v_1)}\\
&\quad+\|F_3u_m-F_3u\|_{L^{r'}(\Omega,\omega_2)}+M_2\|F_4u_m-F_4u\|_{L^{s'}(\Omega,v_2)}.
\end{aligned}$$

Therefore, using (3.4), (3.5), (3.7) and (3.9), we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous (and this implies that A is hemicontinuous).

Therefore, by Theorem 3.1, the operator equation $Au=T$ has a solution $u\in X$ and it is a solution for problem (P).

Step 8. Let us now prove the uniqueness of the solution. Suppose that $u_1,u_2\in X$ are two solutions of problem (P). Then

$$\begin{aligned}
&\int_\Omega |\Delta u_i|^{p-2}\Delta u_i\Delta\varphi\omega_1dx+\int_\Omega|\Delta u_i|^{q-2}\Delta u_i\Delta\varphi v_1dx\\
&\quad+\int_\Omega|\nabla u_i|^{r-2}\langle\nabla u_i,\nabla\varphi\rangle\omega_2dx+\int_\Omega|\nabla u_i|^{s-2}\langle\nabla u_i,\nabla\varphi\rangle v_2dx\\
=&\int_\Omega f\varphi dx+\int_\Omega\langle G,\nabla\varphi\rangle dx,
\end{aligned}$$

for all $\varphi\in X$, and $i=1,2$. Hence,

$$\begin{aligned}
&\int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-1}\Delta u_2 \Big)\Delta\varphi\,\omega_1 dx \\
&\quad + \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-1}\Delta u_2 \Big)\Delta\varphi\, v_1 dx \\
&\quad + \int_\Omega \langle |\nabla u_1|^{r-2}\nabla u_1 - |\nabla u_2|^{r-2}\nabla u_2, \nabla\varphi\rangle \omega_2 dx \\
&\quad + \int_\Omega \langle |\nabla u_1|^{s-2}\nabla u_1 - |\nabla u_2|^{s-2}\nabla u_2, \nabla\varphi\rangle v_2 dx = 0.
\end{aligned}$$

Therefore, we obtain for $\varphi = u_1 - u_2$ and by Lemma 2.1(b) and $2<r<p<\infty$

$$\begin{aligned}
0 =& \int_\Omega \Big(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2 \Big)\Delta(u_1-u_2)\omega_1 dx \\
&+ \int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2 \Big)\Delta(u_1-u_2) v_1 dx \\
&+ \int_\Omega \langle |\nabla u_1|^{r-2}\nabla u_1 - |\nabla u_2|^{r-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle \omega_2 dx \\
&+ \int_\Omega \langle |\nabla u_1|^{s-2}\nabla u_1 - |\nabla u_2|^{s-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle v_2 dx \\
\geq& \beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2| \Big)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 dx \\
&+ \beta_q \int_\Omega \Big(|\Delta u_1| + |\Delta u_2| \Big)^{q-2} |\Delta u_1 - \Delta u_2|^2 v_1 dx \\
&+ \beta_r \int_\Omega \Big(|\nabla u_1| + |\nabla u_2| \Big)^{r-2} |\nabla u_1 - \nabla u_2|^2 \omega_2 dx \\
&+ \beta_s \int_\Omega \Big(|\nabla u_1| + |\nabla u_2| \Big)^{s-2} |\nabla u_1 - \nabla u_2|^2 v_2 dx \\
\geq& \beta_p \int_\Omega \Big(|\Delta u_1| + |\Delta u_2| \Big)^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 dx \\
&+ \beta_r \int_\Omega \Big(|\nabla u_1| + |\nabla u_2| \Big)^{r-2} |\nabla u_1 - \nabla u_2|^2 \omega_2 dx \\
\geq& \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^{p-2} |\Delta u_1 - \Delta u_2|^2 \omega_1 dx \\
&+ \beta_r \int_\Omega |\nabla u_1 - \nabla u_2|^{r-2} |\nabla u_1 - \nabla u_2|^2 \omega_2 dx \\
=& \beta_p \int_\Omega |\Delta u_1 - \Delta u_2|^p \omega_1 dx + \beta_r \int_\Omega |\nabla u_1 - \nabla u_2|^r \omega_2 dx.
\end{aligned}$$

Hence

$$\|\nabla u_1 - \nabla u_2\|_{L^r(\Omega,\omega_2)} = \|\Delta u_1 - \Delta u_2\|_{L^p(\Omega,\omega_1)} = 0.$$

Since $u_1, u_2 \in X$, then $u_1 = u_2$ a.e..

Step 9. Estimate for $\|u\|_X$. Since $u\in X$ is solution for problem (P), then for $\varphi = u$ in Definition 2.5, we have

$$\begin{aligned}&\int_\Omega |\Delta u|^p \omega_1 dx + \int_\Omega |\Delta u|^q v_1 dx + \int_\Omega |\nabla u|^r \omega_2 dx + \int_\Omega |\nabla u|^s v_2 dx \\ =&\int_\Omega f u dx + \int_\Omega \langle G, \nabla u\rangle dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_\Omega |\Delta u|^p \omega_1 dx + \int_\Omega |\nabla u|^r \omega_2 dx \\ \le&\int_\Omega |\Delta u|^p \omega_1 dx + \int_\Omega |\Delta u|^q v_1 dx + \int_\Omega |\nabla u|^r \omega_2 dx + \int_\Omega |\nabla u|^s v_2 dx \\ =&\int_\Omega f u dx + \int_\Omega \langle G, \nabla u\rangle dx \\ \le& C_\Omega \|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)} \||\nabla u|\|_{L^r(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)} \||\nabla u|\|_{L^r(\Omega,\omega_2)} \\ \le& \Big(C_\Omega \|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)}\Big) \|u\|_X \\ =& M\|u\|_X,\end{aligned}$$

where

$$M = C_\Omega \|f/\omega_2\|_{L^{r'}(\Omega,\omega_2)} + \||G|/\omega_2\|_{L^{r'}(\Omega,\omega_2)}.$$

With that we obtain

$$\int_\Omega |\Delta u|^p \omega_1 dx \le M\|u\|_X \quad \text{and} \quad \int_\Omega |\nabla u|^r \omega_2 dx \le M\|u\|_X.$$

Therefore, by Young's inequality, we obtain

$$\begin{aligned}\|u\|_X =&\|\Delta u\|_{L^p(\Omega,\omega_1)} + \||\nabla u|\|_{L^r(\Omega,\omega_2)} \\ \le& M^{1/p}\|u\|_X^{1/p} + M^{1/r}\|u\|_X^{1/r} \\ \le& \frac{M^{p'/p}}{p'} + \frac{\|u\|_X}{p} + \frac{M^{r'/r}}{r'} + \frac{\|u\|_X}{r} \\ =& \frac{M^{p'/p}}{p'} + \frac{M^{r'/r}}{r'} + \left(\frac{1}{p} + \frac{1}{r}\right)\|u\|_X.\end{aligned}$$

Hence, since

$$0 < \frac{1}{p} + \frac{1}{r} < 1,$$

we obtain

$$\|u\|_X \leq C_{p,r}\left(\frac{M^{p'-1}}{p'}+\frac{M^{r'-1}}{r'}\right),$$

where $C_{p,r}=pr/(pr-p-r)$. This completes the proof of Theorem 1.1. □

Example 3.1. Let $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. Consider $q=s=2$, $r=3$ and $p=4$. Let $\omega_1(x,y)=(x^2+y^2)^{5/2}$, $\omega_2(x,y)=(x^2+y^2)^{-1/2}$, $v_1(x,y)=(x^2+y^2)^2$ and $v_2(x,y)=(x^2+y^2)^{-1/3}$ (we have $\omega_1\in A_4$ and $\omega_2\in A_3$). Let us consider the partial differential operator

$$Lu(x,y)=\Delta\Big[\omega_1(x,y)|\Delta u|^2\Delta u+v_1(x,y)\Delta u\Big]-\operatorname{div}\Big[\omega_2(x,y)|\nabla u|\nabla u+v_2(x,y)\nabla u\Big].$$

Therefore, by Theorem 1.1, the problem

$$(P)\quad\begin{cases} Lu(x)=\dfrac{\cos(xy)}{\sqrt{x^2+y^2}}-\dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{\sqrt{x^2+y^2}}\right) & \text{in } \Omega,\\ u(x)=\Delta u(x)=0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u\in X=W^{2,4}(\Omega,\omega_1)\cap W_0^{1,3}(\Omega,\omega_2)$.

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