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Stability of Viscoelastic Wave Equation with Structural δ-Evolution in $\mathbb{R}^n$

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Abstract. The aim of this paper is to study the Cauchy problem for the viscoelastic wave equation for structural δ-evolution models. By using the energy method in the Fourier spaces, we obtain the decay estimates of the solution to considered problem.

Key Words: Viscoelatic wave equation, Fourier transform, Lyapunov functions, Decay rates.
AMS Subject Classifications: 35L05, 35B35

1 Introduction

Cavalcanti et al. [3] studied the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t g(t-s)\Delta u(s)ds - \gamma\Delta u_t = 0, \tag{1.1}$$

with $x \in \Omega$, $t > 0$, $\rho > 0$. They proved a global existence result for $\gamma \geq 0$, and an exponential decay for $\gamma > 0$. This last result has been extended to a situation, where a source term is competing with the strong damping mechanism and the one induced by the viscosity. For more details see [9]. The authors combined well known methods with perturbation techniques to show that a solution with positive small energy exists globally and decay to the rest state exponentially.

In any spaces dimension, the paper [13] treated the viscoelastic wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.2}$$

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where g, being positive and nonincreasing, is the relaxation function which describes the material in consideration and $u_0 = u(x,0)$ and $u_1 = u_t(x,0)$ are given data. By using the energy method in the Fourier spaces, the general decay estimates of the solution is shown. In [17], the author considered the following equation

$$\rho(x)\left(|u'|^{q-2}u'\right)' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t g(t-s)\Delta_x u(s)ds = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.3}$$

where $q, n \geq 2$ and M is a positive C^1-function satisfying for

$$s \geq 0, \quad m_0 > 0, \quad m_1 \geq 0, \quad \gamma \geq 1, \quad M(s) = m_0 + m_1 s^{\gamma}.$$

In order to compensate the lack of Poincaré's inequality in $\mathbb{R}^n$ and for wider class of relaxation functions, the author used the weighted spaces to establish a very general decay rate of solutions of viscoelastic wave equations in Kirchhoff-type.

In [5], the author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincaré's inequality. The same problem treated in [5], was considered in [7], where it id considered a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. The used conditions on the relaxation function g and its derivative g' are different from the usual conditions.

Recently, in [8], the authors considered the weak-viscoelastic case in the following problem,

$$u'' - \Delta u - \Delta u' + \alpha(t)\int_0^t g(t-s)\Delta u(s,x)ds = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_*^+, \tag{1.4}$$

where $n \geq 2$. The energy decay results were established for weak-viscoelastic wave equation in $\mathbb{R}^n$, which depends on the behavior of both α and g. The main idea of the proof was to construct an appropriate Lyapunov function of the system obtained after taking the Fourier transform.

To extend previous results, we study the decay rate of the solution to the Cauchy problem for structural damped δ-evolution with memory term in Fourier spaces

$$u_{tt} + (-\Delta)^{\delta} u - \int_0^t g(t-s)(-\Delta)^{\delta} u(s)ds = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.5}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{and} \quad \delta > 1. \tag{1.6}$$

The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.

2 Preliminaries and assumptions and Asymptotic stability

We assume that the function g satisfies the following conditions:
A1: $g:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying,

$$g(0)>0, \quad 1-\int_0^\infty g(t)dt = k>0, \quad g'(t)\le 0, \quad \forall t\in\mathbb{R}^+. \tag{2.1}$$

A2: There exists a positive nonincreasing differentiable function $\alpha(t)$ satisfying

$$g'(t)+\alpha(t)g(t)\le 0, \quad \forall t\in\mathbb{R}^+, \tag{2.2}$$

and

$$\alpha(t)>0, \quad \alpha'(t)\le 0, \quad \forall t>0. \tag{2.3}$$

For any real (complex)-valued function $h(t)$, we define

$$(g*h)(t)=\int_0^t g(t-\tau)h(\tau)d\tau,$$
$$(g\circ h)(t)=\int_0^t g(t-\tau)|h(t)-h(\tau)|^2 d\tau.$$

For a later use, we have the following lemma, which is useful in obtaining our estimate of solutions in the Fourier space.

Lemma 2.1 ([8]). *For any $f\in C^1(\mathbb{R}^+)$ and any $h\in H^1(\mathbb{R}^+)$, we have*

$$\begin{aligned} Re(f*h)(t)\bar{h}_t(t) = &-\frac{1}{2}f(t)|h(t)|^2+\frac{1}{2}(f'\circ h)(t) \\ &-\frac{1}{2}\frac{d}{dt}\left\{(f\circ h)(t)-\left(\int_0^t f(\tau)d\tau\right)|h(t)|^2\right\}, \end{aligned} \tag{2.4}$$

and

$$\left|\int_0^t f(t-s)(h(s)-h(t))ds\right|^2 \le \int_0^t |f(s)|ds \int_0^t |f|(t-s)|h(t)-h(s)|^2 ds. \tag{2.5}$$

The following Lemma is very useful in the sequel.

Lemma 2.2 ([15]). *Assume that $\beta(t)>0$ for all $t\ge 0$, $l,p\ge 2$. Then we have*

$$\left\| |\xi|^l \exp\left\{-c|\xi|^2\int_0^t \beta(\tau)d\tau\right\}\right\|_{L^p} \le C\left(1+\int_0^t \beta(\tau)d\tau\right)^{-\frac{l}{2}-\frac{N}{2p}},$$

where C is a positive constant.

Our aim is to obtain the decay estimates of the solution to the problem (1.5)-(1.6). After Fourier transformation of (1.5)-(1.6), we get the following system

$$\begin{cases} \hat{u}_{tt} + |\xi|^{2\delta}\Big(\hat{u} - \displaystyle\int_0^t g(t-s)\hat{u}(s)ds\Big) = 0, & \xi \in \mathbb{R}^n, \quad t \geq 0, \\ \hat{u}(\xi,0) = \hat{u}_0(x), \quad \hat{u}_t(\xi,0) = \hat{u}_1(x), & \xi \in \mathbb{R}^n. \end{cases} \tag{2.6}$$

The energy associated to (2.6) is given by

$$\hat{E}(\xi,t) = \frac{1}{2}\left\{|\hat{u}_t|^2 + |\xi|^{2\delta}|\hat{u}|^2\right\}(\xi,t). \tag{2.7}$$

Lemma 2.3. *The modified energy associated to* (2.6) *is defined as follows.*

$$\hat{\mathcal{E}}(\xi,t) = \frac{1}{2}\left\{|\hat{u}_t|^2 + \left(1 - \int_0^t g(s)ds\right)|\xi|^{2\delta}|\hat{u}|^2 + |\xi|^{2\delta}(g\circ\hat{u})\right\}(\xi,t), \tag{2.8}$$

and the modified energy $\hat{\mathcal{E}}(\xi,t)$ is non-increasing and satisfies for all $t \geq 0$

$$\frac{d\hat{\mathcal{E}}(\xi,t)}{dt} = |\xi|^{2\delta}(g'\circ\hat{u})(\xi,t) - |\xi|^{2\delta}g(t)|\hat{u}(\xi,t)|^2 \leq 0. \tag{2.9}$$

Proof. Multiplying the equation in (2.6) by $\bar{\hat{u}}_t$ and taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}\left\{|\hat{u}_t(\xi,t)|^2 + |\xi|^{2\delta}|\hat{u}(\xi,t)|^2\right\} = Re\left\{|\xi|^{2\delta}\bar{\hat{u}}_t\int_0^t g(s)\hat{u}(\xi,t-s)ds\right\}. \tag{2.10}$$

Using Lemma 2.1, we easily see that

$$\begin{aligned} &Re\left\{|\xi|^{2\delta}\bar{\hat{u}}_t\int_0^t g(s)\hat{u}(\xi,t-s)ds\right\} \\ =&-\frac{1}{2}|\xi|^{2\delta}g(t)|\hat{u}(\xi,t)|^2 + \frac{1}{2}|\xi|^{2\delta}(g'\circ\hat{u})(\xi,t) \\ &-|\xi|^{2\delta}\frac{1}{2}\frac{d}{dt}\left\{(g\circ\hat{u})(\xi,t) - \left(\int_0^t g(\tau)d\tau\right)|\hat{u}(\xi,t)|^2\right\}. \end{aligned}$$

Substituting this last equality into (2.10), the identity (2.9) then follows.

On the other hand, using (**A1**) there exists a positive constant $c_1 > 0$ such that, for all $t \geq 0$ and for all $\xi \in \mathbb{R}^n$, we have

$$\hat{E}(\xi,t) \leq c_1\hat{\mathcal{E}}(\xi,t). \tag{2.11}$$

Thus, we complete the proof. □

We will prove the following exponential stability result.

Proposition 2.1. *Let $\hat{u}(\xi,t)$ be the solution of* (2.6). *Then there exist two positive constants C, β_0 such that*

$$\hat{E}(\xi,t) \le \hat{E}(\xi,0)\exp\left\{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_0^t \alpha(s)ds\right\}, \quad \forall t\ge t_0>0. \tag{2.12}$$

To prove Proposition 2.1, the key point is to apply the multiplier techniques in Fourier spaces in order to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functional.

Proof. Multiplying (2.6) by $\bar{\hat{u}}$ and taking the real part, we get

$$\begin{aligned}&\frac{d}{dt}Re(\hat{u}_t\bar{\hat{u}}) + \left(1-\int_0^t g(s)ds\right)|\xi|^{2\delta}|\hat{u}|^2 - |\hat{u}_t|^2\\ &\quad + |\xi|^{2\delta}Re\left\{\bar{\hat{u}}(t)\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right\} = 0.\end{aligned}$$

Applying Young's inequality, we obtain, for any $\varepsilon_1>0$,

$$\begin{aligned}&\frac{d}{dt}Re(\hat{u}_t\bar{\hat{u}}) + \left(1-\varepsilon_1-\int_0^t g(s)ds\right)|\xi|^{2\delta}|\hat{u}|^2 - |\hat{u}_t|^2\\ \le&\frac{1}{4\varepsilon_1}|\xi|^{2\delta}(1-k)(g\circ\hat{u})(\xi,t).\end{aligned} \tag{2.13}$$

Next, the existence of the memory term forces us to make the first modification of the energy by multiplying (2.6) by $-\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds$ and taking the real part, we have that

$$\begin{aligned}&-\frac{d}{dt}Re\left(\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) + Re\left(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(st))ds\right)\\ &+|\hat{u}_t|^2\int_0^t g(s)ds - |\xi|^{2\delta}Re\left(\hat{u}(t)\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)\\ &+|\xi|^{2\delta}Re\left\{\left(\hat{u}\int_0^t g(t-s)ds\right)\left(\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)\right\}\\ &-|\xi|^{2\delta}\left|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right|^2 = 0.\end{aligned}$$

Young's inequality gives, for any $\varepsilon_2>0$,

$$\begin{aligned}&-\frac{d}{dt}Re\left(\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) + |\hat{u}_t|^2\int_0^t g(s)ds\\ \le&\varepsilon_2|\hat{u}_t|^2 - \frac{g(0)}{4\varepsilon_2}\int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2ds + \varepsilon_2(2-k)|\xi|^{2\delta}|\hat{u}|^2\\ &+\left(|\xi|^{2\delta}\left\{\frac{(1-k)}{4\varepsilon_2}+(1-k)^2+\frac{1}{4\varepsilon_2}\right\}\right)(g\circ\hat{u})(t), \quad \forall t\ge 0.\end{aligned} \tag{2.14}$$

Then, for any $t \geq t_0$, the estimate (2.14) can be rewritten as follows

$$\begin{aligned}&-\frac{d}{dt}Re\left(\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)+\left(\int_0^{t_0} g(s)ds-\varepsilon_2\right)|\hat{u}_t|^2\\ \leq&\varepsilon_2(2-k)|\xi|^{2\delta}|\hat{u}|^2-\frac{g(0)}{4\varepsilon_2}(g'\circ\hat{u})(t)+|\xi|^{2\delta}\left\{\frac{(2-k)}{4\varepsilon_2}+(1-k)^2\right\}(g\circ\hat{u})(t). \qquad (2.15)\end{aligned}$$

Now, we define the functional $\mathcal{L}(\xi,t)$ as follows.

$$\mathcal{L}(\xi,t)=\frac{g_0}{2}Re(\hat{u}_t\bar{\hat{u}})-Re\left(\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right), \qquad (2.16)$$

where $g_0=\int_0^{t_0} g(s)ds$. Taking the derivative of $\mathcal{L}(\xi,t)$ with respect to t and exploiting the estimates (2.13) and (2.15), we arrive to

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}(\xi,t)+\frac{g_0}{2}(k-\varepsilon_1-\varepsilon_2(2-k)|\xi|^{2\delta}|\hat{u}|^2+\left\{\frac{g_0}{2}-\varepsilon_2\right\}|\hat{u}_t|^2\\ \leq&-c_2(g'\circ\hat{u})(t)+c_3|\xi|^{2\delta}(g\circ\hat{u})(t), \quad \forall t\geq 0, \qquad (2.17)\end{aligned}$$

where c_2 and c_3 are two positive constants depending on ε_1 and ε_2. In the last estimate we have used the fact that $1-\int_0^t g(s)ds \geq k$ and $g_0=\int_0^{t_0} g(s)ds \leq \int_0^t g(s)ds$ for any $t\geq t_0$.

Now, we choose $\varepsilon_1 < k$ in (2.17) and ε_2 small enough such that $\varepsilon_2 < \min\left(\frac{g_0}{2},\frac{l-\varepsilon_1}{2-k}\right)$. Then there exist positive constants λ_i, $(i=1,2,3)$ positive constants such that

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}(\xi,t)+\lambda_1|\xi|^{2\delta}|\hat{u}|^2+\lambda_2|\hat{u}_t|^2+\lambda_3|\xi|^{2\delta}(g\circ\hat{u})(t)\\ \leq&-c_2(g'\circ\hat{u})(t)+(c_3+\lambda_3)|\xi|^{2\delta}(g\circ\hat{u})(t), \quad \forall t\geq 0. \qquad (2.18)\end{aligned}$$

Define the Lyapunov functional $\mathcal{F}(\xi,t)$ as follows.

$$\mathcal{F}(\xi,t):=\eta\left(1+|\xi|^{2\delta}\right)\mathcal{E}(\xi,t)+|\xi|^{2\delta}\mathcal{L}(\xi,t), \qquad (2.19)$$

for a large positive constant η that has to be chosen later. It is straightforward to see that for η large enough, we can find two positive constants β_1 and β_2 such that

$$\beta_1\left(1+|\xi|^{2\delta}\right)\mathcal{E}(\xi,t)\leq\mathcal{F}(\xi,t)\leq\beta_2\left(1+|\xi|^{2\delta}\right)\mathcal{E}(\xi,t), \quad \forall t\geq 0. \qquad (2.20)$$

On the other hand, taking the derivative of $\mathcal{F}(\xi,t)$ with respect to t and using (2.9) and (2.18), we have

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}(\xi,t)+\lambda_1|\xi|^{4\delta}|\hat{u}|^2+\lambda_2|\xi|^{2\delta}|\hat{u}_t|^2+\lambda_3|\xi|^{4\delta}(g\circ\hat{u})(t)\\ \leq&(\eta-c_2)|\xi|^{2\delta}(g'\circ\hat{u})(t)+(c_3+\lambda_3)|\xi|^{4\delta}(g\circ\hat{u})(t), \quad \forall t\geq 0. \qquad (2.21)\end{aligned}$$

At this point, we choose η large enough such that (2.20) holds and $\eta - c_2 > 0$. Consequently (2.21) yields

$$\frac{d}{dt}\mathcal{F}(\xi,t) + \lambda_0|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t) \le c_4|\xi|^{4\delta}(g\circ\hat{u})(t), \quad \forall t \ge t_0, \tag{2.22}$$

for some $\lambda_0, c_4 > 0$.

Multiplying (2.22) by $\alpha(t)$ and making use of (2.2), we get

$$\begin{aligned}&\alpha(t)\frac{d}{dt}\mathcal{F}(\xi,t) + \lambda_0\alpha(t)|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t)\\ \le& c_4\alpha(t)|\xi|^{4\delta}(g\circ\hat{u})(t)\\ \le& c_4|\xi|^{4\delta}(-g'\circ\hat{u})(t), \quad \forall t \ge t_0.\end{aligned}$$

Exploiting (2.9), we have

$$\alpha(t)\frac{d}{dt}\mathcal{F}(\xi,t) + \lambda_0\alpha(t)|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t) \le -c_4|\xi|^{2\delta}\frac{d}{dt}\hat{\mathcal{E}}(\xi,t), \quad \forall t \ge t_0. \tag{2.23}$$

This gives

$$\frac{d}{dt}\left\{\alpha(t)\mathcal{F}(\xi,t) + c_4|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t)\right\} - \alpha'(t)\mathcal{F}(\xi,t) + \lambda_0\alpha(t)|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t) \le 0, \quad \forall t \ge t_0. \tag{2.24}$$

Recalling that $\alpha'(t) \le 0$ and setting

$$\mathcal{K}(\xi,t) := \alpha(t)\mathcal{F}(\xi,t) + c_4|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t), \tag{2.25}$$

we get

$$\frac{d}{dt}\mathcal{K}(\xi,t) + \lambda_0\alpha(t)|\xi|^{2\delta}\hat{\mathcal{E}}(\xi,t) \le 0, \quad \forall t \ge t_0. \tag{2.26}$$

On the other hand, since $\alpha(t)$ is bounded, we deduce that

$$\mathcal{K}(\xi,t) \sim \left(1 + |\xi|^{2\delta}\right)\hat{\mathcal{E}}(\xi,t). \tag{2.27}$$

These last two estimates lead to

$$\frac{d}{dt}\mathcal{K}(\xi,t) + \beta_0\frac{|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\alpha(t)\mathcal{K}(\xi,t) \le 0, \quad \forall t \ge t_0, \tag{2.28}$$

for some $\beta_0 > 0$. Integrating (2.28) with respect to t yields

$$\mathcal{K}(\xi,t) \le \mathcal{K}(\xi,t_0)\exp\left\{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_{t_0}^{t}\alpha(t)ds\right\}. \tag{2.29}$$

Thus, using (2.27), we obtain

$$\begin{aligned}\hat{\mathcal{E}}(\xi,t) \leq& C\hat{\mathcal{E}}(\xi,t_0)\exp\left\{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_{t_0}^t \alpha(t)ds\right\}\\ \leq& C\hat{\mathcal{E}}(\xi,0)\exp\left\{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_0^t \alpha(t)ds\right\}. \end{aligned} \tag{2.30}$$

Finally, the estimate (2.12) holds by combining (2.11) and (2.30). This finishes the proof of Proposition 2.1. □

Thus our main result reads as follows:

Theorem 2.1. *Let σ be a nonnegative integer. Assume that $U_0 = (u_1, u_0')^T \in H^\sigma(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then the solution $U = (u_t, u_x)^T$ of the problem* (1.5) *satisfies, for all $t \geq 0$, the following decay estimates*

$$\|\partial_x^k U(t)\|_{L^2} \leq C\left(1+\int_0^t \alpha(s)ds\right)^{-\frac{\delta k}{2}-\frac{N}{4}}\|U_0\|_{L^1} + Ce^{-c\int_0^t \alpha(s)ds}\|\partial_x^k U_0\|_{L^2}, \tag{2.31}$$

where C and c are two positive constants and $k \leq \sigma$.

Proof. Applying the Plancherel theorem and observing that $|\widehat{U}(\xi,t)|^2$ and $\widehat{E}(\xi,t)$ are equivalent, and making use of (2.12), we obtain

$$\begin{aligned}\|\partial_x^k U(t)\|_2^2 =& \int_{\mathbb{R}^n}|\xi|^{2k}|\widehat{U}(\xi,t)|^2 d\xi\\ \leq& C\int_{\mathbb{R}^n}|\xi|^{2k}e^{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_0^t\alpha(s)ds}|\widehat{U}(\xi,0)|^2d\xi\\ =& C\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_0^t\alpha(s)ds}|\widehat{U}(\xi,0)|^2d\xi\\ &+C\int_{|\xi|\geq 1}|\xi|^{2k}e^{-\frac{\beta_0|\xi|^{2\delta}}{1+|\xi|^{2\delta}}\int_0^t\alpha(s)ds}|\widehat{U}(\xi,0)|^2d\xi\\ =:& I_1 + I_2. \end{aligned} \tag{2.32}$$

Looking carefully to the form of $\rho(\xi) = \frac{|\xi|^{2\delta}}{1+|\xi|^{2\delta}}$, we see that $\rho(\xi) \geq \frac{1}{2}|\xi|^{2\delta}$ for $|\xi| \leq 1$ and $\rho(\xi) \geq \frac{1}{2}$ for $|\xi| \geq 1$.

Consequently, we infer that

$$\begin{aligned} I_1 \leq& C\|\widehat{U}_0\|_{L^\infty}^2\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\frac{1}{2}|\xi|^{2\delta}\int_0^t\alpha(s)ds}d\xi\\ \leq& C\left(1+\int_0^t\alpha(\tau)d\tau\right)^{-k\delta-\frac{n}{2}}\|U_0\|_{L^1}^2, \end{aligned} \tag{2.33}$$

where we have used Lemma 2.2

On the other hand,

$$I_2 \le Ce^{-\frac{1}{2}\int_0^t \alpha(s)ds} \int_{|\xi|\ge 1} |\xi|^{2k} |\widehat{U}_0(\xi)|^2 d\xi \le Ce^{-\frac{1}{2}\int_0^t \alpha(s)ds} \|\partial_x^k U_0\|_{L^2}^2. \tag{2.34}$$

Collecting (2.33) and (2.34), then the estimate (2.31) holds. This completes the proof of Theorem 2.1. □

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