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Some Estimates of Bilinear Hausdorff Operators on Stratified Groups

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Abstract. In this paper, we give four kinds of sharp estimates of two variants of bilinear Hausdorff operators on stratified groups, involving weighted Lebesgue spaces, classical Morrey spaces and central Morrey spaces. Meanwhile, some necessary and sufficient conditions of boundness are obtained.

Key Words: Stratified group, Hausdorff operator, bilinear, sharp estimate, weighted Lebesgue space, Morrey space.

AMS Subject Classifications: 42B35, 43A80, 47H60, 47A30

1 Introduction

The investigation on Hausdorff operators is a branch of the classical analysis and it is closely related to Fourier analysis and complex analysis. The Hausdorff summability methods was introduced [18, 19] long time ago to solve some classical problems. Readers can refer to [31] for the Hausdorff summability and its application and [17] for the Hausdorff means and its application on the summation of series. For more details of the background and the historical development of the operators, one can see [24]. During modern time, studies focusing on the Hausdorff operators have never stopped and the operator became its modern versions which were initiated by Siskakis in the work on complex analysis [30] and Georgakis [13], Liflyand and Móricz [22, 23] in their work on the Fourier transform setting. Readers can also refer to [3, 21, 27], for example, for more results about Hausdorff operators right after them.

The classical 1-dimension Hausdorff operator is defined by (see [22])

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\Big(\frac{x}{t}\Big)dt, \quad x \in \mathbb{R},$$

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where Φ is a locally integrable function in $(0,\infty)$. Using variables substitution, the above formula is rewritten as the following equality

$$h_\Phi(f)(x)=\int_0^\infty \frac{\Phi(x/t)}{t}f(t)dt.$$

We can take appropriate function Φ to obtain many classical operators in analysis as its special cases. These operators include Hardy operators, Cesàro operators, Hilbert operators, Hardy-Littlewood-Pólya operaotrs and others. We shall list [10,26,30] here for instance. Three extensions of the 1-dimensional Hausdorff operator in $\mathbb{R}^n$ were introduced and studied in [4,21] respectively. One of them is the operator [4]

$$H_\Phi f(x)=\int_{\mathbb{R}^n}\frac{\Phi(|y|^{-1}x)}{|y|^n}f(y)dy,$$

where Φ is a radial function defined on $\mathbb{R}^+$. With $\Phi_1(t)=t^{-n}\chi_{(1,\infty)}(t)$, $\Phi_2=t^{-n}\chi_{(0,1]}(t)$, H_Φ becomes the high dimensional Hardy operator $\mathcal{H}$ and its adjoint operator which is defined as

$$\mathcal{H}^*(f)(x)=\int_{|y|\geq|x|}f(y)dy.$$

We have noticed that H_Φ can map functions on $\mathbb{R}^{nm}$ to functions on $\mathbb{R}^n$ in some ways. In [5], Chen, Fan and Zhang introduced such a extension of H_Φ, i.e.,

$$T_\Phi(F)(x)=\int_{\mathbb{R}^{nm}}\frac{\Phi(|u|^{-1}x)}{|u|^{nm}}F(u)du,$$

where $x\in\mathbb{R}^n$. Then taking $m=2$, $u=(u_1,u_2)$, $u_i\in\mathbb{R}^n$, $i=1,2$, and $F(u_1,u_2)=f_1(u_1)f_2(u_2)$, we have

$$T_\Phi(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\frac{\Phi(|u|^{-1}x)}{|u|^{2n}}f_1(u_1)f_2(u_2)du.$$

The above operator is called bilinear Hausdorff operator and can reduce to bilinear Hardy operator (see [10]) when we take

$$\Phi(t)=\frac{1}{\Omega^{2n}}t^{-2n}\chi_{(1,\infty)}(t).$$

Meanwhile, they [5] defined another version of the multilinear Hausdorff operator and the following is the bilinear case

$$S_\Psi(f_1,f_2)(x)=\int_{\mathbb{R}^{2n}}\frac{\Psi(|u_1|^{-1}x,|u_2|^{-1}x)}{|u_1|^n\,|u_2|^n}f_1(u_1)f_2(u_2)du,$$

where $\Psi(s_1,s_2)$ is a locally integrable function on $\mathbb{R}^+\times\mathbb{R}^+$. Let

$$\Psi(s_1,s_2)=\Psi_1(s_1,s_2)=(s_1s_2)^{-n}\chi_{s_1^{-2}+s_2^{-2}<1}(s_1,s_2),$$

the operator $S_\Psi(f_1,f_2)$ also become the bilinear Hardy operator $\mathcal{H}^2(f_1,f_2)$ which is defined by

$$\frac{1}{\Omega^{2n}\,|x|^{2n}}\int_{|(y_1,y_2)|<|x|} f_1(y_1)f_2(y_2)dy_1dy_2, \quad x\in\mathbb{R}^{2n}\setminus\{0\}\,.$$

In [5], Chen et al. obtained the boundedness of the Hausdorff operators on weighted Lebesgue spaces, with optimal bound, and Herz spaces. Immediately after [5], Hussain gained the boundedness of T_Φ and S_Ψ on central Morrey spaces in his doctoral thesis [20]. Studies also involve the necessary and sufficient conditions of boundedness. Gao and Fan investigated the necessary and sufficient conditions and the best constants of Hausdorff operators on weighted Lebesgue spaces and central Morrey spaces in [11]. For more results about the Hausdorff operators on these function spaces in $\mathbb{R}^n$, one can refer to [4, 6, 7, 32] etc. and the references therein. Recently, studies on the Hausdorff operators on abstract groups have raised. In 2015, Guo, Sun and Zhao obtained the sharp bounds for strong (p,p) estimates of high-dimensional and multilinear Hausdorff operators on Heisenberg groups in [16]. Only three years later, Guo went further and gained the best constants of two Hausdorff operators in Morrey spaces and CBMO spaces in [15]. More and newer result nearly connected with this article should be referred in [29, 33] etc..

Particularly, all the results above are aimed at the function spaces on $\mathbb{R}^n$ and Heisenberg groups $\mathbb{H}^n$. Instead, we are going to study the Hausdorff operator on stratified group G. As a more general underlying space of the Euclidean space, the Heisenberg group, and many other spaces, the Stratified groups have received a boost with newly applications in subelliptic estimates, multiplier theorems, index formulae, nonlinear problems, potential theory, and symbolic calculi tracing full symbols of operators in recent years [2, 8]. In this paper, we obtain some results of the stratified group extensions of T_Φ and S_Ψ(whose definitions are placed in the next section). The sharp weighted Lebesgue estimates and (classical and central) Morrey estimates of the first operator and the central Morrey estimates of the second operator are obtained. Meanwhile, we also work out the necessary and sufficient conditions of boundedness, except the operator on $L^{p,\lambda}(G)$.

The structure of this paper is simple. In Section 2, we give a brief introduction to the stratified groups and list our key definition and main results gained in this paper. In Section 3, we give the proofs of all the theorems.

2 Notations and main results

Stratified lie group is the homogeneous group whose lie algebra is stratified (cf. [9]). A lie group G is called homogeneous if it is nilpotent, connected and simply connected, and its lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r : r>0\}$ which is a family of algebra automorphisms of $\mathfrak{g}$ of the form $\delta_r=\exp(A\log r)$. Exponential map is the isomorphism from $\mathfrak{g}$ onto G. The stratified algebra $\mathfrak{g}$ is endowed with a vector space decomposition $\mathfrak{g}=V_1\bigoplus\cdots\bigoplus V_m$ such that

$$[V_1,V_k]=V_{k+1},\quad 1\le k<m,\quad \text{and}\quad [V_1,V_m]=0.$$

In this case, V_1 generates $\mathfrak{g}$ as an algebra and the natural dilations on $\mathfrak{g}$ are given by

$$\delta_r\left(\sum_{j=1}^{m} X_j\right)=\sum_{j=1}^{m} r^j X_j, \quad X_j\in V_j.$$

Assuming that G is a lie group of dimension n and has an underlying manifold $\mathbb{R}^n$. G inherits dilations from $\mathfrak{g}$ with the following form

$$rx=(r^{d_1}x_1,\cdots,r^{d_n}x_n),$$

for $x\in G$ and $r>0$, where $1\le d_1\le\cdots\le d_n$. Therefore, map $x\to rx$ is an automorphsim of G. The inverse of any $x\in G$ is $-x$. The group law must have the form

$$xy=(p_1(x,y),\cdots,p_n(x,y))$$

for some polynomials $p_1,\cdots,p_n$ in $x_1,\cdots,x_n,y_1,\cdots,y_n$.

The number $Q=\sum_{j=1}^{n} d_j$ is called the homogeneous dimension of G. We fix a homogeneous norm $|\cdot|$ which satisfies that $|\cdot|$ is smooth away from 0, $|ax|=a|x|$ for all $x\in G$ and $a>0$, $|x^{-1}|=|x|$ for all $x\in G$, and $|x|=0$ if and only if $x=0$. Therefore, the homogeneous norm induces a quasi-distance d which is defined by $d(x,y)=|x^{-1}y|$. Then we record the definition of the ball of radius r centered at x as $B(x,r)$ which means the set $\{y\in G: d(x,y)<r\}$. And the sphere is the set $S(x,r)=\{y\in G: d(x,y)=r\}$. The biinvariant Haar measure on G can be exported by the Lebesgue measure on $\mathfrak{g}$. We denote the Haar measure of any measurable set $E\subset G$ by $|E|$. Then we have

$$|\delta_r(E)|=r^Q|E| \quad \text{and} \quad |B(x,r)|=r^Q|B(x,1)|.$$

We record $|B(0,1)|$ as Ω_Q and $|S(0,1)|$ as ω_Q, respectively. Consider the products of stratified group $G\times G$. The norm on $G\times G$ is defined as

$$|(x_1,x_2)|=\sqrt{|x_1|^2+|x_2|^2}.$$

However, using same notations of the norms here won't bring confusion. Then the distance is derived as

$$d(p,q)=d(-q+p,0)=|-q+p|=\left|(y_1^{-1}x_1,y_2^{-1}x_2)\right|,$$

where p is the point (x_1,x_2) and q is (y_1,y_2). We can easily check that the distance is well defined. Then, two variants bilinear Hausdorff operators on the product space of G are defined as following.

Definition 2.1. Let f_1,f_2 be locally integrable functions on G. The bilinear Hausdorff operator T_Φ is defined by

$$T_\Phi(f_1,f_2)(x)=\int_{G\times G}\frac{\Phi(\delta_{|u|^{-1}}x)}{|u|^{2Q}}f_1(u_1)f_2(u_2)du,$$

where Φ is a locally integrable function on G.

Definition 2.2. The second bilinear extension of the Hausdorff operator S_Ψ is defined by

$$S_\Psi(f_1,f_2)(x)=\int_{G\times G}\frac{\Psi(\delta_{|u_1|^{-1}}x,\delta_{|u_2|^{-1}}x)}{|u_1|^Q\,|u_2|^Q}f_1(u_1)f_2(u_2)du,$$

where Ψ is a locally integrable function on $G\times G$.

As what we investigate are the two operators with radial kernel functions, we take

$$\Phi(t)=\Phi(|t|)\quad\text{ and }\quad \Psi(s_1,s_2)=\Psi(|s_1|,|s_2|)$$

for convenience. Then, we will give our main results.

Theorem 2.1. *Let $1<p_1,p_2<\infty$, $1\le p<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $\alpha_i<pn(1-\frac{1}{p_i})$, $i=1,2$, and $\alpha=\alpha_1+\alpha_2$. If Φ is a radial function which satisfies that*

$$K_1=\int_{G\times G}\frac{\left|\Phi(\delta_{|u|^{-1}}e)\right|}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}}du<+\infty,$$

where $|e|=1$, we have

$$\|T_\Phi(f_1,f_2)\|_{L^p(G,|x|^\alpha dx)}\le K_1\|f_1\|_{L^{p_1}(G,|x|^{\frac{\alpha_1p_1}{p}}dx)}\|f_2\|_{L^{p_2}(G,|x|^{\frac{\alpha_2p_2}{p}}dx)} \tag{2.1}$$

for $f_1\in L^{p_1}(G,|x|^{\frac{\alpha_1p_1}{p}}dx)$ and $f_2\in L^{p_2}(G,|x|^{\frac{\alpha_2p_2}{p}}dx)$. Moreover, when Φ is a nonnegative radial function, T_Φ is a bounded operator which maps $L^{p_1}(G,|x|^{\frac{\alpha_1p_1}{p}}dx)\times L^{p_2}(G,|x|^{\frac{\alpha_2p_2}{p}}dx)$ to $L^p(G,|x|^\alpha dx)$ if and only if $K_1<+\infty$. Finally, if Φ is a nonnegative radial function and $K_1<+\infty$, the constant K_1 is a sharp estimate.

For purposes of the following theorems, two definitions should be extended. In 1938, Morrey [28] introduced the $L^{q,\lambda}(\mathbb{R}^n)$ spaces in order to investigate the local behavior of solutions to second order elliptic partial differential equations. And in 2000, Alvarez, Lakey and Guzmán-Partida [1] introduced the notion of central Morrey spaces $\dot{B}^{q,\lambda}(\mathbb{R}^n)$. We will give the definition of these two spaces in stratified groups G. Accordingly, they are recorded as $L^{p,\lambda}(G)$ and $\dot{B}^{q,\lambda}(G)$ respectively. It is worth noting that Lu, Yang and Yuan have introduced the Morrey spaces to quasi-metric measure spaces, a class of more general spaces, in [25] which provide us some useful details to refer. And Ruan et al. [29] also defined the weighted central Morrey spaces on Heisenberg groups recently. In fact, Guliyev et al. gave the definition of generalized Morrey spaces on Heisenberg groups in [14] and Guo gave the definition of central Morrey spaces in [15] before [29]. But there are some differences in details between those definitions. However, we call $L^{p,\lambda}(G)$ and $\dot{B}^{q,\lambda}(G)$ in the following as Morrey spaces and central Morrey spaces respectively in keeping with our main references, though our definitions are different from [14,15].

Definition 2.3. Let $1 \leq q < \infty$ and $1/q \leq \lambda$. The classical Morrey spaces $L^{q,\lambda}(G)$ on stratified groups is defined by

$$L^{q,\lambda} = \{f \in L^q_{loc}(G) : \|f\|_{L^{q,\lambda}(G)} < \infty\},$$

where

$$\|f\|_{L^{q,\lambda}(G)} = \sup_{a\in G, R>0} \left(\frac{1}{|B(a,R)|^{1+\lambda q}} \int_{B(a,R)} |f(x)|^q \, dx \right)^{1/q}.$$

It is easy to verify that $L^{p,-1/p}(G) = L^p(G)$ and $L^{p,0}(G) = L^\infty(G)$. If $\lambda > 0$, $L^{p,\lambda}(G) = 0$. So, we assume that $-1/q < \lambda \leq 0$. These properties are similar as classical Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$.

Definition 2.4. Let $1 \leq q < \infty$ and $-1/q \leq \lambda \leq 0$. The central homogeneous Morrey space $\dot{B}^{q,\lambda}(G)$ is defined by

$$\dot{B}^{q,\lambda}(G) = \{f \in L^q_{loc}(G) : \|f\|_{\dot{B}^{q,\lambda}(G)} < \infty\},$$

where

$$\|f\|_{\dot{B}^{q,\lambda}(G)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x)|^q \, dx \right)^{1/q}.$$

Remark 2.1. The inhomogeneous central Morrey space $B^{q,\lambda}(G)$ is defined as same as the definiton 3 with the exception that in the previous norm the supremum over $R > 0$ is restricted to $R \geq 1$. It is easy to see that $\dot{B}^{q,\lambda}(G) \subset B^{q,\lambda}(G)$ and $\dot{B}^{q,-1/q}(G) = B^{q,-1/q}(G) = L^q(G)$. If $\lambda < -1/q$, $\dot{B}^{q,\lambda}(G)$ and $B^{q,\lambda}(G)$ reduce to $\{0\}$. If $\lambda_1 < \lambda_2$, we have $B^{q,\lambda_1}(G) \subset B^{q,\lambda_2}(G)$ for $-1/q \leq \lambda \leq 0$. When $1 < q_1 < q_2 < \infty$, the Hölder's inequality yields that $\dot{B}^{q,\lambda}(G) \subset \dot{B}^{q_1,\lambda}(G)$ and $B^{q_2,\lambda}(G) \subset B^{q_1,\lambda}(G)$.

Then we can give the latter three theorems.

Theorem 2.2. *Let* $1 < p_1, p_2 < \infty$, $1 \leq p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $-\frac{1}{p_1} < \lambda_1 < 0$, $-\frac{1}{p_2} < \lambda_2 < 0$, *and* $\lambda = \lambda_1 + \lambda_2$. *If* Φ *is a radial fnction which satisfies that*

$$K_2 = \int_G \frac{\left|\Phi(\delta_{|u|^{-1}} e)\right|}{|u|^{2Q}} |u_1|^{Q\lambda_1} |u_2|^{Q\lambda_2} \, du < +\infty,$$

we have

$$\|T_\Phi(f_1, f_2)\|_{\dot{B}^{p,\lambda}(G)} \leq K_1 \|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)} \|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)}$$

for $f_1 \in \dot{B}^{p_1,\lambda_1}(G)$ *and* $f_2 \in \dot{B}^{p_2,\lambda_2}(G)$. *Moreover, when* Φ *is a nonnegative radial function,* T_Φ *is a bounded operator which maps* $\dot{B}^{p_1,\lambda_1}(G) \times \dot{B}^{p_2,\lambda_2}(G)$ *to* $\dot{B}^{p,\lambda}(G)$ *if and only if* $K_2 < +\infty$. *Finally, if* Φ *is a radial function,* $p_1\lambda_1 = p_2\lambda_2$ *and* $K_2 < +\infty$, *the constant* K_2 *is sharp.*

Theorem 2.3. *Let p_i, p, λ_i λ and K_2 be as Theorem 2.2. If $K_2<+\infty$, for radial functions f_i in $L^{p_i,\lambda_i}(G)$, we have*

$$\|T_\Phi(f_1,f_2)\|_{L^{p,\lambda}(G)}\le K_2\|f_1\|_{L^{p_1,\lambda_1}(G)}\|f_2\|_{L^{p_2,\lambda_2}(G)}.$$

Moreover, if $p_1\lambda_1=p_2\lambda_2$, the constant K_2 is sharp.

Theorem 2.4. *Let p_i, p, λ_i and λ be as Theorem 2.2. If Ψ is a radial fnction which satisfies that*

$$K_3=\omega_Q^2\int_0^\infty\int_0^\infty\Psi(s_1,s_2)s_1^{-Q\lambda_1}s_2^{-Q\lambda_2}ds_1ds_2<+\infty,$$

we have

$$\|S_\Psi(f_1,f_2)\|_{\dot{B}^{p,\lambda}(G)}\le K_3\|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)}\|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)},$$

for $f_i\in\dot{B}^{p_i,\lambda_i}(G)$, $i=1,2$. Moreover, when Ψ is a nonnegative radial function, S_Ψ is a bounded operator which maps $\dot{B}^{p_1,\lambda_1}(G)\times\dot{B}^{p_2,\lambda_2}(G)$ to $\dot{B}^{p,\lambda}(G)$ if and only if $K_3<+\infty$. Finally, if Ψ is a radial function, $p_1\lambda_1=p_2\lambda_2$ and $K_2<+\infty$, the constant K_3 is sharp.

3 Proofs of theorems

Inspiring by [10–12] and others, we finish the following theorems.

3.1 Proof of Theorem 2.1

Firstly, we claim that the operator T_Φ and its restriction to radial functions should have same operator norms from

$$L^{p_1}(G,|x|^{\frac{\alpha_1p_1}{p}}\,dx)\times L^{p_2}(G,|x|^{\frac{\alpha_2p_2}{p}}\,dx)$$

to $L^p(G,|x|^\alpha\,dx)$. For any $f_i\in L^{p_i}(G,|x|^{\frac{\alpha_ip_i}{p}}\,dx)$, we have that

$$g_i(y_i)=\frac{1}{\omega_Q}\int_{|\xi_i|=1}f_i(\delta_{|y_i|}\xi_i)d\xi_i$$

is radial function, where $y_i\in G$, $i=1,2$. By change of variables, $T_\Phi(g_1,g_2)(x)$ is equal to

$$\begin{aligned}
&\int_{G\times G}\frac{\Phi(\delta_{|u|^{-1}}x)}{|u|^{2Q}}g_1(u_1)g_2(u_2)du\\
=&\int_{G\times G}\frac{\Phi(\delta_{|u|^{-1}}x)}{|u|^{2Q}}\prod_{i=1}^2\left(\frac{1}{\omega_Q}\int_{|\xi_i|=1}f_i(\delta_{|u_i|}\xi_i)d\xi_i\right)du\\
=&\int_{|\xi_1|=1}\int_{|\xi_2|=1}\left(\frac{1}{\omega_Q^2}\int_{G\times G}\frac{\Phi(\delta_{|u|^{-1}}x)}{|u|^{2Q}}f_1(\delta_{|u_1|}\xi_1)f_2(\delta_{|u_2|}\xi_2)du\right)d\xi_1d\xi_2\\
=&\int_{G\times G}\frac{\Phi(\delta_{|u|^{-1}}x)}{|u|^{2Q}}f_1(u_1)f_2(u_2)du\\
=&T_\Phi(f_1,f_2)(x).
\end{aligned}$$

Let $\beta_i = \frac{\alpha_i p_i}{p}$, $i = 1,2$. Using Hölder's inequality, we have

$$\begin{aligned}
\|g_1\|_{L^{p_1}(G,|x|^{\beta_1}dx)} =&\frac{1}{\omega_Q}\left(\int_G \left|\int_{|\xi|=1} f_1(\delta_{|x|}\xi)d\xi\right|^{p_1} |x|^{\beta_1}\,dx\right)^{\frac{1}{p_1}}\\
\leq&\frac{1}{\omega_Q}\left(\int_G \left(\int_{|\xi|=1} \left|f_1(\delta_{|x|}\xi)\right|^{p_1} d\xi\right)\left(\int_{|\xi|=1} d\xi\right)^{p_1-1} |x|^{\beta_1}\,dx\right)^{\frac{1}{p_1}}\\
=&\omega_Q^{-\frac{1}{p_1}}\left(\int_{|\eta|=1}\int_0^{+\infty}\int_{|\xi|=1} |f_1(\delta_r\xi)|^{p_1}\,d\xi r^{\beta_1+Q-1}drd\eta\right)^{\frac{1}{p_1}}\\
=&\|f_1\|_{L^{p_1}(G,|x|^{\beta_1}dx)}.
\end{aligned}$$

Similarly, for g_2, we obtain

$$\|g_2\|_{L^{p_2}(G,|x|^{\beta_2}dx)} \leq \|f_2\|_{L^{p_2}(G,|x|^{\beta_2}dx)}.$$

Thus,

$$\frac{\|T_\Phi(f_1,f_2)\|_{L^p(G,|x|^\alpha dx)}}{\|f_1\|_{L^{p_1}(G,|x|^{\beta_1}dx)}\|f_2\|_{L^{p_2}(G,|x|^{\beta_2}dx)}} \leq \frac{\|T_\Phi(g_1,g_2)\|_{L^p(G,|x|^\alpha dx)}}{\|g_1\|_{L^{p_1}(G,|x|^{\beta_1}dx)}\|g_2\|_{L^{p_2}(G,|x|^{\beta_2}dx)}}.$$

So, we could assume that f_i are radial functions in following proof, $i = 1,2$. Then, we make variable substitution to modify the form of T_Φ. By definition,

$$\begin{aligned}
T_\Phi(f_1,f_2)(x) &= \int_G \frac{\Phi(\delta_{|u|^{-1}}x)}{|u|^{2Q}} f_1(u_1)f_2(u_2)du\\
&= \int_G \frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}} f_1(\delta_{|x|}u_1)f_2(\delta_{|x|}u_2)du. \qquad (3.1)
\end{aligned}$$

The norm of bilinear Hausdorff operator $\|T_\Phi(f_1,f_2)\|_{L^p(G,|x|^\alpha dx)}$ is equal to

$$\begin{aligned}
&\left(\int_G \left|\int_{G\times G} \frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}} f_1(\delta_{|x|}u_1)f_2(\delta_{|x|}u_2)du\right|^p |x|^\alpha\,dx\right)^{\frac{1}{p}}\\
=&\left(\int_G \left|\int_{G\times G} \frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}} f_1(\delta_{|u_1|}x)f_2(\delta_{|u_2|}x)du_1du_2\right|^p |x|^\alpha\,dx\right)^{\frac{1}{p}}\\
\leq&\int_{G\times G}\left(\int_G \left|\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}\right|^p \left|f_1(\delta_{|u_1|}x)\right|^p \left|f_2(\delta_{|u_2|}x)\right|^p |x|^{\alpha_1+\alpha_2}\,dx\right)^{\frac{1}{p}} du\\
\leq&\int_{G\times G}\left|\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}\right|\prod_{i=1}^2\left(\int_G \left|f_i(\delta_{|z_i|}x)\right|^{p_i} |x|^{\beta_i}\,dx\right)^{\frac{1}{p_i}} du.
\end{aligned}$$

In above inequalities, we use Minkowski's inequality and Hölder's inequality and prove that

$$\|T_\Phi(f_1,f_2)\|_{L^p(G,|x|^\alpha dx)} \le K_1 \|f_1\|_{L^{p_1}(G,|x|^{\beta_1}dx)} \|f_2\|_{L^{p_2}(G,|x|^{\beta_2}dx)}. \tag{3.2}$$

Let's come up to the necessary and sufficient conditions. In fact, (3.2) implies that T_Φ is a bounded operator when Φ is a radial function and $K_1<+\infty$ and what we should prove in following is the necessity. At first, we assume that T_Φ is a bounded operator which maps $L^{p_1}(G,|x|^{\frac{\alpha_1 p_1}{p}}\,dx)\times L^{p_2}(G,|x|^{\frac{\alpha_2 p_2}{p}}\,dx)$ to $L^p(G,|x|^\alpha\,dx)$. For

$$0<\varepsilon<\min\left\{1,\frac{(p_1-1)Q}{p_2},\frac{Q}{p_2'}\right\},$$

we define

$$f_1^\varepsilon(u_1)=\begin{cases}0, & |u_1|\le\dfrac{\sqrt{2}}{2},\\ |u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}, & |u_1|>\dfrac{\sqrt{2}}{2},\end{cases}$$
$$f_2^\varepsilon(u_2)=\begin{cases}0, & |u_2|\le\dfrac{\sqrt{2}}{2},\\ |u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}, & |u_2|>\dfrac{\sqrt{2}}{2}.\end{cases}$$

Easily, we have

$$\begin{aligned}\|f_1^\varepsilon\|^{p_1}_{L^{p_1}(G,|x|^{\beta_1}dx)}&=\int_G|f_1^\varepsilon(u_1)|^{p_1}|u_1|^{\frac{p_1\alpha_1}{p}}du_1=\int_{|u_1|>\frac{\sqrt{2}}{2}}|u_1|^{-Q-p_2\varepsilon}du_1\\&=\omega_Q\int_{\frac{\sqrt{2}}{2}}^{+\infty}\rho_1^{-p_2\varepsilon-1}d\rho_1=\frac{\omega_Q}{p_2\varepsilon}\left(\frac{\sqrt{2}}{2}\right)^{-p_2\varepsilon}.\end{aligned}$$

Then we have

$$\|f_2^\varepsilon\|^{p_2}_{L^{p_2}(G,|x|^{\beta_2}dx)}=\frac{\omega_Q}{p_2\varepsilon}\left(\frac{\sqrt{2}}{2}\right)^{-p_2\varepsilon},$$

similarly. Because of the definitions of f_1^ε and f_2^ε, when $|x|\le 1$, we have that $T_\Phi(f_1^\varepsilon,f_2^\varepsilon)(x)=0$. And when $|x|>1$, we have that $T_\Phi(f_1^\varepsilon,f_2^\varepsilon)(x)$ is equal to

$$\begin{aligned}&\int_{G\times G}\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}f_1^\varepsilon(\delta_{|x|}u_1)f_2^\varepsilon(\delta_{|x|}u_2)du\\=&|x|^{-\frac{Q+\alpha+p_2\varepsilon}{p}}\int_{|u_1|>\frac{\sqrt{2}}{2|x|};|u_2|>\frac{\sqrt{2}}{2|x|}}\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}du_1du_2.\end{aligned}$$

Let's record the domain

$$|u_1| > \frac{\sqrt{2}}{2|x|}, \quad |u_2| > \frac{\sqrt{2}}{2|x|} \quad \text{as } | D_{|x|}.$$

We obtain that $\|T_\Phi(f_1^\varepsilon, f_2^\varepsilon)\|_{L^p(G,|x|^\alpha dx)}$ is equal to

$$\begin{aligned}
&\left(\int_{|x|>1}\left(|x|^{-\frac{Q+\alpha+p_2\varepsilon}{p}}\int_{D_{|x|}}\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}du_1du_2\right)^p|x|^\alpha dx\right)^{\frac{1}{p}}\\
\geq&\left(\int_{|x|>\frac{1}{\varepsilon}}\left(|x|^{\frac{-Q-p_2\varepsilon}{p}}\int_{D_{\frac{1}{\varepsilon}}}\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}du_1du_2\right)^p dx\right)^{\frac{1}{p}}\\
=&\left(\int_{|x|>\frac{1}{\varepsilon}}|x|^{-Q-p_2\varepsilon}dx\right)^{\frac{1}{p}}\int_{D_{\frac{1}{\varepsilon}}}\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}du_1du_2. \qquad (3.3)
\end{aligned}$$

By some simple calculation, we have

$$\begin{aligned}
\left(\int_{|x|>\frac{1}{\varepsilon}}|x|^{-Q-p_2\varepsilon}dx\right)^{\frac{1}{p}} &= \omega_Q^{\frac{1}{p}}\int_{\frac{1}{\varepsilon}}^{+\infty}r^{-p_2\varepsilon-1}dr\\
&=(\varepsilon^\varepsilon)^{\frac{p_2}{p}}\left(\frac{\sqrt{2}}{2}\right)^{\frac{p_2\varepsilon}{p}}\|f_1^\varepsilon\|^{p_1}_{L^{p_1}(G,|x|^{\beta_1}dx)}\|f_2^\varepsilon\|^{p_2}_{L^{p_2}(G,|x|^{\beta_2}dx)}. \qquad (3.4)
\end{aligned}$$

Considering that T_Φ is a bounded operator and $\Phi \geq 0$, we know that there exists a constant $C > 0$ such that

$$0 \leq \int_{D_{\frac{1}{\varepsilon}}}\frac{|\Phi(\delta_{|u|^{-1}}e)|}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}du_1du_2 \leq C.$$

Let $\varepsilon \to 0^+$, then $K_1 < +\infty$.

At last, we assume that Φ is a nonnegative radial function and $K_1 < +\infty$. Inequality (3.2) and Eqs. (3.3) and (3.4) yields that

$$(\varepsilon^\varepsilon)^{\frac{p_2}{p}}\left(\frac{\sqrt{2}}{2}\right)^{\frac{p_2\varepsilon}{p}}\int_{D_{\frac{1}{\varepsilon}}}\frac{|\Phi(\delta_{|u|^{-1}}e)|}{|u|^{2Q}}|u_1|^{-\frac{Q}{p_1}-\frac{\alpha_1}{p}-\frac{p_2\varepsilon}{p_1}}|u_2|^{-\frac{Q}{p_2}-\frac{\alpha_2}{p}-\varepsilon}du_1du_2 \leq \|T_\Phi\| \leq K_1,$$

for

$$0 < \varepsilon < \min\left\{1, \frac{(p_1-1)Q}{p_2}, \frac{Q}{p_2'}\right\}.$$

So, K_1 is the sharp estimate of T_Φ. $\square$

3.2 Proof of Theorem 2.2

To prove, we need an analogous result about restriction of the bilinear Hausdorff operator on radial functions in $\dot{B}^{p_i,\lambda_i}(G)$, $i=1,2$. Let $f_i\in\dot{B}^{p_i,\lambda_i}(G)$, and g_i be the same as in Theorem 2.1, $i=1,2$. Then

$$T_\Phi(f_1,f_2)(x)=T_\Phi(g_1,g_2)(x).$$

So, what we need to prove is

$$\|g_i\|_{\dot{B}^{p_i,\lambda_i}(G)}\le\|f_i\|_{\dot{B}^{p_i,\lambda_i}(G)},\quad j=1,2.$$

In fact, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|\frac{1}{\omega_Q}\int_{|\xi_i|=1}f_i(\delta_{|y_i|}\xi_i)d\xi_i\right|^p dy_i\right)^{1/p}\\
\le&\frac{1}{\omega_Q}\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\int_{|\xi_i|=1}\left|f_i(\delta_{|y_i|}\xi_i)\right|^p d\xi_i\left(\int_{|\xi|=1}d\xi\right)^{p-1}dy_i\right)^{1/p}\\
=&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\frac{1}{\omega_Q}\int_{|\xi_i|=1}\left|f_i(\delta_{|y_i|}\xi_i)\right|^p d\xi_i dy_i\right)^{1/p}\\
=&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|f_i(y_i)|^p dy_i\right)^{1/p}.
\end{aligned}$$

Therefore, the similar result is obtained

$$\frac{\|T_\Phi(f_1,f_2)\|_{\dot{B}^{p,\lambda}(G)}}{\|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)}\|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)}}\le\frac{\|T_\Phi(g_1,g_2)\|_{\dot{B}^{p,\lambda}(G)}}{\|g_1\|_{\dot{B}^{p_1,\lambda_1}(G)}\|g_2\|_{\dot{B}^{p_2,\lambda_2}(G)}}.$$

Without loss of generality, we assume that f_1 and f_2 are radial functions in the rest of the proof. By Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|T_\Phi(f_1,f_2)(x)|^p dx\right)^{\frac{1}{p}}\\
=&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|\int_G\frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}}f_1(\delta_{|x|}u_1)f_2(\delta_{|x|}u_2)du\right|^p dx\right)^{\frac{1}{p}}\\
\le&\int_G\frac{\left|\Phi(\delta_{|u|^{-1}}e)\right|}{|u|^{2Q}}\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|f_1(\delta_{|u_1|}x)f_2(\delta_{|u_2|}x)|^p dx\right)^{1/p}du
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_G \frac{\left|\Phi(\delta_{|u|^{-1}}e)\right|}{|u|^{2Q}} \left(\frac{1}{|B(0,R)|^{1+\lambda_1 p_1}} \int_{B(0,R)} \left|f_1(\delta_{|u_1|}x)\right|^{p_1} dx\right)^{1/p_1} \\
&\quad \times \left(\frac{1}{|B(0,R)|^{1+\lambda_2 p_2}} \int_{B(0,R)} \left|f_2(\delta_{|u_2|}x)\right|^{p_2} dx\right)^{1/p_2} du \\
&= \int_G \frac{\left|\Phi(\delta_{|u|^{-1}}e)\right|}{|u|^{2Q}} |u_1|^{Q\lambda_1} |u_2|^{Q\lambda_2} \left(\frac{1}{|B(0,|u_1|\,R)|^{1+\lambda_1 p_1}} \int_{B(0,|u_1|R)} |f_1(x)|^{p_1} dx\right)^{1/p_1} \\
&\quad \times \left(\frac{1}{|B(0,|u_2|\,R)|^{1+\lambda_2 p_2}} \int_{B(0,|u_2|R)} |f_2(x)|^{p_2} dx\right)^{1/p_2} du \\
&\leq \int_G \frac{\left|\Phi(\delta_{|u|^{-1}}e)\right|}{|u|^{2Q}} |u_1|^{Q\lambda_1} |u_2|^{Q\lambda_2} du \|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)} \|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)}.
\end{aligned}
$$

If

$$K_2 = \int_G \frac{\left|\Phi(\delta_{|u|^{-1}}e)\right|}{|u|^{2Q}} |u_1|^{Q\lambda_1} |u_2|^{Q\lambda_2} du < +\infty,$$

we have

$$\|T_\Phi(f_1,f_2)\|_{\dot{B}^{p,\lambda}(G)} \leq K_2 \|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)} \|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)}.$$

On the other hand, we study the assumption that T_Φ is bounded and $\Phi > 0$. Firstly, we take $\tilde{f}_1(x) = |x|^{Q\lambda_1}$, $\tilde{f}_1(x) = |x|^{Q\lambda_2}$. It is easy to verify that

$$\|\tilde{f}_i\|_{\dot{B}^{p_i,\lambda_i}(G)} = \frac{1}{\Omega_Q^{\lambda_i}} \left(\frac{1}{1+\lambda_i p_i}\right)^{1/p_i} \qquad \text{and} \qquad \tilde{f}_i \in \dot{B}^{p_i,\lambda_i}(G),$$

$i = 1,2$. Then, we need some calculation, i.e.,

$$
\begin{aligned}
T_\Phi(\tilde{f}_1,\tilde{f}_2)(x) &= \int_G \frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}} \tilde{f}_1(\delta_{|x|}u_1)\tilde{f}_2(\delta_{|x|}u_2) du \\
&= \int_G \frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}} \left|\delta_{|x|}u_1\right|^{Q\lambda_1} \left|\delta_{|x|}u_2\right|^{Q\lambda_2} du \\
&= |x|^{Q\lambda_1} |x|^{Q\lambda_2} \int_G \frac{\Phi(\delta_{|u|^{-1}}e)}{|u|^{2Q}} |u_1|^{Q\lambda_1} |u_2|^{Q\lambda_2} du \\
&= K_2 \tilde{f}_1 \tilde{f}_2,
\end{aligned}
$$

$$\|T_\Phi(\tilde{f}_1,\tilde{f}_2)\|_{\dot{B}^{p,\lambda}(G)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda p}} \int_{B(0,R)} \left|T_\Phi(\tilde{f}_1,\tilde{f}_2)(x)\right|^p dx\right)^{1/p}$$

$$
\begin{aligned}
&= \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda p}} \int_{B(0,R)} K_2 \, |x|^{Q\lambda p} \, dx \right)^{1/p} \\
&= K_2 \sup_{R>0} \left(\frac{\omega_Q}{(\Omega_Q R^Q)^{1+\lambda p}} \int_0^R r^{Q\lambda p+Q-1} dr \right)^{1/p} \\
&= K_2 \frac{1}{\Omega_Q^{\lambda}} \left(\frac{1}{1+\lambda p} \right)^{1/p}.
\end{aligned}
$$

Because of the boundedness of T_Φ, $K_2 < +\infty$. Hence, the necessary and sufficient conditions are obtained.

Finally, by virtue of $\lambda_1 p_1 = \lambda_2 p_2$, an easy computation shows that

$$
\lambda p = \frac{\lambda_1 p_1 p_2 + \lambda_2 p_1 p_2}{p_1 + p_2} = \lambda_1 p_1 = \lambda_2 p_2.
$$

So,

$$
\begin{aligned}
\|T_\Phi(\tilde{f}_1, \tilde{f}_2)\|_{\dot{B}^{p,\lambda}(G)} =& K_2 \frac{1}{\Omega_Q^{\lambda_1+\lambda_2}} \left(\frac{1}{1+\lambda p} \right)^{1/p_1+1/p_2} \\
=& K_2 \|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)} \|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)}.
\end{aligned}
$$

It shows that the constant K_2 is sharp. □

Let $f_1 \in L^{\lambda_1,p_1}(G)$ and $f_2 \in L^{\lambda_2,p_2}(G)$ be radial functions, we can easily prove the Theorem 2.3 by the same method used in the proof of Theorem 2.2.

3.3 Proof of Theorem 2.4

By the polar transformation, $S_\Psi(f_1, f_2)(x)$ is equal to

$$
\begin{aligned}
&\int_G \frac{\Psi(|x|/|u_1|, |x|/|u_2|)}{|u_1|^Q \, |u_2|^Q} f_1(u_1) f_2(u_2) du \\
=& \int_0^\infty \int_0^\infty \frac{\Psi(|x|/t_1, |x|/t_2)}{t_1 t_2} \\
&\quad \times \int_{|u_1'|=1} f_1(\delta_{t_1} u_1') du_1' \int_{|u_2'|=1} f_2(\delta_{t_2} u_2') du_2' dt_1 dt_2 \\
=& \int_0^\infty \int_0^\infty \frac{\Psi(s_1, s_2)}{s_1 s_2} \prod_{i=1}^{2} \int_{|u_i'|=1} f_i(\delta_{|x|/s_i} u_i') du_i' ds_1 ds_2.
\end{aligned}
$$

By Minkowski's inequality and Höder's inequality, we have

$$
\begin{aligned}
&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}|S_{\Psi}(f_1,f_2)(x)|^p\,dx\right)^{\frac{1}{p}}\\
=&\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|\int_0^\infty\int_0^\infty\frac{\Psi(s_1,s_2)}{s_1s_2}\prod_{i=1}^{2}\int_{|u_i'|=1}f_i(\delta_{|x|/s_i}u_i')du_i'ds_1ds_2\right|^p dx\right)^{\frac{1}{p}}\\
\leq&\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}\\
&\times\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|\prod_{i=1}^{2}\int_{|u_i'|=1}f_i(\delta_{|x|/s_i}u_i')du_i'\right|^p dx\right)^{\frac{1}{p}}ds_1ds_2\\
\leq&\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}\\
&\times\prod_{i=1}^{2}\left(\frac{1}{|B(0,R)|^{1+\lambda_ip_i}}\int_{B(0,R)}\left|\int_{|u_i'|=1}f_i(\delta_{|x|/s_i}u_i')du_i'\right|^{p_i}dx\right)^{\frac{1}{p_i}}ds_1ds_2\\
=&\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}\\
&\times\prod_{i=1}^{2}\left(\frac{s_i^{-Q\lambda_ip_i}}{|B(0,s_i^{-1}R)|^{1+\lambda_ip_i}}\int_{B(0,s_i^{-1}R)}\left|\int_{|u_i'|=1}f_i(\delta_{|x|}u_i')du_i'\right|^{p_i}dx\right)^{\frac{1}{p_i}}ds_1ds_2\\
=&\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}s_1^{-Q\lambda_1}s_2^{-Q\lambda_2}\\
&\times\prod_{i=1}^{2}\left(\frac{1}{|B(0,s_i^{-1}R)|^{1+\lambda_ip_i}}\int_{B(0,s_i^{-1}R)}\left|\int_{|u_i'|=1}f_i(\delta_{|x|}u_i')du_i'\right|^{p_i}dx\right)^{\frac{1}{p_i}}ds_1ds_2\\
\leq&\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}s_1^{-Q\lambda_1}s_2^{-Q\lambda_2}\\
&\times\prod_{i=1}^{2}\omega_Q^{\frac{1}{p_i'}}\left(\frac{1}{|B(0,s_i^{-1}R)|^{1+\lambda_ip_i}}\int_{B(0,s_i^{-1}R)}\int_{|u_i'|=1}\left|f_i(\delta_{|x|}u_i')\right|^{p_i}du_i'dx\right)^{\frac{1}{p_i}}ds_1ds_2\\
=&\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}s_1^{-Q\lambda_1}s_2^{-Q\lambda_2}\\
&\times\prod_{i=1}^{2}\omega_Q^{\frac{1}{p_i'}+\frac{1}{p_i}}\left(\frac{1}{|B(0,s_i^{-1}R)|^{1+\lambda_ip_i}}\int_0^{\frac{R}{s_i}}\int_{|u_i'|=1}\left|f_i(\delta_{t_i}u_i')\right|^{p_i}du_i'dt_i\right)^{\frac{1}{p_i}}ds_1ds_2\\
=&\omega_Q^2\int_0^\infty\int_0^\infty\frac{|\Psi(s_1,s_2)|}{s_1s_2}s_1^{-Q\lambda_1}s_2^{-Q\lambda_2}\\
&\times\prod_{i=1}^{2}\left(\frac{1}{|B(0,s_i^{-1}R)|^{1+\lambda_ip_i}}\int_{B(0,s_i^{-1}R)}|f_i(x)|^{p_i}dx\right)^{\frac{1}{p_i}}ds_1ds_2
\end{aligned}
$$

$$\leq \omega_Q^2 \int_0^\infty \int_0^\infty \frac{|\Psi(s_1,s_2)|}{s_1 s_2} s_1^{-Q\lambda_1} s_2^{-Q\lambda_2} ds_1 ds_2 \|f_1\|_{\dot{B}^{p_1,\lambda_1}(G)} \|f_2\|_{\dot{B}^{p_2,\lambda_2}(G)}.$$

On the other hand, we take

$$\tilde{f}_1(x) = |x|^{Q\lambda_1}, \quad \tilde{f}_1(x) = |x|^{Q\lambda_2}.$$

Then,

$$\begin{aligned}
S_\Psi(\tilde{f}_1,\tilde{f}_2)(x) &= \int_0^\infty \int_0^\infty \frac{\Psi(s_1,s_2)}{s_1 s_2} \prod_{i=1}^2 \int_{|u_i'|=1} \tilde{f}_i(\delta_{|x|/s_i} u_i') du_i' ds_1 ds_2 \\
&= \int_0^\infty \int_0^\infty \frac{\Psi(s_1,s_2)}{s_1 s_2} \prod_{i=1}^2 \int_{|u_i'|=1} \left|\delta_{|x|/s_i} u_i'\right|^{Q\lambda_i} du_i' ds_1 ds_2 \\
&= |x|^{Q\lambda_1} |x|^{Q\lambda_2} \omega_Q^2 \int_0^\infty \int_0^\infty \frac{|\Psi(s_1,s_2)|}{s_1 s_2} s_1^{-Q\lambda_1} s_2^{-Q\lambda_2} ds_1 ds_2 \\
&= K_3 \tilde{f}_1 \tilde{f}_2.
\end{aligned}$$

Using the similar method as in the proof of Theorem 2.2, one can yield that when Φ is a nonnegative radial function, S_Ψ is a bounded operator which maps $\dot{B}^{p_1,\lambda_1}(G) \times \dot{B}^{p_2,\lambda_2}(G)$ to $\dot{B}^{p,\lambda}(G)$ if and only if $K_3 < +\infty$. And in the above conditions,

$$\|S_\Psi(\tilde{f}_1,\tilde{f}_2)\|_{\dot{B}^{p,\lambda}(G)} = K_3 \|\tilde{f}_1\|_{\dot{B}^{p_1,\lambda_1}(G)} \|\tilde{f}_2\|_{\dot{B}^{p_2,\lambda_2}(G)},$$

while $\lambda_1 p_1 = \lambda_2 p_2$. Therefore, we complete the proof of Theorem 2.4. □

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