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Direct Result for a Summation-Integral Type Modification of Szász–Mirakjan Operators

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Abstract. In this paper, study of direct result for a summation-integral type modification of Szász–Mirakjan operators is carried out. Calculation of moments, density result and a Voronvskaja-type result are also obtained.

Key Words: Szász–Mirakjan operators, K-functional, modulus of smoothness, Voronovskaja-type result.

AMS Subject Classifications: 41A10, 41A25, 41A36, 40A30

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1 Introduction

In 1941, G. M. Mirakjan [9] defined the operators $SM_n : C_2[0,\infty) \to C[0,\infty)$ for any $x \in [0,\infty)$ and for any $n \in \mathbb{N}$ given by,

$$SM_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, \qquad 0 \le x < \infty. \tag{1.2}$$

and

$$C_2[0,\infty) = \left\{ f \in C[0,\infty) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

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The operators $(SM_n)_{n\in\mathbb{N}}$ are named Szász-Mirakjan operators, where $s_{n,k}$'s are Szász basis functions. They were extensively studied in 1950 by O. Szász [15].

Durrmeyer [3] defined the summation-integral type approximation process, using the Bernstein polynomials, as

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} b_{n,k}(x)\left(\int_0^1 b_{n,k}(t)f(t)dt\right), \tag{1.3}$$

where Bernstein polynomial are given by

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

and

$$b_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k},$$

$0 \le x \le 1$, $k = 0,1,\cdots,n$ and $n \in \mathbb{N}$.

Derriennic [2] studied the operators given by (1.3) extensively. Motivated by Derriennic, Sahai and Prasad [14] studied many properties of the modified Lupaş operators of the type

$$M_n(f;x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\left(\int_0^{\infty} p_{n,k}(t)f(t)dt\right), \tag{1.4}$$

where

$$p_{n,k}(x) = \left(\begin{array}{c} n+k-1 \\ k \end{array}\right) \frac{x^k}{(1+x)^{n+k}},$$

$0 \le x < \infty$, $k = 0,1,2,\cdots$ and $n \in \mathbb{N}$.

Mazhar and Totik [8] introduced two Durrmeyer type modifications of Szász-Mirakjan operators (1.1) as

$$\bar{S}_n(f;x) = f(0)s_{n,0}(x) + n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)dt \tag{1.5}$$

and

$$S_n(f;x) = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k}(t)f(t)dt, \tag{1.6}$$

where $s_{n,k}$'s are as given by (1.2).

Various properties, like global approximation in weight spaces, uniform approximation, simultaneous approximation, weighted approximations, of these operators, their generalizations and modifications are studied over the years. We can mention some important studies of this type (see [4–7, 10–12]).

In 2015, Mishra et al. [13] introduced Szász-Mirakjan-Durrmeyer-type generalization of (1.6) given by

$$S_n^*(f;x) = b_n \sum_{k=0}^{\infty} s_{b_n,k}(x) \int_0^{\infty} s_{b_n,k}(t) f(t)\, dt, \tag{1.7}$$

where

$$s_{b_n,k}(x) = e^{-b_n x} \frac{(b_n x)^k}{k!}, \quad k = 0,1,2,\cdots, \quad n \in \mathbb{N}, \tag{1.8}$$

(b_n) is an increasing sequence of positive real numbers, $b_n \to \infty$ as $n \to \infty$, $b_1 \geq 1$ and studied the simultaneous approximation properties of the operators (1.7). In this article, we present the study of the direct result related properties of the operators (1.7).

2 Estimation of moments

Let us denote by $\mu^*_{n,m}$, the m^{th} moments of the operators given in (1.7), defined as

$$\mu^*_{n,m}(x) = S_n^*((t-x)^m; x), \quad m = 0,1,2,\cdots. \tag{2.1}$$

Lemma 2.1. *For $e_i(t) = t^i$, $i = 0,1,2,3,4$, the following holds:*

(a) $S_n^*(e_0;x) = 1$,

(b) $S_n^*(e_1;x) = \dfrac{b_n x + 1}{b_n}$,

(c) $S_n^*(e_2;x) = \dfrac{1}{b_n^2}(b_n^2 x^2 + 4 b_n x + 2)$,

(d) $S_n^*(e_3;x) = \dfrac{1}{b_n^3}(b_n^3 x^3 + 9 b_n^2 x^2 + 18 b_n x + 6)$,

(e) $S_n^*(e_4;x) = \dfrac{1}{b_n^4}(b_n^4 x^4 + 16 b_n^3 x^3 + 72 b_n^2 x^2 + 96 b_n x + 24)$.

Proof. By (1.7),

$$\begin{aligned} S_n^*(e_2;x) &= \sum_{k=0}^{\infty} s_{b_n,k}(x) b_n \int_0^{\infty} s_{b_n,k}(t) t^2 dt = \sum_{k=0}^{\infty} s_{b_n,k}(x) b_n \int_0^{\infty} e^{-b_n t} \frac{(b_n t)^k}{k!} t^2 dt \\ &= \sum_{k=0}^{\infty} s_{b_n,k}(x) \frac{(k+1)(k+2)}{b_n^2} = \frac{1}{b_n^2} \sum_{k=0}^{\infty} e^{-b_n x} \frac{(b_n x)^k}{k!} [k(k-1) + 4k + 2] \\ &= \frac{1}{b_n^2}(b_n^2 x^2 + 4 b_n x + 2). \end{aligned}$$

This proves (c). Other relations follow on the same line. □

Consider the Banach lattice

$$C_\gamma[0,\infty)=\{f\in C[0,\infty):|f(t)|\le M(1+t)^\gamma\}$$

for some $M>0$, $\gamma>0$.

Theorem 2.1. $\lim_{n\to\infty}S_n^*(f;x)=f(x)$ *uniformly for* $x\in[0,a]$, *provided* $f\in C_\gamma[0,\infty)$, $\gamma\ge 2$ *and* $a>0$.

Proof. For fix $a>0$, consider the lattice homomorphism $T_a:C[0,\infty)\to C[0,a]$ defined by $T_a(f):=f|_{[0,a]}$ for every $f\in C[0,\infty)$, where $f|_{[0,a]}$ denotes the restriction of the domain of f to the interval $[0,a]$. In this case, we see that, for each $i=0,1,2$ and by (a)-(c) of Lemma 2.1,

$$\lim_{n\to\infty}T_a\left(S_n^*(e_i;x)\right)=T_a(e_i(x))\quad\text{uniformly on }\ [0,a]. \tag{2.2}$$

Thus, by using (2.2) and with the universal Korovkin-type property with respect to positive linear operators (see Theorem 4.1.4 (vi) of [1], pp. 199) we have the result. □

Lemma 2.2. *For the moments defined in* (2.1)*, the following holds:*

(a) $\mu_{n,1}^*(x)=S_n^*((t-x);x)=\dfrac{1}{b_n}$,

(b) $\mu_{n,2}^*(x)=S_n^*((t-x)^2;x)=\dfrac{2b_nx+2}{b_n^2}$,

(c) $\mu_{n,3}^*(x)=S_n^*((t-x)^3;x)=\dfrac{12b_nx+6}{b_n^3}$,

(d) $\mu_{n,4}^*(x)=S_n^*((t-x)^4;x)=\dfrac{12b_n^2x^2+72b_nx+24}{b_n^4}$.

Proof. The results follow from linearity of the operators S_n^* and Lemma 2.1. □

3 Direct result

Let us consider the space $C_B[0,\infty)$ of all continuous and bounded functions on $[0,\infty)$ and for $f\in C_B[0,\infty)$, consider the supremum norm $\|f\|=\sup\{|f(x)|:x\in[0,\infty)\}$. Also, consider the $K-$functional

$$K_2(f;\delta)=\inf_{g\in W^2}\left\{\|f-g\|+\delta\|g''\|\right\}, \tag{3.1}$$

where $\delta>0$ and $W^2=\left\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\right\}$. For a constant $C>0$, the following relationship exists:

$$K_2(f;\delta)\le C\omega_2(f,\sqrt{\delta}), \tag{3.2}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)| \tag{3.3}$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$; and for $f \in C_B[0,\infty)$, let the modulus of continuity be given by

$$\omega_1(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|. \tag{3.4}$$

Theorem 3.1. *For $f \in C_B[0,\infty)$, we have*

$$|S_n^*(f;x) - f(x)| \le \omega_1\left(f,\frac{1}{b_n}\right) + C\omega_2\left(f,\sqrt{\mu_{n,2}^*(x) + \frac{1}{b_n^2}}\right),$$

where C is a positive constant.

Proof. Let the auxiliary operator denoted by $\bar{S}_n^*$ be defined as

$$\bar{S}_n^*(f;x) = S_n^*(f;x) - f\left(\frac{b_n x+1}{b_n}\right) + f(x)$$

for every $x \in [0,\infty)$. It is a linear operator which preserves the linear functions as $\bar{S}_n^*(1;x) = 1$ and $\bar{S}_n^*(t;x) = x$. This gives us $\bar{S}_n^*(t-x;x) = 0$.

For $g \in W^2$, $x \in [0,\infty)$ and by Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Operating $\bar{S}_n^*$ on both the sides,

$$\begin{aligned}
\left|\bar{S}_n^*(g;x) - g(x)\right| &= \left|\bar{S}_n^*\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\
&\le \left|S_n^*\left(\int_x^t (t-u)g''(u)du;x\right)\right| + \left|\int_x^{\frac{b_n x+1}{b_n}} \left(\frac{b_n x+1}{b_n} - u\right) g''(u)du\right| \\
&\le \|g''\| S_n^*((t-x)^2;x) + \|g''\| \left(\frac{b_n x+1}{b_n} - x\right)^2 \\
&= \|g''\| \left[\mu_{n,2}^*(x) + \frac{1}{b_n^2}\right].
\end{aligned}$$

Also, we have $|S_n^*(f;x)| \le \|f\|$. Using these, we get

$$\begin{aligned}
|S_n^*(f;x) - f(x)| &\le \left|\bar{S}_n^*(f-g;x) - (f-g)(x)\right| + \left|\bar{S}_n^*(g;x) - g(x)\right| + \left|f\left(\frac{b_n x+1}{b_n}\right) - f(x)\right| \\
&\le 4\|f-g\| + \|g''\| \left[\mu_{n,2}^*(x) + \frac{1}{b_n^2}\right] + \omega_1\left(f,\frac{1}{b_n}\right).
\end{aligned}$$

Taking infimum on the right hand side for all $g \in W^2$, we get

$$|S_n^*(f;x) - f(x)| \leq 4K_2 \left(f, \frac{1}{4}\left[\mu_{n,2}^*(x) + \frac{1}{b_n^2}\right]\right) + \omega_1\left(f, \frac{1}{b_n}\right).$$

Using (3.2) and $\omega_2(f,\lambda\delta) \leq (\lambda+1)^2\omega_2(f,\delta)$ for $\lambda > 0$, we get

$$|S_n^*(f;x) - f(x)| \leq C\omega_2\left(f, \sqrt{\mu_{n,2}^*(x) + \frac{1}{b_n^2}}\right) + \omega_1\left(f, \frac{1}{b_n}\right).$$

So, we complete the proof. □

4 A Voronovskaja-type result

In this section we prove a Voronovskaja-type theorem for the operators S_n^*.

Lemma 4.1. $\lim_{n\to\infty} b_n^2\mu_{n,4}^*(x) = 12x^2$ *uniformly with respect to* $x \in [0,a]$, $a > 0$.

Proof. Using Lemma 2.2(d), we get the result immediately. □

Theorem 4.1. *For every* $f \in C_\gamma[0,\infty)$ *such that* $f', f'' \in C_\gamma[0,\infty)$, $\gamma \geq 4$, *we have*

$$\lim_{n\to\infty} b_n\left[S_n^*(f;x) - f(x) - \frac{1}{b_n}f'(x)\right] = xf''(x)$$

with respect to $x \in [0,a]$, $(a > 0)$.

Proof. Let $f, f', f'' \in C_\gamma[0,\infty)$ and $x \geq 0$. Define, for $t \neq x$

$$\Psi(t,x) = \frac{1}{(t-x)^2}\left[f(t) - f(x) - (t-x)f'(x) - \frac{1}{2}(t-x)^2 f''(x)\right],$$

and $\Psi(x,x) = 0$. Then the function $\Psi(\cdot,x) \in C_\gamma[0,\infty)$. Hence, by Taylor's theorem we get

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + (t-x)^2\Psi(t,x).$$

Now from Lemma 2.2(a)-(b)

$$b_n[S_n^*(f;x) - f(x)] = b_n f'(x)\mu_{n,1}^*(x) + \frac{1}{2}b_n f''(x)\mu_{n,2}^*(x) + b_n S_n^*((t-x)^2\Psi(t,x)). \quad (4.1)$$

If we apply the Cauchy-Schwarz inequality to the third term on the right hand side of (4.1), then

$$b_n S_n^*((t-x)^2\Psi(t,x);x) \leq \left(b_n^2\mu_{n,4}^*(x)\right)^{\frac{1}{2}} \left(S_n^*(\Psi^2(t,x);x)\right)^{\frac{1}{2}}.$$

Now $\Psi^2(\cdot,x)\in C_\gamma[0,\infty)$, using Theorem 2.1, we have $S_n^*(\Psi^2(t,x);x)\to\Psi^2(x,x)=0$, as $n\to\infty$ and using Lemma 4.1, this third term on the right tends to zero for $x\in[0,a]$ and we get

$$\lim_{n\to\infty} b_n\,[S_n^*(f;x)-f(x)]=f^{'}(x)+xf^{''}(x),$$

for $x\in[0,a]$, $(a>0)$. □

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