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On Sharpening of a Theorem of Ankeny and Rivlin

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Abstract. Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n,

$$M(p,R) =: \max_{|z|=R\geq 0} |p(z)| \quad \text{and} \quad M(p,1) =: \|p\|.$$

Then according to a well-known result of Ankeny and Rivlin [1], we have for $R\geq 1$,

$$M(p,R)\leq \left(\frac{R^n+1}{2}\right)\|p\|.$$

This inequality has been sharpened by Govil [4], who proved that for $R\geq 1$,

$$M(p,R)\leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2-4|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\|p\|}{\|p\|+2|a_n|} - \ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right)\right\}.$$

In this paper, we sharpen the above inequality of Govil [4], which in turn sharpens the inequality of Ankeny and Rivlin [1].

Key Words: Inequalities, polynomials, zeros.

AMS Subject Classifications: 15A18, 30C10, 30C15, 30A10

1 Introduction

Let $p(z)=\sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n, and for $R\geq 0$, let

$$M(p,R) =: \max_{|z|=R} |p(z)|.$$

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We will denote $M(p,1)$ by $\|p\|$. Then, by the maximum modulus principle $M(p,R)$ is a strictly increasing function of R and is defined for $0 \leq R < \infty$. Also, it is a simple deduction from the maximum modulus principle (see [11, pp. 158, Problem 269]) that for $R \geq 1$,

$$M(p,R) \leq R^n \|p\|. \tag{1.1}$$

The result is best possible and equality holds if and only if $p(z) = \lambda z^n$, λ being a complex number.

For polynomials of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [1] sharpened inequality (1.1), by proving following result.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \Big(\frac{R^n+1}{2}\Big) \|p\|, \quad R \geq 1. \tag{1.2}$$

The above inequality is sharp and equality holds for polynomials having all their zeros on the unit circle.

Several papers and research monographs have been written on this subject (see, for example Frappier, Rahman and Ruscheweyh [2], Gardner, Govil and Weems [3], Govil [5], Govil, Qazi and Rahman [6], Milovanović, Mitrinović and Rassias [8], Nwaeze [10], Rahman and Schmeisser [13,14], Sharma and Singh [15], and Zireh [16]).

A refinement of the above inequality (1.2) was given by Govil [4], who proved

Theorem 1.2. *If $p(z)$ is a polynomial of degree n, and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \leq \left(\frac{R^n+1}{2}\right) \|p\| - \frac{n}{2} \left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right) \bigg\{ &\frac{(R-1)\|p\|}{\|p\| + 2|a_n|} \\ &- \ln\left(1 + \frac{(R-1)\|p\|}{\|p\| + 2|a_n|}\right)\bigg\}. \end{aligned} \tag{1.3}$$

The result is best possible and the equality holds for $p(z) = (\lambda + \mu z^n)$, where λ and μ are complex numbers with $|\lambda| = |\mu|$.

2 Main results

In this paper, we prove the following result which sharpens the above Theorem 1.2 due to Govil [4], and so in turn Theorem 1.1 due to Ankeny and Rivlin [1].

Theorem 2.1. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$ and any N, $1 \leq N \leq n$,*

$$M(p,R) \leq \frac{(R^n+1)}{2} \|p\| - \frac{n}{2} \|p\| \left(1 - \frac{2|a_n|}{\|p\|}\right) h(N), \tag{2.1}$$

where

$$h(N) = (R-1) - \left(1 + \frac{2|a_n|}{\|p\|}\right) \ln\left(1 + \frac{(R-1)\,\|p\|}{\|p\| + 2|a_n|}\right) \qquad \textit{for}\ \ N=1, \tag{2.2a}$$

$$\begin{aligned} h(N) = {} & \left(\frac{R^N-1}{N}\right) + \sum_{k=1}^{N-1} (-1)^k \left(\frac{R^{N-k}-1}{N-k}\right)\left(\frac{2\,|a_n|}{\|p\|} + 1\right)\left(\frac{2\,|a_n|}{\|p\|}\right)^{k-1} \\ & + (-1)^N \left(\frac{2\,|a_n|}{\|p\|} + 1\right)\left(\frac{2\,|a_n|}{\|p\|}\right)^{N-1} \ln\left(1 + \frac{(R-1)\,\|p\|}{\|p\| + 2|a_n|}\right) \qquad \textit{for}\ \ N \geq 2. \end{aligned} \tag{2.2b}$$

Remark 2.1. It has been proved in Lemma 3.7 that for any N, $1 \leq N \leq n$, we have

$$h(N) = \int_1^R \frac{(r-1)r^{N-1}}{(r + \frac{2|a_n|}{\|p\|})} dr$$

for $N \geq 1$, which clearly implies that $h(N)$ is a non-negative and an increasing function of N for $N \geq 1$. Using this property of $h(N)$ in inequality (2.1) of Theorem 2.1, we get that for all $N \geq 1$,

$$M(p,R) \leq \frac{(R^n+1)}{2}\,\|p\| - \frac{n}{2}\,\|p\|\left(1 - \frac{2\,|a_n|}{\|p\|}\right) h(N) \leq \frac{(R^n+1)}{2}\,\|p\|,$$

which is Theorem 1.1 due to Ankeny and Rivlin [1]. Thus our Theorem 2.1, clearly generalizes and in fact sharpens Theorem 1.1 due to Ankeny and Rivlin [1].

In Theorem 2.1, it is sufficient to consider the case when the polynomial is of degree $n \geq 1$, because if $n = 0$ then the polynomial $p(z)$ is a constant, and therefore $M(p,R) = \|p\|$ for all R, and so the Theorem holds trivially.

Remark 2.2. If in particular, we take $N = 1$ in Theorem 2.1, we get

$$M(p,R) \leq \frac{(R^n+1)}{2}\,\|p\| - \frac{n}{2}\,\|p\|\left(1 - \frac{2\,|a_n|}{\|p\|}\right) h(1), \tag{2.3}$$

and this when combined with the fact that

$$h(1) = \int_1^R \frac{(r-1)}{(r + \frac{2|a_n|}{\|p\|})} dr = (R-1) - \left(1 + \frac{2\,|a_n|}{\|p\|}\right) \ln\left(1 + \frac{(R-1)\,\|p\|}{\|p\| + 2|a_n|}\right), \tag{2.4}$$

easily gives

$$\begin{aligned} M(p,R) \leq {} & \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\,\|p\|}{\|p\| + 2|a_n|}\right. \\ & \left. - \ln\left(1 + \frac{(R-1)\,\|p\|}{\|p\| + 2|a_n|}\right)\right\}, \end{aligned}$$

which is Theorem 1.2 due to Govil [4].

Similarly, if we take $N=2$ in Theorem 2.1, we get the following result which gives an improvement of Theorem 1.2 due to Govil [4].

Corollary 2.1. *If $p(z)$ is a polynomial of degree $n \geq 2$ and $p(z) \neq 0$ for $|z|<1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \leq& \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\|p\|\left(1-\frac{2|a_n|}{\|p\|}\right) \times \left[\frac{(R^2-1)}{2} - (R-1)\left(1+\frac{2|a_n|}{\|p\|}\right)\right. \\ &\left. + \left(1+\frac{2|a_n|}{\|p\|}\right)\left(\frac{2|a_n|}{\|p\|}\right)\ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right)\right]. \end{aligned}$$

Proof. To prove the above Corollary, we take $N=2$ in Theorem 2.1, which gives

$$M(p,R) \leq \frac{(R^n+1)}{2}\|p\| - \frac{n}{2}\|p\|\left(1-\frac{2|a_n|}{\|p\|}\right)h(2). \tag{2.5}$$

It is easy to see that

$$\begin{aligned} h(2) =& \int_1^R \frac{(r-1)r}{(r+\frac{2|a_n|}{\|p\|})}dr \\ =&\frac{(R^2-1)}{2} - (R-1)\left(1+\frac{2|a_n|}{\|p\|}\right) \\ &+ \left(1+\frac{2|a_n|}{\|p\|}\right)\left(\frac{2|a_n|}{\|p\|}\right)\ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right). \end{aligned} \tag{2.6}$$

Substituting the above value of $h(2)$ in inequality (2.5) will complete the proof of Corollary 2.1. □

Remark 2.3. Since by Remark 2.2, we have $h(N) \geq 0$ and $h(2) \geq h(1) \geq 0$, hence in view of (2.3) and (2.5), we have

$$\begin{aligned} M(p,R) \leq& \frac{(R^n+1)}{2}\|p\| - \frac{n}{2}\|p\|\left(1-\frac{2|a_n|}{\|p\|}\right)h(2) \\ \leq& \frac{(R^n+1)}{2}\|p\| - \frac{n}{2}\|p\|\left(1-\frac{2|a_n|}{\|p\|}\right)h(1), \end{aligned}$$

showing that the Corollary 2.1 always gives an improvement of Theorem 1.2, due to Govil [4]. Also, it is easy to see that the improvement in the bound is in fact

$$\frac{n}{2}\|p\|\left(1-\frac{2|a_n|}{\|p\|}\right)[h(2)-h(1)], \tag{2.7}$$

where $h(1)$ and $h(2)$ are as given by (2.4) and (2.6) respectively.

It may be remarked that in some cases this improvement can be quite significant, and this has been shown by means of an example given in Section 5.

Also, note that if in the proof of Corollary 2.1, instead of taking $N=2$ we take $N=3$, and use the fact that $h(3) \geq h(2)$, then the bound in Corollary 2.1 can be further improved. Further, as is easy to see, this process in fact can be continued in order to obtain better and better bounds, and obviously the best bound by this process will be obtained when $N=n$, the degree of the polynomial.

3 Lemmas

For the proof of theorem, we need the following lemmas. Our first lemma is a well-known generalization of Schwarz's lemma (see for example [9, pp. 167]).

Lemma 3.1. *If $f(z)$ is analytic inside and on the circle $|z|=1$, $f(0)=a$, where $|a|<\|f\|$, then*

$$|f(z)| \leq \|f\| \left(\frac{\|f\|\,|z|+|a|}{|a|\,|z|+\|f\|} \right). \tag{3.1}$$

Lemma 3.2. *If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then for $|z|=R\geq 1$,*

$$|p(z)| \leq \left(\frac{|a_n|\,R+\|p\|}{\|p\|\,R+|a_n|} \right) \|p\|\,R^n. \tag{3.2}$$

The proof follows easily on applying Lemma 3.1 to the function $T(z)=z^n p(1/z)$ and noting that $\|T\|=\|p\|$ (for details see [12, Lemma 2]).

From Lemma 3.2, one immediately gets

Lemma 3.3. *If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then for $|z|=R\geq 1$,*

$$|p(z)| \leq \left(1-\frac{(\|p\|-|a_n|)(R-1)}{\|p\|\,R+|a_n|} \right) \|p\|\,R^n. \tag{3.3}$$

The following result is the well known Erdös conjecture, proved for the first time by Lax [7].

Lemma 3.4. *If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, and $p(z)\neq 0$ for $|z|<1$, then*

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{2}\,\|p\|. \tag{3.4}$$

Lemma 3.5 ([4, Lemma 5])**.** *If $p(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n and $r\geq 1$, then*

$$\left(1-\frac{(x-n\,|a_n|)(r-1)}{rx+n\,|a_n|} \right) x \tag{3.5}$$

is an increasing function of x, for $x>0$.

Lemma 3.6. *Let*

$$I(N)=\int_1^R \frac{(r-1)(r^{N-1})}{r+a}dr,$$

where $a=2|a_n|/\|p\|$. *Then for* $N\geq 2$,

$$\begin{aligned} I(N)=&\left(\frac{R^N-1}{N}\right)+\sum_{k=1}^{N-1}(-1)^k\left(\frac{R^{N-k}-1}{N-k}\right)(a+1)a^{k-1} \\ &+(-1)^N(a+1)a^{N-1}\ln\left(\frac{R+a}{1+a}\right)=h(N), \end{aligned} \tag{3.6}$$

and for $N=1$,

$$I(N)=(R-1)-(1+a)\ln\left(1+\frac{R-1}{1+a}\right)=h(1). \tag{3.7}$$

Proof. We begin with the proof of (3.6) and for this we define the function

$$f(N)=\int_1^R\frac{r^N}{r+a}dr,$$

which is defined for $N\geq 0$. Here $a=2|a_n|/\|p\|$. Then, it is easy to see that $I(N)=f(N)-f(N-1)$ for $N\geq 1$.

Note that for $N\geq 2$,

$$\begin{aligned} f(N)+af(N-1)&=\int_1^R\frac{r^N+ar^{N-1}}{r+a}dr \\ =\int_1^R\frac{r^{N-1}(r+a)}{r+a}dr&=\frac{R^N-1}{N}=g(N). \end{aligned}$$

Therefore

$$f(N)=g(N)-af(N-1)\quad \text{for}\quad N\geq 1, \tag{3.8}$$

and on solving the recurrence relation (3.8), we get

$$f(N)=\sum_{k=0}^{N-1}(-1)^k g(N-k)a^k+(-1)^N a^N f(0),\quad N\geq 1, \tag{3.9}$$

where

$$f(0)=\int_1^R\frac{1}{r+a}dr=\ln\Big(\frac{R+a}{1+a}\Big).$$

If we substitute the value of $f(0)$ in (3.9), we clearly get

$$f(N)=\sum_{k=0}^{N-1}(-1)^k g(N-k)a^k+(-1)^N a^N\ln\left(\frac{R+a}{1+a}\right),\quad N\geq 1. \tag{3.10}$$

Now, on using $I(N) = f(N) - f(N-1)$ and $g(N) = \frac{R^N-1}{N}$, we get that for $N \geq 2$,

$$I(N) = \left(\frac{R^N-1}{N}\right) + \sum_{k=1}^{N-1}(-1)^k\left(\frac{R^{N-k}-1}{N-k}\right)(a+1)a^{k-1} + (-1)^N(a+1)a^{N-1}\ln\left(\frac{R+a}{1+a}\right),$$

where $a = 2|a_n|/\|p\|$, and on substituting this value of a in the above, we get that for every $N, 1 \leq N \leq n$,

$$I(N) = \left(\frac{R^N-1}{N}\right) + \sum_{k=1}^{N-1}(-1)^k\left(\frac{R^{N-k}-1}{N-k}\right)\left(\frac{2|a_n|}{\|p\|}+1\right)\left(\frac{2|a_n|}{\|p\|}\right)^{k-1} + (-1)^N\left(\frac{2|a_n|}{\|p\|}+1\right)\left(\frac{2|a_n|}{\|p\|}\right)^{N-1}\ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right), \quad \text{if} \quad N \geq 2. \tag{3.11}$$

The above is same as $h(N)$, and the inequality (3.6) is thus proved.

To prove (3.7), note that

$$I(1) = \int_1^R \frac{(r-1)}{r+a}dr,$$

and this when evaluated gives

$$I(1) = (R-1) - (1+a)\ln\left(1+\frac{(R-1)}{1+a}\right),$$

which is $h(1)$, and thus (3.7) is proved. □

Since by the above Lemma 3.6, we have

$$h(N) = I(N) = \int_1^R \frac{(r-1)r^{N-1}}{(r+a)}dr \quad \text{for all} \quad N \geq 1,$$

we immediately get the following:

Lemma 3.7. *If $h(N)$ is as defined in Theorem 2.1, then for $R \geq 1$, we have*

$$h(N) = I(N) = \int_1^R \frac{(r-1)r^{N-1}}{(r+a)}dr,$$

for all $N \geq 1$. Further, $h(N)$ is a nonnegative and increasing function of N.

4 Proof of the theorem

Proof. As has also been mentioned in Section 2, it is sufficient to prove our Theorem 2.1 when the polynomial is of degree $n \geq 1$, because if $n = 0$, then the polynomial $p(z)$ is constant, so $M(p, R) = \|p\|$, for all R, and the theorem holds trivially.

Since $p(z)$ is of degree $n \geq 1$, hence for each θ, where $0 \leq \theta < 2\pi$, we have,

$$\begin{aligned} |p(Re^{i\theta}) - p(e^{i\theta})| &= \left| \int_1^R p'(re^{i\theta}) e^{i\theta} dr \right| \leq \int_1^R |p'(re^{i\theta})| dr \\ &\leq \int_1^R r^{n-1} \left\{ 1 - \frac{(\|p'\| - n|a_n|)(r-1)}{n|a_n| + r\,\|p'\|} \right\} \|p'\|\, dr, \end{aligned} \tag{4.1}$$

by applying Lemma 3.3 to the polynomial $p'(z)$, which is of degree $n-1$.

Now, by Lemma 3.5, the integrand in (4.1) is an increasing function of $\|p'\|$, therefore on applying Lemma 3.4 to inequality (4.1), we get that for $0 \leq \theta < 2\pi$,

$$\begin{aligned} |p(Re^{i\theta}) - p(e^{i\theta})| &\leq \int_1^R r^{n-1} \left\{ 1 - \frac{(\frac{n}{2}\|p\| - n|a_n|)(r-1)}{n|a_n| + \frac{rn}{2}\|p\|} \right\} \frac{n}{2}\|p\|\, dr \\ &= \frac{(R^n - 1)}{2}\|p\| - \frac{n}{2}\|p\| \left(1 - \frac{2\,|a_n|}{\|p\|} \right) \int_1^R \frac{(r-1)r^{n-1}}{(r + \frac{2|a_n|}{\|p\|})} dr \\ &= \frac{(R^n - 1)}{2}\|p\| - \frac{n}{2}\|p\|\,(1-a) \int_1^R \frac{(r-1)r^{n-1}}{(r+a)} dr, \end{aligned} \tag{4.2}$$

where $a = 2\,|a_n| / \|p\|$.

Note that the integral $\int_1^R \frac{(r-1)r^{N-1}}{(r+a)} dr$ is a nonnegative and increasing function of N for $1 \leq N \leq n$, therefore for every N, where $1 \leq N \leq n$, we have

$$\int_1^R \frac{(r-1)r^{N-1}}{(r+a)} dr \leq \int_1^R \frac{(r-1)r^{n-1}}{(r+a)} dr. \tag{4.3}$$

If we use the above inequality (4.3) in (4.2), we get that for every N, where $1 \leq N \leq n$,

$$|p(Re^{i\theta}) - p(e^{i\theta})| \leq \frac{(R^n - 1)}{2}\|p\| - \frac{n}{2}\|p\|\,(1-a) \int_1^R \frac{(r-1)(r^{N-1})}{(r+a)} dr, \tag{4.4}$$

which clearly gives that for every N, where $1 \leq N \leq n$,

$$|p(Re^{i\theta})| \leq \frac{(R^n + 1)}{2}\|p\| - \frac{n}{2}\|p\|\,(1-a) \int_1^R \frac{(r-1)(r^{N-1})}{(r+a)} dr. \tag{4.5}$$

Since by Lemma 3.7, we have

$$\int_1^R \frac{(r-1)r^{N-1}}{r+a} dr = I(N) = h(N) \quad \text{for} \quad 1 \leq N \leq n,$$

the inequality (4.5) is in fact equivalent to that for $0 \leq \theta < 2\pi$, and every N, where $1 \leq N \leq n$,

$$|p(Re^{i\theta})| \leq \frac{(R^n+1)}{2}\,\|p\| - \frac{n}{2}\,\|p\| \left(1 - \frac{2\,|a_n|}{\|p\|}\right) h(N), \tag{4.6}$$

which is clearly equivalent to

$$M(p,R) \leq \frac{(R^n+1)}{2}\,\|p\| - \frac{n}{2}\,\|p\| \left(1 - \frac{2\,|a_n|}{\|p\|}\right) h(N) \quad \text{for } 1 \leq N \leq n, \tag{4.7}$$

and Theorem 2.1 is thus proved. □

5 Computation

Consider the polynomial $p(z) = (z-2)^2$. Then $p(z) \neq 0$ for $|z| < 1$ and $\|p\| = 9$. If we take $R = 3$, then $M(p,3) = 25$, and as is easy to see that by using Theorem 1.2, we get

$$M(p,3) \leq 45 - 7 * \left(2 - \frac{11}{9}\ln(29/11)\right) \simeq 39.29, \tag{5.1}$$

while if we use Corollary 2.1 of Theorem 2.1, then we get

$$M(p,3) \leq 45 - 7 * \left(4 - \frac{22}{9} + \frac{22}{81}\ln(29/11)\right) \simeq 32.26, \tag{5.2}$$

which is smaller than the bound 39.29, obtained by using Theorem 1.2. Thus the Corollary 2.1 of Theorem 2.1 gives an improvement over Theorem 1.2, due to Govil [4], and in fact the bound obtained by Corollary 2.1 is about 82% of the bound obtained from Theorem 1.2. In fact, as is easy to see, by making an appropriate choice of the polynomial $p(z)$, N and R, this improvement can be made bigger and bigger.

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