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# Local Well-Posedness for the Compressible Nematic Liquid Crystals Flow with Vacuum

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

**Abstract.** In this paper we prove the local well-posedness of strong solutions to the compressible nematic liquid crystals flow with vacuum in a bounded domain $\Omega \subset \mathbb{R}^3$.

**Key Words**: Liquid crystals, vacuum, Local well-posedness, strong solution.

**AMS Subject Classifications**: 76D03, 35Q30, 35Q35

## 1 Introduction

In this paper we consider the following simplified version of the Ericksen-Leslie system modeling the flow of compressible nematic liquid crystals (see [2,3]):

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0, \tag{1.1a}$$
$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u = -\nabla d \cdot \Delta d, \tag{1.1b}$$
$$\partial_t d + u \cdot \nabla d - \Delta d = d|\nabla d|^2 \quad \text{in } \Omega \times (0,T), \tag{1.1c}$$

with boundary and initial conditions:

$$u = 0, \quad \frac{\partial d}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.2a}$$
$$(\rho, u, d)(\cdot, 0) = (\rho_0, u_0, d_0)(\cdot) \qquad \text{in } \Omega \subset \mathbb{R}^3, \tag{1.2b}$$

where $\rho \geq 0$ is the density of the fluid, $u \in \mathbb{R}^3$ represents velocity field of the fluid, $d \in \mathbb{S}^2$ represents the macroscopic average of the nematic liquid crystals orientation field. The

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parameters $\mu$ and $\lambda$ are shear viscosity and the bulk viscosity coefficients of the fluid, respectively, satisfying the physical conditions:

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

We assume that the pressure $p$ satisfies the $\gamma$-law, i.e., $p =: a\rho^\gamma$ with constants $a > 0$ and $\gamma > 1$. The domain $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, and $n$ is the unit outward normal vector to $\partial\Omega$.

Below let us review some results to the system (1.1a)-(1.1c) briefly. Ding et al. [2] first introduced the system (1.1a)-(1.1c) and studied the low Mach number limit of it, see [9–11] on the recent progress on this topic. Huang, Wang, and Wen [3] (see also [1,5]) showed the local well-posedness of strong solutions with vacuum under the following compatibility condition:

$$-\mu\Delta u_0 - (\lambda + \mu)\nabla \mathrm{div}\, u_0 + \nabla(a\rho_0^\gamma) + \nabla d_0 \cdot \Delta d_0 = \sqrt{\rho_0} g \tag{1.3}$$

for some $g \in L^2(\Omega)$. Jiang, Jiang, and Wang [4] (see also [6]) proved the global existence of weak solutions in $\mathbb{R}^2$. Lin, Lai and Wang [7] established the existence of global weak solutions with finite energy and density satisfying the renormalized continuity equation, provided the initial orientation director field lies in the hemisphere $\mathbb{S}^2_+$.

The purpose of this paper is to establish the local well-posedness of strong solutions of the compressible nematic liquid crystal model (1.1a)-(1.1c) without the compatibility condition (1.3).

We will prove

**Theorem 1.1.** *Let* $0 \leq \rho_0 \in W^{1,q}$, $(3 < q < 6)$, $u_0 \in H^1_0$, $d_0 \in H^2$ *with* $|d_0| = 1$. *Then the problem* (1.1a)-(1.2b) *has a unique local strong solution* $(\rho, u, d)$ *satisfying*

$$\begin{cases} \rho \in C([0,T];L^2) \cap L^\infty(0,T;W^{1,q}), & \partial_t\rho \in L^\infty(0,T;L^2), \\ \rho u \in C([0,T];L^2),\ u \in L^\infty(0,T;H^1_0) \cap L^2(0,T;H^2), & \sqrt{\rho}\partial_t u \in L^2(0,T;L^2), \\ \sqrt{t} u \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,q}), & \sqrt{t}\partial_t u \in L^2(0,T;H^1_0), \\ d \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3),\ \partial_t d \in L^2(0,T;H^1), & \sqrt{t}\partial_t d \in L^\infty(0,T;H^1), \end{cases} \tag{1.4}$$

*for some* $0 < T \leq \infty$.

We will prove Theorem 1.1 in the following way: For $\delta > 0$, we choose $0 < \delta \leq \rho_0^\delta \in H^2$ and $u_0^\delta \in H^1_0 \cap H^2$ satisfying

$$\rho_0^\delta \to \rho_0 \quad \text{in } W^{1,q} \quad \text{and} \quad u_0^\delta \to u_0 \quad \text{in } H^1_0 \quad \text{as } \delta \to 0. \tag{1.5}$$

Then it is easy to verify that the problem (1.1a)-(1.2b) has a unique local strong solution $(\rho^\delta, u^\delta, d^\delta)$ in $[0, T_\delta)$.

We further define

$$\begin{aligned} M(t) =: &1 + \sup_{0\le s\le t}\Big\{\|\rho(\cdot,s)\|_{W^{1,q}} + \|u(\cdot,s)\|_{H^1} + \sqrt{s}\|\sqrt{\rho}u_t(\cdot,s)\|_{L^2} \\ &+ \|d(\cdot,s)\|_{H^2} + \sqrt{s}\|d_t(\cdot,s)\|_{H^1}\Big\} + \|u\|_{L^2(0,t;H^2)} \\ &+ \|\sqrt{\rho}u_t\|_{L^2(0,t;L^2)} + \|\sqrt{s}\nabla u_t\|_{L^2(0,t;L^2)} \\ &+ \|d\|_{L^2(0,t;H^3)} + \|d_t\|_{L^2(0,t;H^1)} \end{aligned} \tag{1.6}$$

and prove

**Theorem 1.2.** *For any $t \in [0, T_\delta)$, we have that*

$$M(t) \le C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\} \tag{1.7}$$

*for some nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.*

It follows from (1.7) that (see [8]):

$$M(t) \le C \tag{1.8}$$

and thus the proof of existence part of Theorem 1.1 is complete by taking $\delta \to 0$ and the standard compactness principle. We present the proof of Theorem 1.2 in Section 2 and the uniqueness part of Theorem 1.1 in Section 3.

## 2 Proof of Theorem 1.2

Below, for the sake of notational simplicity, we shall drop the superscript "$\delta$" of $\rho^\delta$, $u^\delta$ and $d^\delta$. We also ignore to write down the domain $\Omega$ in the subsequent integrals.

Testing (1.1b) by $u$ and using (1.1a), we see that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2\mathrm{d}x + \mu\int|\nabla u|^2\mathrm{d}x + (\lambda+\mu)\int(\mathrm{div}\, u)^2\mathrm{d}x \\ =&\int p\,\mathrm{div}\, u\mathrm{d}x - \int(u\cdot\nabla)d\cdot\Delta d\,\mathrm{d}x. \end{aligned} \tag{2.1}$$

Testing (1.1c) by $-\Delta d - d|\nabla d|^2$ and using $d\cdot d_t = 0$ and $d\cdot\nabla d = 0$, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d|^2\mathrm{d}x + \int|\Delta d + d|\nabla d|^2|^2\mathrm{d}x = \int(u\cdot\nabla)d\cdot\Delta d\mathrm{d}x. \tag{2.2}$$

Summing (2.1) and (2.2) up, and rewriting the continuity equation (1.1a) as

$$p_t + u\cdot\nabla p + \gamma p\mathrm{div}\, u = 0, \tag{2.3}$$

we infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(\rho|u|^2+|\nabla d|^2)\mathrm{d}x+\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2+|\Delta d+d|\nabla d|^2|^2)\mathrm{d}x$$
$$=\int p\mathrm{div}\,u\mathrm{d}x=-\frac{1}{\gamma-1}\int p_t\mathrm{d}x,$$

which gives

$$\int\left(\rho|u|^2+|\nabla d|^2+\frac{2a}{\gamma-1}\rho^\gamma\right)\mathrm{d}x+2\int_0^T\int(|\nabla u|^2+|\Delta d+d|\nabla d|^2|^2)\mathrm{d}x\mathrm{d}t$$
$$\leq\int\left(\rho_0|u_0|^2+|\nabla d_0|^2+\frac{2a}{\gamma-1}\rho_0^\gamma\right)\mathrm{d}x. \tag{2.4}$$

Testing (1.1c) by $d_t$ and using $d\cdot d_t=0$, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d|^2\mathrm{d}x+\int|d_t|^2\mathrm{d}x=-\int u\cdot\nabla d\cdot d_t\mathrm{d}x$$
$$\leq\|u\|_{L^6}\|\nabla d\|_{L^3}\|d_t\|_{L^2}\leq C(M)\|d_t\|_{L^2}$$
$$\leq\frac{1}{2}\|d_t\|_{L^2}^2+C(M),$$

which gives

$$\int_0^t\|d_t\|_{L^2}^2\mathrm{d}s\leq C_0(M_0)+C(M)t. \tag{2.5}$$

Applying $\nabla$ to (1.1c), we see that

$$\nabla d_t-\nabla\Delta d+\nabla(u\cdot\nabla d)=\nabla(d|\nabla d|^2). \tag{2.6}$$

Testing (2.6) by $\nabla d_t$, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Delta d|^2\mathrm{d}x+\int|\nabla d_t|^2\mathrm{d}x$$
$$=\int(\nabla(d|\nabla d|^2)-\nabla(u\cdot\nabla d))\nabla d_t\mathrm{d}x$$
$$\leq C(\|\nabla d\|_{L^6}^3+\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2}^2+\|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}+\|u\|_{L^6}\|\nabla^2 d\|_{L^3})\|\nabla d_t\|_{L^2}$$
$$\leq C(M)(1+\|\nabla d\|_{L^\infty}+\|\nabla^2 d\|_{L^3})\|\nabla d_t\|_{L^2}$$
$$\leq C(M)(1+\|d\|_{H^3}^{\frac{1}{2}})\|\nabla d_t\|_{L^2}$$
$$\leq\frac{1}{2}\|\nabla d_t\|_{L^2}^2+C(M)+C(M)\|d\|_{H^3},$$

which leads to

$$\int|\Delta d|^2\mathrm{d}x+\int_0^t\|\nabla d_t\|_{L^2}^2\mathrm{d}s\leq C_0(M_0)+C(M)t^{\frac{1}{2}}. \tag{2.7}$$

Here we have used the Gagliardo-Nirenberg inequalities

$$\|\nabla d\|_{L^\infty}^2 \leq C\|\nabla^2 d\|_{L^2}\|d\|_{H^3}, \tag{2.8a}$$

$$\|\nabla^2 d\|_{L^3}^2 \leq C\|\nabla^2 d\|_{L^2}\|d\|_{H^3}. \tag{2.8b}$$

Eq. (1.1b) can be written as

$$-\mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = f =: -\rho\partial_t u - \rho u\cdot\nabla u - \nabla p(\rho) - \nabla d\cdot\Delta d. \tag{2.9}$$

Then we have

$$\begin{aligned}
\|u\|_{W^{2,q}} \leq& C\|f\|_{L^q} \leq C\|\rho\partial_t u\|_{L^q} + C\|\rho u\cdot\nabla u\|_{L^q} + C\|\nabla p\|_{L^q} + C\|\nabla d\|_{L^\infty}\|\Delta d\|_{L^q}\\
\leq& C\|\rho\|_{L^\infty}^{\frac{5q-6}{4q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|u_t\|_{L^6}^{\frac{3q-6}{2q}} + C(M)\|u\|_{L^\infty}\|\nabla u\|_{L^q} + C(M)\|d\|_{H^3}^{\frac{3}{2}}\\
\leq& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C(M)\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{3}{2}} + C(M)\|d\|_{H^3}^{\frac{3}{2}}\\
\leq& C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}} + C(M)\|u\|_{H^2}^{\frac{3}{2}} + C(M)\|d\|_{H^3}^{\frac{3}{2}},
\end{aligned}$$

which gives

$$\begin{aligned}
\int_0^t \|u\|_{W^{2,q}}\mathrm{d}s \leq& C(M)\int_0^t \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\|\nabla u_t\|_{L^2}^{\frac{3q-6}{2q}}\mathrm{d}s + C(M)\int_0^t\|u\|_{H^2}^{\frac{3}{2}}\mathrm{d}s + C(M)\int_0^t\|d\|_{H^3}^{\frac{3}{2}}\mathrm{d}s\\
\leq& C(M)\int_0^t s^{-\frac{3q-6}{4q}}(\sqrt{s}\|\nabla u_t\|_{L^2})^{\frac{3q-6}{2q}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-q}{2q}}\mathrm{d}s\\
&+ C(M)\left(\int_0^t \mathrm{d}s\right)^{\frac{1}{4}}\left(\int_0^t\|u\|_{H^2}^2\mathrm{d}s\right)^{\frac{3}{4}} + C(M)\left(\int_0^t \mathrm{d}s\right)^{\frac{1}{4}}\left(\int_0^t\|d\|_{H^3}^2\mathrm{d}s\right)^{\frac{3}{4}}\\
\leq& C(M)\left(\int_0^t s^{-\frac{3q-6}{2q}}\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t s\|\nabla u_t\|_{L^2}^2\mathrm{d}s\right)^{\frac{3q-6}{4q}}\left(\int_0^t\|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}s\right)^{\frac{6-q}{4q}} + C(M)t^{\frac{1}{4}}\\
\leq& C(M)t^{\frac{6-q}{4q}} + C(M)t^{\frac{1}{4}} \leq C(M)t^{\frac{6-q}{4q}}
\end{aligned} \tag{2.10}$$

for all $0 < t \leq 1$.

Using the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^\infty} \leq C\|\nabla u\|_{L^2}^{\frac{2q-6}{5q-6}}\|u\|_{W^{2,q}}^{\frac{3q}{5q-6}}, \tag{2.11}$$

we observe that

$$\begin{aligned}
&\int_0^t \|\nabla u\|_{L^\infty}\mathrm{d}s \leq C(M)\int_0^t\|u\|_{W^{2,q}}^{\frac{3q}{5q-6}}\mathrm{d}s\\
\leq& C\left(\int_0^t\mathrm{d}s\right)^{\frac{2q-6}{5q-6}}\left(\int_0^t\|u\|_{W^{2,q}}\mathrm{d}s\right)^{\frac{3q}{5q-6}}\\
\leq& C(M)t^{\frac{2q-6}{5q-6}}\cdot t^{\frac{6-q}{4q}\cdot\frac{3q}{5q-6}} = C(M)t^{\frac{6-q}{4q}}.
\end{aligned} \tag{2.12}$$

Testing (1.1a) by $\rho^{m-1}$, we see that

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho^m \mathrm{d}x = -\int \mathrm{div}\,(\rho u)\rho^{m-1}\mathrm{d}x = \int \rho u \nabla \rho^{m-1}\mathrm{d}x = -\frac{m-1}{m}\int \rho^m \mathrm{div}\, u \mathrm{d}x,$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^m} \le \|\mathrm{div}\, u\|_{L^\infty}\|\rho\|_{L^m},$$

and thus

$$\begin{aligned}\|\rho\|_{L^m} \le& \|\rho_0\|_{L^m}\exp\left(\int_0^t \|\mathrm{div}\, u\|_{L^\infty}\mathrm{d}s\right)\\ \le& \|\rho_0\|_{L^m}\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}, \quad 2\le m<\infty. \end{aligned} \tag{2.13}$$

For $m=\infty$, (2.13) still holds.

Taking $\nabla$ to (1.1a), testing the result by $|\nabla\rho|^{q-2}\nabla\rho$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^q} \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q} + C\|\rho\|_{L^\infty}\|\nabla \mathrm{div}\, u\|_{L^q},$$

which implies

$$\begin{aligned}\|\nabla\rho\|_{L^q} \le& C\left(\|\nabla\rho_0\|_{L^q} + \int_0^t \|\rho\|_{L^\infty}\|\nabla\mathrm{div}\, u\|_{L^q}\mathrm{d}s\right)\exp\left(\int_0^t \|\nabla u\|_{L^\infty}\mathrm{d}s\right)\\ \le& C\left(1+C(M)t^{\frac{6-q}{4q}}\right)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}\\ \le& C_0(M_0)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}. \end{aligned} \tag{2.14}$$

Testing (1.1b) by $u_t$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}\, u)^2)\mathrm{d}x + \int \rho|u_t|^2\mathrm{d}x\\ =& -\int \rho u\cdot\nabla u\cdot u_t\mathrm{d}x + \int p\mathrm{div}\, u_t\mathrm{d}x + \int\left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right):\nabla u_t\mathrm{d}x\\ =:& I_1+I_2+I_3. \end{aligned} \tag{2.15}$$

We bound $I_1$, $I_2$ and $I_3$ as follows.

$$\begin{aligned}|I_1| \le& \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\ \le& C(M)\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\\ \le& C(M)\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{H^2}^{\frac{1}{2}}\\ \le& \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C(M)\|u\|_{H^2}, \end{aligned}$$

$$
\begin{aligned}
I_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x - \int p_t \mathrm{div}\, u\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x + \int (u\cdot\nabla p + \gamma p\mathrm{div}\, u)\mathrm{div}\, u\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x + (\|u\|_{L^6}\|\nabla p\|_{L^3} + \gamma\|p\|_{L^\infty}\|\mathrm{div}\, u\|_{L^2})\|\mathrm{div}\, u\|_{L^2} \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int p\mathrm{div}\, u\mathrm{d}x + C(M).
\end{aligned}
$$

Here we have used the equality (2.3). And for the term $I_3$, it holds

$$
\begin{aligned}
I_3 =&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x - \int \partial_t\left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x \\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x + C\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}\|\nabla u\|_{L^2} \\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u\mathrm{d}x + C(M)\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}.
\end{aligned}
$$

Inserting the above estimates into (2.15), integrating over $(0,t)$,and using (2.7), we have

$$
\begin{aligned}
&\|\nabla u\|_{L^2}^2 + \int_0^t \|\sqrt{\rho}u_t\|_{L^2}^2\mathrm{d}s \\
\le&C_0(M_0) + C(M)t^{\frac{1}{2}} + C(M)t + C\|\nabla d\|_{L^4}^2 \\
\le&C_0(M_0) + C(M)t^{\frac{1}{2}} + C\|\nabla^2 d\|_{L^2} \\
\le&C_0(M_0) + C(M)t^{\frac{1}{2}} \le C_0(M_0)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}.
\end{aligned} \tag{2.16}
$$

Applying $\partial_t$ to (1.1b) and using (1.1a), we infer that

$$
\begin{aligned}
&\rho\partial_t^2 u + \rho u\cdot\nabla u_t - \mu\Delta u_t - (\lambda+\mu)\nabla\mathrm{div}\, u_t \\
=&-\nabla p_t + \mathrm{div}\,(\rho u)(u_t + u\cdot\nabla u) - \rho u_t\cdot\nabla u \\
&-\partial_t\mathrm{div}\,\left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right).
\end{aligned} \tag{2.17}
$$

Testing (2.23) by $u_t$ and using (1.1a), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u_t|^2\mathrm{d}x + \int (\mu|\nabla u_t|^2 + (\lambda+\mu)(\mathrm{div}\, u_t)^2)\mathrm{d}x \\
=&\int p_t\mathrm{div}\, u_t\mathrm{d}x - \int \rho u\nabla|u_t|^2\mathrm{d}x - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\mathrm{d}x \\
&-\int \rho u_t\cdot\nabla u\cdot u_t\mathrm{d}x + \int \partial_t\left(\nabla d\odot\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla u_t\mathrm{d}x \\
=:&\sum_{i=4}^{8} I_i.
\end{aligned} \tag{2.18}
$$

We bound $I_i$ $(i=4,\cdots,8)$ as follows.

$$\begin{aligned}
|I_4|&=\left|\int(u\cdot\nabla p+\gamma p\mathrm{div}\,u)\mathrm{div}\,u_t\mathrm{d}x\right|\\
&\le\big(\|u\|_{L^6}\|\nabla p\|_{L^3}+\gamma\|p\|_{L^\infty}\|\mathrm{div}\,u\|_{L^2}\big)\|\mathrm{div}\,u_t\|_{L^2}\\
&\le C(M)\|\nabla u_t\|_{L^2}\le\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M),\\
|I_5|&\le\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\le C(M)\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\le C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac12}\|\nabla u_t\|_{L^2}\le C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\\
&\le\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|\sqrt{\rho}u_t\|_{L^2}^2,\\
|I_6|&\le\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}^2\|u_t\|_{L^6}+\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla^2u\|_{L^2}\|u_t\|_{L^6}\\
&\quad+\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\le C(M)(\|\nabla u\|_{L^3}^2+\|u\|_{H^2})\|\nabla u_t\|_{L^2}\\
&\le C(M)(\|\nabla u\|_{L^2}\|u\|_{H^2}+\|u\|_{H^2})\|\nabla u_t\|_{L^2}\\
&\le C(M)\|u\|_{H^2}\|\nabla u_t\|_{L^2}\le\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|u\|_{H^2}^2,\\
|I_7|&\le\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^4}^2\le\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t\|_{L^6}^{\frac32}\\
&\le C(M)\|\sqrt{\rho}u_t\|_{L^2}^{\frac12}\|\nabla u_t\|_{L^2}^{\frac32}\le\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|\sqrt{\rho}u_t\|_{L^2}^2,\\
|I_8|&\le C\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2\\
&\le\frac{\mu}{16}\|\nabla u_t\|_{L^2}^2+C(M)\|d\|_{H^3}\|\nabla d_t\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (2.18) gives

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u_t|^2\mathrm{d}x+\frac{11}{16}\mu\int|\nabla u_t|^2\mathrm{d}x\\
\le&C(M)+C(M)\|\sqrt{\rho}u_t\|_{L^2}^2+C(M)\|u\|_{H^2}^2+C(M)\|d\|_{H^3}\|\nabla d_t\|_{L^2}^2.
\end{aligned}\tag{2.19}$$

Multiplying the above inequality by $t$, we obtain

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(t\int\rho u_t^2\mathrm{d}x\right)+\frac{11}{16}\mu t\int|\nabla u_t|^2\mathrm{d}x\\
\le&\frac12\int\rho|u_t|^2\mathrm{d}x+C(M)t+C(M)t\int\rho|u_t|^2\mathrm{d}x+C(M)t\|u\|_{H^2}^2\\
&+C(M)\|d\|_{H^3}\cdot t\|\nabla d_t\|_{L^2}^2,
\end{aligned}\tag{2.20}$$

which implies that

$$t\int\rho|u_t|^2\mathrm{d}x+\int_0^t s\|\nabla u_t\|_{L^2}^2\mathrm{d}s\le C_0(M_0)\exp\left\{t^{\frac{6-q}{4q}}C(M)\right\}.\tag{2.21}$$

It follows from (2.9) that

$$
\begin{aligned}
\|u\|_{H^2} \leq & C\|f\|_{L^2} \leq C\|\rho u_t + \rho u\cdot\nabla u + \nabla p + \nabla d\cdot\Delta d\|_{L^2}\\
\leq & C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\\
& + C\|\nabla p\|_{L^2} + C\|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2}\\
\leq & C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|\nabla u\|_{L^3} + C(M) + C(M)\|\nabla d\|_{L^\infty}\\
\leq & C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|\nabla u\|_{L^2}^{\frac{1}{2}}\cdot\|u\|_{H^2}^{\frac{1}{2}} + C(M) + C(M)\|\nabla d\|_{L^\infty},
\end{aligned}
$$

which yields

$$
\|u\|_{H^2} \leq C(M) + C\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2} + C(M)\|\nabla d\|_{L^\infty},
$$

whence

$$
\|u\|_{L^2(0,t;H^2)} \leq C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}. \tag{2.22}
$$

Taking $\partial_t$ to (1.1c), testing the result by $d_t$, and using $d\cdot d_t=0$, we observe that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|d_t|^2\mathrm{d}x + \int|\nabla d_t|^2\mathrm{d}x\\
=&-\int u_t\cdot\nabla d\cdot d_t\mathrm{d}x - \int u\cdot\nabla d_t\cdot d_t\mathrm{d}x + \int|d_t|^2|\nabla d|^2\mathrm{d}x\\
\leq&\|u_t\|_{L^6}\|\nabla d\|_{L^3}\|d_t\|_{L^2} + \|u\|_{L^6}\|\nabla d_t\|_{L^2}\|d_t\|_{L^3} + \|\nabla d\|_{L^6}^2\|d_t\|_{L^6}\|d_t\|_{L^2}\\
\leq&C\|\nabla d\|_{L^3}\|\nabla u_t\|_{L^2}\|d_t\|_{L^2} + C(M)\|\nabla d_t\|_{L^2}\|d_t\|_{L^3} + C(M)\|d_t\|_{L^6}\|d_t\|_{L^2}\\
\leq&\frac{1}{2}\|\nabla d_t\|_{L^2}^2 + C\|\nabla d\|_{L^3}\|\nabla u_t\|_{L^2}\|d_t\|_{L^2} + C(M)\|d_t\|_{L^2}^2,
\end{aligned}
$$

which gives

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|d_t|^2\mathrm{d}x + \frac{1}{2}\int|\nabla d_t|^2\mathrm{d}x \leq C\|\nabla d\|_{L^3}\|\nabla u_t\|_{L^2}\|d_t\|_{L^2} + C(M)\|d_t\|_{L^2}^2.
$$

Multiplying the above inequality by $t$, using (2.21), (2.5), and (2.7), we obtain

$$
t\int|d_t|^2\mathrm{d}x + \int_0^t s\|\nabla d_t\|_{L^2}^2\mathrm{d}s \leq C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}. \tag{2.23}
$$

Taking $\partial_t$ to (1.1c), testing the result by $-\Delta d_t$, and using (2.5) and (2.7), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d_t|^2\mathrm{d}x + \int|\Delta d_t|^2\mathrm{d}x\\
=&\int(u_t\cdot\nabla d + u\cdot\nabla d_t)\Delta d_t\mathrm{d}x - \int(d_t|\nabla d|^2 + d\partial_t|\nabla d|^2)\Delta d_t\mathrm{d}x\\
\leq&(\|u_t\|_{L^6}\|\nabla d\|_{L^3} + \|u\|_{L^6}\|\nabla d_t\|_{L^3})\|\Delta d_t\|_{L^2}\\
&+(\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|\nabla d\|_{L^6}\|\nabla d_t\|_{L^3})\|\Delta d_t\|_{L^2}\\
\leq&\frac{1}{2}\|\Delta d_t\|_{L^2}^2 + C\|\nabla d\|_{L^3}^2\|\nabla u_t\|_{L^2}^2 + C(M)(\|\nabla d_t\|_{L^2}^2 + \|d_t\|_{L^2}^2).
\end{aligned}
$$

Multiplying the above inequality by $t$, using (2.21), (2.5), (2.7), and (2.23), we have

$$t\int |\nabla d_t|^2 \mathrm{d}x \le C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}. \tag{2.24}$$

It follows from (1.1c) and (2.7) that

$$\begin{aligned}\|d\|_{H^3} \le& C\|d\|_{H^1} + \|\nabla\Delta d\|_{L^2}\\ \le& C\|d\|_{H^1} + \|\nabla(d_t + u\cdot\nabla d - d|\nabla d|^2)\|_{L^2}\\ \le& C + C\|\nabla d_t\|_{L^2} + C\|u\|_{L^6}\|\nabla^2 d\|_{L^3} + C\|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty}\\ &+ C\|\nabla d\|_{L^6}^3 + C\|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2}\\ \le& C + C\|\nabla d_t\|_{L^2} + C(M)\|d\|_{H^3}^{\frac{1}{2}} + C(M),\end{aligned}$$

which implies

$$\|d\|_{H^3} \le C + C(M) + C\|\nabla d_t\|_{L^2},$$

whence

$$\begin{aligned}\|d\|_{L^2(0,t;H^3)} \le& Ct + C(M)t + C(M)t^{\frac{1}{2}} \le C(M)t^{\frac{1}{2}}\\ \le& C_0(M_0)\exp\Big\{t^{\frac{6-q}{4q}}C(M)\Big\}.\end{aligned} \tag{2.25}$$

Combining (2.16) and (2.21)-(2.25), we conclude that (1.7) holds true. This completes the proof of Theorem 1.2. □

## 3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since the existence part has been given, we only need to show the uniqueness part. Let $(\rho_i, u_i, d_i)$, $(i = 1, 2)$ be the two strong solutions satisfying (1.4) with the same initial data.

We denote

$$(\rho, u, d) =: (\rho_1 - \rho_2, u_1 - u_2, d_1 - d_2).$$

Then it is easy to verify that

$$\partial_t\rho + u_2\cdot\nabla\rho + \rho\,\mathrm{div}\, u_2 + \rho_1\mathrm{div}\, u + u\cdot\nabla\rho_1 = 0, \tag{3.1}$$

$$\begin{aligned}&\rho_1\partial_t u + \rho_1 u_1\cdot\nabla u - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}\, u\\ &= -\rho_1 u\cdot\nabla u_2 - \rho(\partial_t u_2 + u_2\cdot\nabla u_2) - \nabla(p(\rho_1) - p(\rho_2))\\ &\quad - \mathrm{div}\left(\nabla d_1\odot\nabla d_1 - \nabla d_2\odot\nabla d_2 - \frac{1}{2}|\nabla d_1|^2\mathbb{I}_3 + \frac{1}{2}|\nabla d_2|^2\mathbb{I}_3\right),\end{aligned} \tag{3.2}$$

$$\partial_t d + u_1\nabla d + u\cdot\nabla d_2 - \Delta d = d_1|\nabla d_1|^2 - d_2|\nabla d_2|^2. \tag{3.3}$$

Testing (3.1) by $\rho$ and using (1.4), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^2\mathrm{d}x =& -\int(u_2\nabla\rho+\rho\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u+u\cdot\nabla\rho_1)\rho\mathrm{d}x\\ =& -\int\left(\frac{1}{2}\rho^2\mathrm{div}\,u_2+\rho_1\mathrm{div}\,u\rho+u\nabla\rho_1\rho\right)\mathrm{d}x\\ \le& C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}^2+C\|\rho_1\|_{L^\infty}\|\nabla u\|_{L^2}\|\rho\|_{L^2}+C\|u\|_{L^6}\|\nabla\rho_1\|_{L^3}\|\rho\|_{L^2}\\ \le& C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}^2+C\|\nabla u\|_{L^2}\|\rho\|_{L^2},\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^2}\le C\|\nabla u_2\|_{L^\infty}\|\rho\|_{L^2}+C\|\nabla u\|_{L^2}. \tag{3.4}$$

By the Gronwall inequality, we get

$$\|\rho\|_{L^2}\le C\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s. \tag{3.5}$$

Testing (3.2) by $u$, using (1.1a), (1.4), and (3.5), we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\rho_1|u|^2\mathrm{d}x+\int_0^t\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x\mathrm{d}s\right)\\ &\quad+\frac{1}{2}\int(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}\,u)^2)\mathrm{d}x\\ \le& C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C\|\partial_t u_2\|_{L^6}\|u\|_{L^6}\|\rho\|_{L^{\frac{3}{2}}}+C\|u_2\|_{L^6}\|\nabla u_2\|_{L^6}\|u\|_{L^6}\|\rho\|_{L^2}\\ &\quad+C\|p(\rho_1)-p(\rho_2)\|_{L^2}\|\nabla u\|_{L^2}+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|\nabla u\|_{L^2}\\ \le& C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C\|\nabla u_{2t}\|_{L^2}\|\nabla u\|_{L^2}\|\rho\|_{L^2}+C\|u_2\|_{H^2}\|\nabla u\|_{L^2}\|\rho\|_{L^2}\\ &\quad+C\|\rho\|_{L^2}\|\nabla u\|_{L^2}+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|\nabla u\|_{L^2}\\ \le& C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C(\|\nabla u_{2t}\|_{L^2}+\|u\|_{H^2}+1)\|\nabla u\|_{L^2}\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s\\ &\quad+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla u\|_{L^2}\|\nabla d\|_{L^2}\\ \le&\frac{\mu}{16}\|\nabla u\|_{L^2}^2+C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C(\|\nabla u_{2t}\|_{L^2}^2+\|u\|_{H^2}^2+1)\left(\int_0^t\|\nabla u\|_{L^2}\mathrm{d}s\right)^2\\ &\quad+C(\|\nabla d_1\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^2)\|\nabla d\|_{L^2}^2\\ \le&\frac{\mu}{16}\|\nabla u\|_{L^2}^2+C\|\nabla u_2\|_{L^\infty}\int\rho_1|u|^2\mathrm{d}x+C(t\|\nabla u_{2t}\|_{L^2}^2+\|u\|_{H^2}^2+1)\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}s\\ &\quad+C(\|\nabla d_1\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^2)\|\nabla d\|_{L^2}^2.\end{aligned} \tag{3.6}$$

Testing (3.3) by $d$ and using (1.4), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|d|^2\mathrm{d}x+\int|\nabla d|^2\mathrm{d}x\\
\le&\|u_1\|_{L^6}\|\nabla d\|_{L^2}\|d\|_{L^3}+\|u\|_{L^6}\|\nabla d_2\|_{L^3}\|d\|_{L^2}+\|d\|_{L^2}^2\|\nabla d_1\|_{L^\infty}^2\\
&\quad+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|d\|_{L^2}\\
\le&C\|\nabla d\|_{L^2}\|d\|_{L^3}+C\|d\|_{L^2}^2+C\|\nabla d_1\|_{L^\infty}^2\|d\|_{L^2}^2\\
&\quad+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|d\|_{L^2}+\frac{\mu}{16}\|\nabla u\|_{L^2}^2. \qquad (3.7)
\end{aligned}
$$

Testing (3.3) by $-\Delta d$ and using (1.4), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla d|^2\mathrm{d}x+\int|\Delta d|^2\mathrm{d}x\\
\le&\|u_1\|_{L^6}\|\nabla d\|_{L^3}\|\Delta d\|_{L^2}+\|u\|_{L^6}\|\nabla d_2\|_{L^3}\|\Delta d\|_{L^2}\\
&\quad+C\|d\|_{L^6}\|\Delta d\|_{L^2}\|\nabla d_1\|_{L^6}^2+C(\|\nabla d_1\|_{L^\infty}+\|\nabla d_2\|_{L^\infty})\|\nabla d\|_{L^2}\|\Delta d\|_{L^2}\\
\le&C_1\mu\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\Delta d\|_{L^2}^2+C\|\nabla d\|_{L^2}^2+C\|d\|_{L^2}^2\\
&\quad+C(\|\nabla d_1\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^2)\|\nabla d\|_{L^2}^2. \qquad (3.8)
\end{aligned}
$$

Doing (3.6)$\times 8C_1$+(3.7)+(3.8) and using the Gronwall inequality, we arrive at

$$
\rho=0,\quad u=0\quad\text{and}\quad d=0. \qquad (3.9)
$$

This completes the proof of Theorem 1.1. □

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