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Shadowing Homoclinic Chains to a Symplectic Critical Manifold

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. We prove the existence of trajectories shadowing chains of heteroclinic orbits to a symplectic normally hyperbolic critical manifold of a Hamiltonian system. The results are quite different for real and complex eigenvalues. General results are applied to Hamiltonian systems depending on a parameter which slowly changes with rate ε. If the frozen autonomous system has a hyperbolic equilibrium possessing transverse homoclinic orbits, we construct trajectories shadowing homoclinic chains with energy having quasirandom jumps of order ε and changing with average rate of order $\varepsilon|\ln\varepsilon|$. This provides a partial multidimensional extension of the results of A. Neishtadt on the destruction of adiabatic invariants for systems with one degree of freedom and a figure 8 separatrix.

Key Words: Hamiltonian system, homoclinic orbit, shadowing.

AMS Subject Classifications: 37Jxx, 37Dxx, 70Hxx

1 Introduction

Consider a smooth Hamiltonian system (M,ω,H) with phase space M, symplectic form ω and Hamiltonian H. Let $v=J\nabla H$ be the Hamiltonian vector field and ϕ^t the phase flow. Suppose H has a connected symplectic nondegenerate critical manifold N. Then any $z\in N$ is a critical point of H with rank $d^2H(z)=\dim M-\dim N$, and the restriction $\omega|_{T_zN}$ is nondegenerate. We also assume that N is normally hyperbolic, i.e., nonzero eigenvalues of the linearization $\Lambda(z)=Dv(z)$ have nonzero real parts.

Denote by

$$E_z=\{\xi\in T_zM:\omega(\xi,\eta)=0\text{ for all }\eta\in T_zN\}$$

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the symplectic complement to T_zN. Since N is symplectic, $T_zM = T_zN \oplus E_z$ and $\omega|_{E_z}$ is nondegenerate. Hence $E_z = E_z^+ \oplus E_z^-$, where $E_z^\pm$ are $\Lambda(z)$-invariant Lagrangian stable and unstable subspaces of E_z corresponding to the eigenvalues with negative and positive real parts respectively.

Let

$$W^\pm(z) = \{x \in M : \lim_{t\to\pm\infty} \phi^t(x) = z\}, \qquad T_zW^\pm(z) = E_z^\pm,$$

be the stable and unstable manifolds of $z \in N$ and

$$W^\pm(N) = \cup_{z\in N} W^\pm(z)$$

the stable and unstable manifolds of N. The intersection $W^+(N) \cap W^-(N) \setminus N$ consists of orbits $\gamma : \mathbb{R} \to M$ homoclinic to N, i.e., heteroclinic from $z_- = \gamma(-\infty)$ to $z_+ = \gamma(+\infty)$. The heteroclinic orbit is called transverse if $T_{\gamma(t)}W^-(z_-) \cap T_{\gamma(t)}W^+(N) = \mathbb{R}\dot\gamma(t)$.

Define a multivalued partially defined symplectic scattering map $\mathcal{F} : N \to N$ by $\mathcal{F}(z_-) = z_+$ if there is a transverse heteroclinic from z_- to z_+. We call a sequence $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$ of transverse heteroclinic orbits a heteroclinic chain if $\sigma_i(+\infty) = \sigma_{i+1}(-\infty) = z_i \in N$. A heteroclinic chain corresponds to an orbit $\mathbf{z} = (z_i)_{i\in\mathbb{Z}}$ of the scattering map. We call the chain strongly nondegenerate if the orbit $\mathbf{z}$ is hyperbolic.

Without loss of generality let $N \subset \Sigma_0 = H^{-1}(0)$. Our goal is to construct, for small μ, orbits $\gamma : \mathbb{R} \to \Sigma_\mu = H^{-1}(\mu)$ shadowing strongly nondegenerate infinite heteroclinic chains. This requires several assumptions which are different for real and complex eigenvalues. For degenerate heteroclinic chains we get weaker results.

Our research is motivated by two classical problems. The first is Poincaré's theory of second species almost collision solutions in celestial mechanics. This application was already discussed in [6,7], so we will be brief. Consider the plane 3 body problem with two small masses of order $\mu \ll 1$. Let the center of mass be at rest and let q_i be the relative positions of small bodies with respect to the large one. Then we obtain the Hamiltonian

$$H_\mu(q,p) = H_0(q,p) + \frac{\mu}{2}|p_1+p_2|^2 - \mu\frac{\alpha_1\alpha_2}{|q_1-q_2|}, \quad q \in (\mathbb{R}^2\setminus\{0\})^2,$$

where

$$H_0 = \sum_{i=1}^2 \left(\frac{|p_i|^2}{2\alpha_i} - \frac{\alpha_i}{|q_i|}\right)$$

is the Hamiltonian of two uncoupled Kepler problems with masses α_i. For $\mu > 0$ there are singularities at double collisions $\Delta = \{q : q_1 = q_2\}$. Fixing an energy level $H_\mu^{-1}(E)$ and performing the Levi-Civita regularization at Δ, we obtain the regularized Hamiltonian $\hat H$ which has a symplectic normally hyperbolic critical manifold $N \subset \hat H^{-1}(0)$ corresponding to Δ. Trajectories of the 3 body problem on $H_\mu^{-1}(E)$ correspond to trajectories of the regularized Hamiltonian on $\Sigma_\mu = \hat H^{-1}(\mu)$. Homoclinic trajectories to N correspond to orbits of the uncoupled Kepler problems with collisions of the small bodies, and trajectories on Σ_μ shadowing heteroclinic chains correspond to almost collision second species solutions. This approach was used in [3,7] to prove the existence of periodic and chaotic second species solutions of the 3 body problem. In the applications to celestial mechanics the critical manifold has multiple real eigenvalues. The shadowing theorem in [7] was proved only in this nongeneric case.

In the second application also complex eigenvalues may appear. Consider a slowly time dependent Hamiltonian system on a symplectic manifold M:

$$\dot{z} = J\partial_z H(z,\tau), \quad \dot{\tau} = \varepsilon \ll 1. \tag{1.1}$$

For small ε the energy $E(t) = H(z(t),\tau(t))$ changes slowly: $\dot{E} = \varepsilon\partial_\tau H$. For $\varepsilon = 0$ we obtain a frozen autonomous system with Hamiltonian $H_\tau(z) = H(z,\tau)$ depending on a parameter τ.

If the frozen system has one degree of freedom and the level curves $\gamma = H_\tau^{-1}(E)$ are closed, then the area inside γ (the Maupertuis action)

$$I(\tau,E) = A(\gamma) = \oint_\gamma p\,dq, \quad \omega = dp\wedge dq,$$

is an adiabatic invariant [1]. For small ε the change of $I(t) = I(\tau(t),E(t))$ on long time intervals is small:

$$|I(t) - I(0)| \le C\varepsilon, \quad 0 \le t \le T/\varepsilon. \tag{1.2}$$

Then the energy changes gradually: (τ,E) approximately follow a level curve $I(\tau,E) =$ const.

However, (1.2) fails for trajectories passing near equilibria, since then the frozen dynamics is slow, and the averaging method does not work. A. Neishtadt [19] considered the case when the plane frozen system has a hyperbolic equilibrium with a figure 8 separatrix–union of two homoclinic loops. The separatrix divides the plane in 3 regions. In the interior of each region there is an adiabatic invariant, so (τ,E) follows its level curves. Neishtadt showed that when a trajectory crosses the separatrix, the adiabatic invariant, and hence also the energy, have jumps of order ε. Then large measure of trajectories have quasirandom behavior, and the energy changes with average speed of order $\varepsilon|\ln\varepsilon|$.

It turns out that there is a partial analog of Neishtadt's result for multidimensional Hamiltonian systems such that the frozen system has a hyperbolic equilibrium $z_0(\tau)$ possessing several transverse homoclinic orbits $\gamma_\tau^k : \mathbb{R} \to M$, $k \in K$. Under certain conditions there exist multibump trajectories shadowing homoclinic chains $(\gamma_{\tau_i}^{k_i})$ with energy having quasirandom jumps of order ε and growing with average rate $\sim \varepsilon|\ln\varepsilon|$. For real eigenvalues we need $\#K \ge 2$. For complex eigenvalues, generically $\#K = \infty$, see [10,15].

Let us show how to reduce the problem to a general theorem on shadowing heteroclinic chains to a normally hyperbolic symplectic critical manifold. For simplicity suppose that $H(z,\tau)$ is periodic in $\tau \in \mathbb{T}$. Replacing H by $H - H(z_0(\tau),\tau)$ we may assume that

$$H(z_0(\tau),\tau) = 0, \quad \partial_z H(z_0(\tau),\tau) = 0. \tag{1.3}$$

Consider an autonomous Hamiltonian system

$$\dot{z} = J\partial_z H(z,\tau), \quad \dot{\tau} = \varepsilon, \quad \dot{h} = -\partial_\tau H(z,\tau), \tag{1.4}$$

on the extended phase space $\hat{M} = M \times \mathbb{T} \times \mathbb{R}$ with Hamiltonian

$$\hat{H}_\varepsilon(z,\tau,h) = H(z,\tau) + \varepsilon h$$

and symplectic structure

$$\hat{\omega} = \omega + dh \wedge d\tau.$$

For $\varepsilon > 0$ trajectories of the Hamiltonian system (1.1) are in one-to-one correspondence with trajectories of the extended system (1.4) on the energy level $\hat{H}_\varepsilon^{-1}(0)$. For $\varepsilon = 0$ the extended system has a 2-dimensional normally hyperbolic symplectic critical manifold

$$\hat{N} = \{(z_0(\tau),\tau,h) : \tau \in \mathbb{T},\ h \in \mathbb{R}\}.$$

The Hamiltonian $\hat{H}_\varepsilon$ depends on ε, so we replace it by the Hamiltonian

$$\tilde{H}(z,\tau,h) = \frac{H(z,\tau)}{h} \tag{1.5}$$

on $\tilde{M} = \hat{M} \cap \{h > 0\}$. Since

$$\tilde{\Sigma}_{-\varepsilon} = \tilde{H}^{-1}(-\varepsilon) = \hat{H}_\varepsilon^{-1}(0) \cap \{h > 0\},$$

trajectories of the systems with Hamiltonian $\hat{H}_\varepsilon$ and with Hamiltonian $\tilde{H}$ on $\tilde{\Sigma}_{-\varepsilon}$ are the same. Time parametrizations of trajectories are of course different. The Hamiltonian $\tilde{H}$ has a nondegenerate symplectic critical manifold

$$\tilde{N} = \hat{N} \cap \{h > 0\},$$

and homoclinics of the frozen system define families of homoclinics to the manifold N.

After proving general theorems on shadowing heteroclinic chains to a symplectic critical manifold we will apply them to the system $(\tilde{M},\hat{\omega},\tilde{H})$ and obtain quasirandom trajectories of a slowly time dependent system. Note that for $H = o(\varepsilon)$ this approach does not work since the Hamiltonian (1.5) becomes singular. We briefly discuss this case in the Appendix.

Our results for slowly time dependent systems are related to the work of V. Gelfreich and D. Turaev [17]. They constructed quasirandom trajectories under the assumption that for each (τ,E) in an open set the frozen system has hyperbolic periodic orbits $\gamma^1_{\tau,E}$ and $\gamma^2_{\tau,E}$ joined by transverse heteroclinics in $H_\tau^{-1}(E)$. Then the frozen system has a uniformly hyperbolic chaotic invariant set consisting of trajectories shadowing chains of these heteroclinics. For ε small, while a trajectory stays close to $\gamma^k_{\tau,E}$, the Maupertuis action

$$I_k(\tau,E) = A(\gamma^k_{\tau,E}) = \oint_{\gamma^k_{\tau,E}} p\,dq$$

is a local adiabatic invariant. Gelfreich and Turaev constructed trajectories making many revolutions near $\gamma^1_{\tau,E}$, then many revolutions near $\gamma^2_{\tau,E}$, and so on. For such trajectories (τ, E) first approximately follow a level curve of $I_1(\tau, E)$, then a level curve of $I_2(\tau, E)$, and so on. Gelfreich and Turaev proved that if I_1 and I_2 are functionally independent, there exist trajectories with energy changing in a quasirandom way with average rate of order ε. However, this result does not work near a homoclinic set of an equilibrium, where dynamics of the frozen system is slow.

The shadowing theorems we prove have roots in many classical results in dynamical systems and calculus of variations which are too numerous to mention. Maybe the most important for us were the Turayev–Shilnikov theorem [23] and the works of P. Rabinowitz [12], E. Sere [21] and many others on the existence of multibump homoclinics by variational methods. Some ideas used in this paper were developed over the years in collaboration with Paul Rabinowitz. In particular [8] was a foundation to the present research. However we do not use global variational methods as in [8], since transversality of heteroclinics is assumed.

Phenomena similar to the ones studied in this paper appear in the problem of Arnold's diffusion for nearly integrable Hamiltonian systems near a multiple resonance [2, 11, 14, 20, 26]. Our research is also closely related to the theory of scattering maps [13] and of separatrix maps [25].

In this paper we use local variational methods, more precisely generating functions of symplectic relations and discrete action functionals. For Tonelli Hamiltonians one can use global methods of Aubry–Mather theory [2, 11, 20]. However for general Hamiltonians considered in this paper only local variational methods work.

Next we formulate and prove general shadowing theorems for systems with a normally hyperbolic symplectic critical manifold. In the last section these results are applied to slowly time dependent systems.

2 Main results

Let N^{2m} be a connected symplectic normally hyperbolic critical manifold of a Hamiltonian system (M^{2m+2k}, ω, H). We assume $N \subset \Sigma_0 = H^{-1}(0)$. Define projections $\pi_\pm : W^\pm(N) \to N$ by $\pi_\pm(x) = z$ if $x \in W^\pm(z)$:

$$\pi_\pm(x) = \lim_{t\to\pm\infty} \phi^t(x).$$

Following [13], define a scattering relation $\mathcal{R} \subset N \times N$ setting $(z_-, z_+) \in \mathcal{R}$ if there is an orbit heteroclinic from z_- to z_+, i.e., there is $a \in W^-(N) \cap W^+(N) \setminus N$ such that $\pi_\pm(a) = z_\pm$. If $\mathcal{R}$ is locally a graph, then it defines a branch of the symplectic scattering map $\mathcal{F} : N \to N$. The general theory of scattering maps was developed in [13]. However, our case is different because the manifold N is critical and the energy level Σ_0 containing N is not a manifold, so the results in [13] do not apply directly.

If a heteroclinic orbit $\gamma(t)=\phi^t(a)$, $\gamma(\pm\infty)=c_\pm\in N$ is transverse, then the following equivalent conditions hold. Let v be the Hamiltonian vector field.

- $T_aW^-(c_-)\cap T_aW^+(N)=\mathbb{R}v(a)$.
- $T_aW^+(c_+)\cap T_aW^-(N)=\mathbb{R}v(a)$.
- The symplectic form ω defines a nondegenerate modulo $\mathbb{R}v(a)$ bilinear form on $T_aW^-(c_-)\times T_aW^+(c_+)$.
- There exist Lagrangian submanifolds $L^\pm\subset N$ containing $c_\pm$ such that the Lagrangian manifolds $W^\pm(L^\pm)=\cup_{z\in L^\pm}W^\pm(z)$ intersect transversely in Σ_0 along $\gamma(\mathbb{R})$:
$$T_aW^+(L^+)\cap T_aW^-(L^-)=\mathbb{R}v(a).$$

Then the scattering map $\mathcal{F}$ has a well defined smooth branch $f:V^-\to V^+$, where $V^\pm\subset N$ are neighborhoods of $c_\pm$. Let $(x_\pm,y_\pm)\in\mathbb{R}^{2m}$ be local symplectic coordinates in $V^\pm$ such that $\omega|_{V^\pm}=dy_\pm\wedge dx_\pm$ and

$$L^+=\{y_+=b_+\}=B_+\times\{b_+\},\quad L^-=\{x_-=a_-\}=\{a_-\}\times B_-,$$

where $c_\pm=(a_\pm,b_\pm)$ and $B_\pm$ are small balls in $\mathbb{R}^m$ centered at a_+ and b_- respectively. Then for (x_-,y_+) in a neighborhood of (a_-,b_+), the Lagrangian manifolds $W^-(\{x_-\}\times B_-)$ and $W^+(B_+\times\{y_+\})$ intersect transversely in Σ_0 along a heteroclinic trajectory $\sigma(x_-,y_+)$ joining the points $z_-=(x_-,y_-)$ with $f(z_-)=z_+=(x_+,y_+)$. We represent f by a generating function $S(x_-,y_+)$:

$$f(x_-,y_-)=(x_+,y_+)\quad\Leftrightarrow\quad dS(x_-,y_+)=y_-\,dx_-+x_+\,dy_+. \tag{2.1}$$

Introducing local branches of the scattering map near transverse heteroclinic orbits, we represent $\mathcal{F}$ by a collection of symplectic diffeomorphisms $f_k:V_k^-\to V_k^+$ of open sets in N. The map f_k has a generating function S_k defined on an open set in $\mathbb{R}^{2m}$.

An orbit of $\mathcal{F}$ is a pair of sequences $\mathbf{k}=(k_i)_{i\in\mathbb{Z}}$, $\mathbf{z}=(z_i)_{i\in\mathbb{Z}}$, where $z_i\in V_i=V_{k_{i-1}}^+\cap V_{k_i}^-$ and $z_{i+1}=f_{k_i}(z_i)$. It defines a chain $\sigma=(\sigma_i)_{i\in\mathbb{Z}}$ of transverse heteroclinic orbits σ_i connecting z_i with z_{i+1}. An orbit of $\mathcal{F}$ is a critical point of the formal discrete action functional

$$\mathcal{A}(\mathbf{z})=\mathcal{A}_{\mathbf{k}}(\mathbf{z})=\sum_{i\in\mathbb{Z}}(S_{k_i}(x_i,y_{i+1})-\langle x_i,y_i\rangle),\quad z_i=(x_i,y_i). \tag{2.2}$$

Although the functional is formal, its derivative is a well defined sequence in $l_\infty(\mathbb{R}^{2m})$. It is well known that the orbit $\mathbf{z}$ is hyperbolic (has nonzero Lyapunov exponents) iff the Hessian $\mathcal{A}''(\mathbf{z})$ has a bounded inverse in l_∞. Then we call the chain $\sigma=(\sigma_i)_{i\in\mathbb{Z}}$ strongly nondegenerate. To shadow the chain σ by a trajectory on $\Sigma_\mu=H^{-1}(\mu)$ with small $\mu\neq 0$, we need more conditions.

Let $\Lambda(z) = Dv(z)$ and $\Lambda_\pm(z) = \Lambda|_{E_z^\pm}$. Let

$$\alpha(z) = \min\{|\mathrm{Re}\,\lambda| : \lambda \in \mathrm{Spec}\,\Lambda_\pm(z)\} > 0. \tag{2.3}$$

Then

$$\|e^{\pm t\Lambda_\pm(z)}\| \le Ce^{-t\alpha(z)}, \quad t > 0.$$

We call an eigenvalue λ of $\Lambda_\pm(z)$ leading if $|\mathrm{Re}\,\lambda| = \alpha(z)$. Generically leading eigenvalues are simple. Then there are 2 cases:

- Real case: $\pm\alpha(z)$ are real simple leading eigenvalues.
- Complex case: $\pm\alpha(z) \pm i\beta(z)$, $\beta(z) > 0$, are complex simple leading eigenvalues.

We assume that one of the cases hold for all $z \in N$.

Recall that the strong stable and unstable manifolds $W^\pm_{\mathrm{strong}}(z)$ are invariant manifolds in $W^\pm(z)$ tangent to the eigenspace $E^\pm_{\mathrm{strong}}(z)$ associated to the nonleading eigenvalues with $|\mathrm{Re}\,\lambda| > \alpha(z)$. In the real case $\dim W^\pm_{\mathrm{strong}}(z) = \dim W^\pm(z) - 1$, and in the complex case $\dim W^\pm_{\mathrm{strong}}(z) = \dim W^\pm(z) - 2$.

We call a heteroclinic orbit $\gamma : \mathbb{R} \to W^-(z_-) \cap W^+(z_+)$ leading if it does not lie in $W^+_{\mathrm{strong}}(z_+) \cup W^-_{\mathrm{strong}}(z_-)$. Generic heteroclinics are leading.

The results in the real and complex case are different. The real case was studied in [7] under the assumption that the eigenvalues have maximal multiplicity. For N a single hyperbolic equilibrium with real eigenvalues the result was discovered much earlier by Turayev and Shilnikov [23], and the proofs (with different generality) were given in [8,24]. For N a hyperbolic equilibrium with complex eigenvalues of a system with two degrees of freedom the problem was studied by Devaney [15]. In [10], variational methods were used to extend the results of [15] to the case of nontransverse homoclinics.

First consider the real case. Then the flow on $W^\pm(z)$ looks like a node: for any $a \in W^\pm(z)$ there exist the limits

$$\xi_\pm(a) = \alpha(z)^{-1} \lim_{t\to\pm\infty} e^{\pm t\alpha(z)} v(\phi^t(a)), \tag{2.4}$$

and $\xi_\pm(a) = 0$ iff $a \in W^\pm_{\mathrm{strong}}(z)$. For $a \notin W^\pm_{\mathrm{strong}}(z)$, $\xi_\pm(a)$ are eigenvectors associated to $\mp\alpha(z)$. We fix leading eigenvectors $\zeta_\pm(z) \in E_z^\pm$, smoothly depending on z, such that $\omega(\zeta_-(z), \zeta_+(z)) = 1$. Then define smooth functions $s_\pm$ on $W^\pm(z)$ by

$$\xi_\pm(a) = s_\pm(a)\zeta_\pm(z). \tag{2.5}$$

In local symplectic coordinates such that $\omega|_{E_z} = dp \wedge dq$ and

$$H(z,q,p) = -\alpha(z)p_1q_1 + O_2(p_2,\cdots,p_k,q_2,\cdots,q_k) + O_3(p,q),$$

we have $s_+ = -q_1 + O_2(q)$ and $s_- = p_1 + O_2(p)$.

Remark 2.1. To deal with infinite heteroclinic chains we need some uniformity assumptions. In general the collection $\{f_k\}$ is countable, but we single out a finite subcollection $\mathcal{F} = \{f_k\}_{k\in K}$. We assume that the sets $\overline{V_k^-}$ are compact, the maps f_k can be extended to their neighborhoods, and heteroclinics joining $z \in \overline{V_k^-}$ with $f_k(z)$ are leading.

For a leading heteroclinic orbit γ set

$$\rho_\pm(\gamma) = \operatorname{sgn} s_\pm(\gamma(0)). \tag{2.6}$$

Note that $\rho_\pm(\gamma)$ does not depend on the choice of the point $\gamma(0)$ on $\gamma(\mathbb{R})$.

Let $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$ be a leading heteroclinic chain corresponding to an orbit $(c_i)_{i\in\mathbb{Z}}$ of the scattering map. We call σ positive (negative) if

$$\rho_+(\sigma_i)\rho_-(\sigma_{i+1}) > 0, \quad (\rho_+(\sigma_i)\rho_-(\sigma_{i+1}) < 0) \quad \text{for all } \ i.$$

Positive heteroclinic chains can be shadowed by orbits with small positive energy, and negative chains with small negative energy.

Let $\pi : U \to N$ be a smooth retraction of a tubular neighborhood U of N such that $W^\pm_{\text{loc}}(z) \subset \pi^{-1}(z)$.

Theorem 2.1. *Suppose leading eigenvalues are real and simple. There is $\mu_0 > 0$ such that for any $\mu \in (0,\mu_0]$ and any strongly nondegenerate positive heteroclinic chain $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$ there exists an orbit $\gamma : \mathbb{R} \to \Sigma_\mu$ shadowing the chain σ. More precisely:*

- *There exists a sequence $(t_i)_{i\in\mathbb{Z}}$ such that*[†] *$d(\gamma(t),\sigma_i(\mathbb{R})) \le C\sqrt{\mu}$ for $t_i \le t \le t_{i+1}$ and*
$$t_{i+1} - t_i = \frac{|\ln\mu|}{\alpha(z_i)} + \mathcal{O}(1).$$
- *$d(\gamma(t),N)$ has a local minimum $d(\gamma(t_i),N) \le C\sqrt{\mu}$ at $t = t_i$ and the sequence $z_i = \pi(\gamma(t_i))$ shadows the orbit $(c_i)_{i\in\mathbb{Z}}$ of the scattering map: $d(c_i,z_i) \le C\mu|\ln\mu|$.*
- *Except inside the neighborhood U of N, γ is $\mathcal{O}(\mu|\ln\mu|)$-shadowing σ: there exist sequences $t_{i-1} < s_i^+ < t_i < s_i^- < t_{i+1}$ such that $\gamma(s_i^\pm) \in \partial U$, $\gamma([s_i^+,s_i^-]) \subset U$ and $d(\gamma(t),\sigma_i(\mathbb{R})) \le C\mu|\ln\mu|$ for $s_i^- \le t \le s_{i+1}^+$. We have*
$$0 < s_i^- - s_i^+ - \frac{|\ln\mu|}{\alpha(z_i)} \le C, \quad s_{i+1}^+ - s_i^- \le C. \tag{2.7}$$

If the chain σ is negative, then shadowing orbits exist on Σ_μ with $\mu \in [-\mu_0, 0)$. If the chain is not positive or negative, then in general there are no shadowing orbits satisfying conditions in the theorem.

Theorem 2.1 is a generalization of the main result in [7]. In [7] it is assumed that $\Lambda_\pm(z) = \mp\alpha(z)I$, and only periodic heteroclinic chains were considered.

For complex leading eigenvalues there are more shadowing trajectories.

[†] C is a constant independent of μ and the chain σ.

Theorem 2.2. *Suppose the leading eigenvalues are simple and complex. For any integer m_0 there exists $\mu_0 > 0$ such that for any strongly nondegenerate leading heteroclinic chain $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$, any integer sequence $0 \le n_i \le m_0$, and any $\mu \in [-\mu_0, \mu_0] \setminus \{0\}$,*

- *There exists an orbit $\gamma : \mathbb{R} \to \Sigma_\mu$ shadowing the chain σ: there is a sequence $(t_i)_{i\in\mathbb{Z}}$ such that $d(\gamma(t), \sigma_i(\mathbb{R})) \le C\sqrt{\mu}$ for $t_i \le t \le t_{i+1}$.*
- *$d(\gamma(t), N)$ has a local minimum $d(\gamma(t_i), N) \le C\sqrt{\mu}$ at $t = t_i$.*
- *The sequence $z_i = \pi(\gamma(t_i))$ shadows the orbit (c_i) of the scattering map: $d(c_i, z_i) \le C\mu|\ln \mu|$.*
- *Except inside the neighborhood U, γ is $\mathcal{O}(\mu|\ln\mu|)$-shadowing σ: there exist sequences $t_{i-1} < s_i^+ < t_i < s_i^- < t_{i+1}$ such that $\gamma(s_i^\pm) \in \partial U$, $\gamma([s_i^+, s_i^-]) \subset U$ and $d(\gamma(t), \sigma_i(\mathbb{R})) \le C\mu|\ln\mu|$ for $s_i^- \le t \le s_{i+1}^+$.*
- *There is a constant $C > 0$ independent of m_0 such*

$$0 < s_i^- - s_i^+ - \frac{|\ln \mu|}{\alpha(z_i)} + \frac{2\pi n_i}{\beta(z_i)} \le C, \quad s_{i+1}^+ - s_i^- \le C. \tag{2.8}$$

Remark 2.2. In the complex case there exist also shadowing orbits on Σ_0, including multi-bump homoclinic orbits. For two degrees of freedom and N a single equilibrium this was proved in [10]. However we will not discuss this result since it does not apply to slowly time dependent systems.

It follows from the proof that the orbits in Theorems 2.1 and 2.2 are hyperbolic with nonzero Lyapunov exponents. Since μ_0 is independent of the chain, if $\mathcal{F}$ has a compact hyperbolic invariant set, then shadowing orbits form a compact hyperbolic invariant set in Σ_μ. If the heteroclinic chain (and the sequence (n_i) in Theorem 2.2) are periodic, then the shadowing orbits will be periodic.

Unfortunately in our application to slowly time dependent systems the heteroclinic chains are degenerate. So we need weaker results for finite homoclinic chains.

Theorem 2.3. *Suppose the leading eigenvalues are real and simple. Let $\sigma = (\sigma_i)_{i=0}^n$ be a positive heteroclinic chain. Then there is $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ there exists an orbit $\gamma : [0, T] \to \Sigma_\mu$ and a sequence $(t_i)_{i=0}^n$ such that $t_0 = 0$, $t_n = T$, $\pi(\gamma(0)) = c_0$, and all assertions of Theorem 2.1 hold for $i = 0, \cdots, n$.*

In particular,

$$T = \sum_{i=0}^{n-1} \frac{|\ln \mu|}{\alpha(c_i)} + \mathcal{O}(1).$$

In the complex case we have

Theorem 2.4. *Suppose the leading eigenvalues are simple and complex. For any integer m_0 and any leading heteroclinic chain $\sigma = (\sigma_i)_{i=0}^n$ there exists $\mu_0 > 0$ such that for any $\mu \in [-\mu_0, \mu_0] \setminus \{0\}$ and any integer sequence $0 \le n_i \le m_0$ there exists an orbit $\gamma : [0, T] \to \Sigma_\mu$ and a sequence $(t_i)_{i=0}^n$ such that $t_0 = 0$, $t_n = T$, $\pi(\gamma(0)) = c_0$, and all assertions of Theorem* 2.2 *hold for $i = 0, \cdots, n$.*

In particular, by (2.8),

$$T = \sum_{i=0}^{n-1} \left(\frac{|\ln \mu|}{\alpha(c_i)} - \frac{2\pi n_i}{\beta(z_i)} \right) + \mathcal{O}(1).$$

The proofs of Theorems 2.3 and 2.4 are simplified versions of the proofs of Theorems 2.1 and 2.2, so we skip them. These theorems work for finite chains with n independent of μ. But then one can continue the procedure using a version of the continuation lemma, see [25, 26] and [17]. The details will be published in another paper.

3 Proofs of the shadowing theorems

3.1 Generating functions of local symplectic relations

First we describe trajectories passing close to the critical manifold N. Take a small domain $V \subset N$ with symplectic coordinates $z = (x, y)$ and identify V with a domain in $\mathbb{R}^{2m}$. If V is small enough, a tubular neighborhood U of V in M can be identified with

$$U = V \times B_r \times B_r = \{(z, q, p) : z \in V,\ q, p \in B_r\}, \quad B_r = \{q \in \mathbb{R}^k : |q| \le r\}, \tag{3.1}$$

in such a way that

$$E_z^+ = \{z\} \times \mathbb{R}^k \times \{0\}, \quad E_z^- = \{z\} \times \{0\} \times \mathbb{R}^k,$$

and the coordinates in U are symplectic:

$$\omega|_U = dy \wedge dx + dp \wedge dq, \qquad z = (x, y) \in V.$$

Then

$$H(z, q, p) = \langle p, \Lambda_+(z) q \rangle + O_3(q, p). \tag{3.2}$$

The local stable and unstable manifolds $W_{\mathrm{loc}}^\pm(z_0)$ are parameterized by the embeddings $\psi_{z_0}^\pm : B_r \to U$:

$$\psi_{z_0}^+(q) = (g_+(z_0, q), q, h_+(z_0, q)), \quad \psi_{z_0}^-(p) = (g_-(z_0, p), h_-(z_0, p), p),$$

where

$$g_+ = O_2(q), \quad g_- = O_2(p), \quad h_+ = O_2(q), \quad h_- = O_2(p).$$

They can be also represented by generating functions.

Proposition 3.1. *There exist smooth functions*

$$S_+(y_0,x_+,q_+) = \langle x_+,y_0\rangle + O_2(q_+), \tag{3.3a}$$
$$S_-(x_0,y_-,p_-) = \langle x_0,y_-\rangle + O_2(p_-), \tag{3.3b}$$

on open sets in $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^k$ *such that for any* $z_0 = (x_0,y_0) \in V$ *and* $A_\pm = (x_\pm,y_\pm,q_\pm,p_\pm) \in U$,

$$A_+ \in W^+_{\rm loc}(z_0) \;\Leftrightarrow\; p_+ = \frac{\partial S_+}{\partial q_+}, \quad y_+ = \frac{\partial S_+}{\partial x_+}, \quad x_0 = \frac{\partial S_+}{\partial y_0}, \tag{3.4a}$$
$$A_- \in W^-_{\rm loc}(z_0) \;\Leftrightarrow\; q_- = \frac{\partial S_-}{\partial p_-}, \quad x_- = \frac{\partial S_-}{\partial y_-}, \quad y_0 = \frac{\partial S_-}{\partial x_0}. \tag{3.4b}$$

Equivalently,

$$dS_+(y_0,x_+,q_+) = p_+\,dq_+ + y_+\,dx_+ + x_0\,dy_0, \tag{3.5a}$$
$$dS_-(x_0,y_-,p_-) = q_-\,dp_- + x_-\,dy_- + y_0\,dx_0. \tag{3.5b}$$

Define a symplectic relation $\mathcal{L} \subset U \times U$ as follows: $(A_+,A_-) \in \mathcal{L}$ if there exists $z_0 \in V$ such that $A_\pm \in W^\pm(z_0)$. The relation is represented by the generating function

$$F(z_0,Z) = S_+(y_0,x_+,q_+) + S_-(x_0,y_-,p_-) - \langle x_0,y_0\rangle, \quad Z = (x_+,y_-,q_+,p_-),$$

via the equations

$$\frac{\partial F}{\partial z_0} = 0, \quad y_+ = \frac{\partial F}{\partial x_+}, \quad x_- = \frac{\partial F}{\partial y_-}, \quad p_+ = \frac{\partial F}{\partial q_+}, \quad q_- = \frac{\partial F}{\partial p_-}. \tag{3.6}$$

Let

$$L(Z) = \mathrm{Crit}_{z_0} F(z_0,Z) = F(\zeta(Z),Z),$$

which means taking the critical value at the nondegenerate critical point $z_0 = \zeta(Z)$ with respect to z_0. We obtain

Proposition 3.2. *The generating function L defines the symplectic relation $(A_+,A_-) \in \mathcal{L}$ by the equations*

$$dL(Z) = y_+\,dx_+ + x_-\,dy_- + p_+\,dq_+ + q_-\,dp_-.$$

From now on we assume that $r > 0$ is small enough. The next proposition is a minor generalization of Shilnikov's theorem [22], or λ-lemma. The proof is an application of the contraction principle, see [7,16].

Proposition 3.3. *For any $(z_0,q_+,p_-) \in V \times B_r \times B_r$ and $T \geq 1$:*

- *There exists a unique solution* $\gamma : [-T, T] \to U$,
$$\gamma(t) = (z(t), q(t), p(t)), \tag{3.7}$$
satisfying the initial–boundary conditions
$$z(0) = z_0, \quad p(T) = p_-, \quad q(-T) = q_+. \tag{3.8}$$
- γ *smoothly depends on* z_0, q_+, p_-, T.
- *Let* $\gamma(\mp T) = A_\pm = (z_\pm, q_\pm, p_\pm)$. *Then*
$$\begin{aligned} z_+ &= g_+(z_0, q_+) + \mathcal{O}(Te^{-2\alpha T}), &(3.9a)\\ z_- &= g_-(z_0, p_-) + \mathcal{O}(Te^{-2\alpha T}), &(3.9b)\\ p_+ &= h_+(z_+, q_+) + \mathcal{O}(e^{-2\alpha T}), &(3.9c)\\ q_- &= h_-(z_-, p_-) + \mathcal{O}(e^{-2\alpha T}). &(3.9d)\end{aligned}$$

We write for simplicity $\alpha = \alpha(z_0)$. Let
$$B_+ = \psi^+_{z_0}(q_+) \in W^+(z_0), \quad B_- = \psi^-_{z_0}(p_-) \in W^-(z_0). \tag{3.10}$$
Then (3.9a)–(3.9d) imply $d(A_\pm, B_\pm) = \mathcal{O}(Te^{-2\alpha T})$. As $T \to +\infty$, the trajectory γ converges pointwise to the concatenation $\gamma_+ \cdot \gamma_-$ of the asymptotic trajectories $\gamma_+(t) = \phi^t B_+$, $t \geq 0$ and $\gamma_-(t) = \phi^t B_-$, $t \leq 0$.

The flow on $W^\pm_{\rm loc}(z_0)$ satisfies
$$\phi^t B_+ = \psi^+_{z_0}(e^{t\Lambda_+(z_0)} q_+ + \mathcal{O}(e^{-\alpha t}|q_+|^2)), \quad \phi^{-t} B_- = \psi^-_{z_0}(e^{t\Lambda_-(z_0)} p_- + \mathcal{O}(e^{-\alpha t}|p_-|^2)).$$
Hence for the point $\gamma(0) = (z_0, q(0), p(0))$, we have
$$q(0) = e^{T\Lambda_+(z_0)} q_+ + \mathcal{O}(e^{-\alpha t} r^2), \quad p(0) = e^{-T\Lambda_-(z_0)} p_- + \mathcal{O}(e^{-\alpha t} r^2). \tag{3.11}$$
If $Z = (x_+, q_+, y_-, p_-)$ is given, we can solve Eqs. (3.9a)–(3.9b) for $z_0 = \zeta(Z) + \mathcal{O}(Te^{-2\alpha T})$. Then we obtain a symplectic relation $(A_+, A_-) \in \mathcal{L}_T$ if the points are joined by a trajectory $\gamma : [-T, T] \to U$. The relation $\mathcal{L}_T$ is defined by the generating function
$$L_T(Z) = L(Z) + \mathcal{O}(Te^{-2\alpha T}).$$

To construct trajectories with given energy we need to find $H(\gamma(0))$ for the trajectory γ in Proposition 3.3. Up to now it did not matter if the leading eigenvalues were real or complex. Now we have to consider these cases separately.

In the real case the manifolds $W^\pm_{\rm strong}(z_0)$ divide $W^\pm_{\rm loc}(z_0)$ in 2 components. In order to connect the points $A_\pm$ by a trajectory with positive energy, we need to have the points (3.10) in the right components of $W^\pm_{\rm loc}(z_0) \setminus W^\pm_{\rm strong}(z_0)$. Let $s_\pm$ be the function (2.5) and
$$s_+(z_0, q_+) = s_+(\psi^+_{z_0}(q_+)), \quad s_-(z_0, p_-) = s_-(\psi^-_{z_0}(p_-)).$$

Then $W^{\pm}_{\rm strong}(z_0) \subset W^{\pm}_{\rm loc}(z_0)$ are given by the equations $s_\pm = 0$ respectively. A computation [7] shows that

$$H(\gamma(0)) = \alpha e^{-2T\alpha} s_+(z_0, q_+) s_-(z_0, p_-) + \mathcal{O}(e^{-(2\alpha+\nu)T}), \quad \nu > 0.$$

Let Ω be a compact set contained in

$$\{(z_0, q_+, p_-) \in V \times B_r \times B_r : s_+(z_0, q_+) s_-(z_0,, p_-) > 0\}.$$

Later on we take $\Omega = V \times Q \times P$, where $P, Q \subset B_r \setminus B_{r/2}$ are small closed balls. Let $\mu_0 > 0$ be small enough and $\mu \in (0, \mu_0]$. Solving the equation $H(\gamma(0)) = \mu$ for T we obtain:

Proposition 3.4. *For any $\mu \in (0, \mu_0]$ and $(z_0, q_+, p_-) \in \Omega$:*

- *There exist*
$$T = \frac{|\ln \mu|}{2\alpha(z_0)} + \mathcal{O}(1)$$
and a unique solution (3.7) *on $\Sigma_\mu \cap U$ satisfying* (3.8).
- *γ and T smoothly depend on z_0, q_+, p_-, μ.*
- *γ converges to the concatenation $\gamma_+ \cdot \gamma_-$ as $\mu \to 0$.*
- *The boundary points $A_\pm$ of γ satisfy $d(A_\pm, B_\pm) \le C\mu|\ln \mu|$.*
- *$d(\gamma(0), N) = \mathcal{O}(\sqrt{\mu})$.*

Proposition 3.4 was proved in [7] for equal real eigenvalues. In [16] the proof was extended to the generic real case.

We have a symplectic relation: $(A_+, A_-) \in \mathcal{L}_\mu$ if there exists z_0 such that the points $A_\pm$ are joined by a trajectory in Proposition 3.4. The generating function of the relation is

$$L_\mu(Z) = L(Z) + \mathcal{O}(\mu|\ln \mu|), \quad Z = (x_+, y_-, q_+, p_-), \tag{3.12}$$

where L is the generating function in Proposition 3.2.

In the complex case the formula for the energy is more complicated. Let $\Pi_\pm : \mathbb{R}^k \to \mathbb{R}^k$ be the projections to the eigenspaces $E^\pm_{\rm lead}$ associated to the leading eigenvalues $\pm\alpha \pm i\beta$. Set

$$\hat p = \Pi_- p_- \in E^-_{\rm lead}, \qquad \tilde p = p_- - \hat p \in E^-_{\rm strong},$$
$$\hat q = \Pi_+ q_+ \in E^+_{\rm lead}, \qquad \tilde q = q - \hat q \in E^+_{\rm strong}.$$

Lemma 3.1. *We have*

$$H(\gamma(0)) = e^{-2\alpha T}(f(\hat q, \hat p)\cos(2\beta T) + g(\hat q, \hat p)\sin(2\beta T) + \mathcal{O}(r^3) + \mathcal{O}(e^{-\nu T} r^2)), \tag{3.13}$$

where $\nu > 0$ and f, g are linear functions such that

$$\sqrt{f^2 + g^2} = \sqrt{\alpha^2 + \beta^2}|\hat p||\hat q| = u(z_0, q_+, p_-).$$

The proof is a computation similar to the one in [10]. Let $\Lambda_+ = \Lambda_+(z_0)$. We have

$$|e^{T\Lambda_+}\tilde{q}| \le Ce^{-T(\alpha+\nu)}|\tilde{q}|, \quad \nu > 0,$$
$$e^{T\Lambda_+}\hat{q} = e^{-T\alpha}e^{\beta TJ}\hat{q},$$

where $J : E^+_{\rm lead} \to E^+_{\rm lead}$, $J^2 = -I$, is a rotation by $\pi/2$. By (3.2) and (3.11),

$$\begin{aligned} H(\gamma(0)) =& H(z_0, q(0), p(0)) = \langle p(0), \Lambda_+ q(0)\rangle + O_3(p(0), q(0)) \\ =& \langle p_-, e^{T\Lambda_+}\Lambda_+ e^{T\Lambda_+} q_+\rangle + \mathcal{O}(e^{-2\alpha T}r^3) \\ =& \langle \hat{p}, e^{T\Lambda_+}\Lambda_+ e^{T\Lambda_+}\hat{q}\rangle + \langle \tilde{p}, e^{T\Lambda_+}\Lambda_+ e^{T\Lambda_+}\tilde{q}\rangle + \mathcal{O}(e^{-2\alpha T}r^3) \\ =& e^{-2\alpha T}(\langle \hat{p}, e^{2T\beta J}(-\alpha I + \beta J)\hat{q}\rangle + \mathcal{O}(r^3) + \mathcal{O}(r^2 e^{-\nu T})). \end{aligned}$$

Here $e^{2T\beta J}$ is a rotation in the plane $E^+_{\rm lead}$ by the angle $2\beta T$, and $|(-\alpha I + \beta J)\hat{q}| = \sqrt{\alpha^2+\beta^2}|\hat{q}|$. This implies (3.13).

Let $r > 0$ be sufficiently small and let $P, Q \subset B_r \setminus B_{r/2}$ be small closed balls such that for $z_0 \in V$ we have

$$\psi^+_{z_0}(Q) \subset W^+_{\rm loc}(z_0) \setminus W^+_{\rm strong}(z_0), \quad \psi^-_{z_0}(P) \subset W^-_{\rm loc}(z_0) \setminus W^-_{\rm strong}(z_0).$$

Then for all $(z_0, q_+, p_-) \in \Omega = V \times Q \times P$ we can write[‡]

$$H(\gamma(0)) = e^{-2T\alpha(z_0)}(u(z_0, q_+, p_-)\cos(\psi(z_0, q_+, p_-) + 2\beta(z_0)T) + \mathcal{O}(r^2e^{-\nu T}) + \mathcal{O}(r^3)).$$

Then for small $\mu_0 < \min_\Omega u$ and $0 < |\mu| < \mu_0$ the equation $H(\gamma(0)) = \mu$ has $n(\mu)$ nondegenerate solutions $T = T_k(\mu)$, smoothly depending on μ, and $n(\mu) \to +\infty$ as $\mu \to 0$. For $\mu = 0$ the number of solutions T_k is infinite, but we don't consider this case. For m_0 independent of μ there are solutions $T_0 > T_1 > \cdots > T_{m_0}$ satisfying

$$T_k - T_{k-1} = \frac{\pi}{\beta(z_0)} + o(1), \quad T_0 = \frac{|\ln|\mu||}{\alpha(z_0)} + \mathcal{O}(1). \tag{3.14}$$

We obtain:

Proposition 3.5. *For any integer m_0 there exists $\mu_0 > 0$ such that for $\mu \in [-\mu_0, \mu_0] \setminus \{0\}$ and any $(z_0, q_+, p_-) \in \Omega$:*

- *There exists a sequence $T_0 > T_1 > \cdots > T_{m_0}$ satisfying (3.14) such that for any $0 \le k \le m_0$ there is a solution $\gamma : [-T_k, T_k] \to U \cap \Sigma_\mu$ satisfying (3.8).*
- *γ and T_k smoothly depend on z_0, q_+, p_-, μ.*
- *γ converges to the concatenation $\gamma_+ \cdot \gamma_-$ as $\mu \to 0$.*

[‡]Without restricting to Ω we get a multivalued ψ.

- *The boundary points satisfy* $d(A_\pm, B_\pm) \le C\mu|\ln\mu|$.

The symplectic relation $\mathcal{L}_\mu = \mathcal{L}_\mu^k$ between the points $A_+ = \gamma(-T_k)$ and $A_- = \gamma(T_k)$ is given by a generating function $L_\mu = L_\mu^k$ as in (3.12).

The following corollary works both for real and complex cases. Let the sets P and Q be chosen as above. In the complex case we fix an integer $k \in [0, m_0]$ and drop the dependence on k from the notation. Let $\mu \in (0, \mu_0]$ in the real case, and $\mu \in [-\mu_0, 0) \cup (0, \mu_0]$ in the complex case. Let F be the generating function in (3.6).

Corollary 3.1. *The symplectic relation $\mathcal{L}_\mu$ is given by the generating function*

$$F_\mu(z_0, Z) = L_\mu(Z) - L(Z) + F(z_0, Z) = F(z_0, Z) + \mathcal{O}(\mu|\ln\mu|) \tag{3.15}$$

via the equations

$$\frac{\partial F_\mu}{\partial z_0} = 0, \quad y_+ = \frac{\partial F_\mu}{\partial x_+}, \quad x_- = \frac{\partial F_\mu}{\partial y_-}, \quad p_+ = \frac{\partial F_\mu}{\partial q_+}, \quad q_- = \frac{\partial F_\mu}{\partial p_-}.$$

3.2 Discrete variational problem

To formulate a variational problem for shadowing orbits we need to relate the generating functions of the stable and unstable manifolds $W^\pm(N)$ and of the scattering map $\mathcal{F}$.

Let $f : V^- \to V^+$ be a local branch of $\mathcal{F}$ represented by a generating function S as in (2.1). Then to any (x_0, y_1) in a small open set $W \subset \mathbb{R}^{2k}$ there corresponds the transverse heteroclinic $\sigma(x_0, y_1)$ joining $z_0 = (x_0, y_0) \in V^-$ with $z_1 = f(z_0) = (x_1, y_1) \in V^+$. As in (3.1), let $(x_\pm, y_\pm, q_\pm, p_\pm)$ be symplectic coordinates in a neighborhood

$$U_\pm \cong V^\pm \times B_r \times B_r.$$

Let $A_-(x_0, y_1) \in U_-$ be the first intersection point of $\sigma(x_0, y_1)$ with the cross section $|q| = r$, and $A_+(x_0, y_1) \in U_+$ the last intersection point with the cross section $|p| = r$. Let $O_\pm$ be a small neighborhood of $A_\pm(W)$ in $U_\pm$.

We introduce a symplectic relation $\mathcal{R} \subset O_- \times O_+$ as follows: $(B_-, B_+) \in \mathcal{R}$ if there is a trajectory on the energy level Σ_0 joining B_- with B_+ and close to the heteroclinics in $\sigma(W)$. Under certain transversality conditions (nonconjugacy of A_- and A_+ along σ), which one can verify as in [7], to any $X = (y_-, p_-, x_+, q_+)$ in an open set $D \subset \mathbb{R}^{2m+2k}$ there correspond points $B_\pm = (x_\pm, y_\pm, q_\pm, p_\pm) \in O_\pm$ such that $(B_-, B_+) \in \mathcal{R}$. We obtain:

Proposition 3.6. *The relation $\mathcal{R}$ is given by a generating function $R(X)$, $X = (y_-, p_-, x_+, q_+) \in D$, as follows:*

$$(B_-, B_+) \in \mathcal{R} \;\Leftrightarrow\; dR(X) = p_+\, dq_+ + y_+\, dx_+ + x_-\, dy_- + q_-\, dp_-. \tag{3.16}$$

We denote by $B_\pm(X)$ the points corresponding to $X \in D$.

Suppose now that $\mu_0 > 0$ is sufficiently small and let $\mu \in [-\mu_0, \mu_0]$. Proposition 3.6 implies

Corollary 3.2. *For any $\mu \in [-\mu_0, \mu_0]$ and $X = (y_-, p_-, x_+, q_+) \in D$ there exist x_-, p_-, y_+, q_+ such that the points $B_\pm(X,\mu) = (x_\pm, y_\pm, q_\pm, p_\pm) \in \Sigma_\mu$ are joined by a trajectory in Σ_μ, smoothly depending on μ. The symplectic relation $\mathcal{R}_\mu$ between $B_\pm$ is given by a generating function $R_\mu(X) = R(X) + \mathcal{O}(\mu)$:*

$$(B_-, B_+) \in \mathcal{R}_\mu \quad \Leftrightarrow \quad dR_\mu(X) = p_+\, dq_+ + y_+\, dx_+ + x_-\, dy_- + q_-\, dp_-.$$

Let $S_\pm$ be the generating functions (3.3) of the local stable and unstable manifolds $W^\pm(V^\pm)$. Set

$$G_{x_0,y_1}(X) = S_-(x_0, y_-, p_-) - R(X) + S_+(y_1, x_+, , q_+). \tag{3.17}$$

Eqs. (3.5a), (3.5b) and (3.16) imply:

Proposition 3.7.

- *$X \in D$ is a critical point of G_{x_0,y_1} iff $B_-(X) \in W^-(x = x_0)$ and $B_+(X) \in W^+(y = y_1)$. Then the points $B_\pm$ lie on a heteroclinic orbit $\sigma(x_0, y_1)$ joining (x_0, y_0) and (x_1, y_1).*
- *Let $\hat{D} = \{(y_-, p_-, x_+, q_+) \in D : |p_-| = |q_+| = r\}$. For $(x_0, y_1) \in W$ the function G_{x_0,y_1} on $\hat{D}$ has a nondegenerate critical point $X(x_0, y_1) \in \hat{D}$.*
- *The critical value is the generating function of the scattering map:*

$$S(x_0, y_1) = \mathrm{Crit}_X\, G_{x_0,y_1}(X) = G_{x_0,y_1}(X(x_0, y_1)).$$

Next we introduce a discrete action functional whose critical points correspond to heteroclinic chains.

Let $\mathbf{c} = (c_i)_{i\in\mathbb{Z}}$, $c_{i+1} = f_{k_i}(c_i)$, be an orbit of $\mathcal{F}$ and let $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$ be the corresponding heteroclinic chain: $c_i = \sigma_i(-\infty)$ and $c_{i+1} = \sigma_i(+\infty)$. In the symplectic coordinates $z_i = (x_i, y_i)$ in a neighborhood $V_i \subset N$ of $c_i = (a_i, b_i)$, f_{k_i} is represented by a generating function $S_{k_i}(x_i, y_{i+1})$. Then $\mathbf{c} = (c_i)_{i\in\mathbb{Z}}$ is a critical point of the action functional (2.2).

In a neighborhood $U_i \cong V_i \times B_r \times B_r$ of V_i in M we will use symplectic coordinates (x_i, y_i, q_i, p_i) as in (3.1). Let

$$A_i^- = \sigma_i(t_i^-) = (\hat{x}_i^-, \hat{y}_i^-, \hat{q}_i^-, \hat{p}_i^-) \in \partial U_i, \quad |\hat{p}_i^-| = r,$$

be the exit point of σ_i from U_i, and

$$A_{i+1}^+ = \sigma_i(t_i^+) = (\hat{x}_{i+1}^+, \hat{y}_{i+1}^+, \hat{q}_{i+1}^+, \hat{p}_{i+1}^+) \in \partial U_{i+1}, \quad |\hat{q}_{i+1}^+| = r,$$

the entrance point of σ_i to U_{i+1}. Take small neighborhoods O_i^- of A_i^- and O_{i+1}^+ of A_{i+1}^+. Let D_i be a small neighborhood of $\hat{X}_i = (\hat{y}_i^-, \hat{p}_i^-, \hat{x}_{i+1}^+, \hat{q}_{i+1}^+)$ and $R_i(X_i)$, $X_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in D_i$, the generating function in Proposition 3.6. Then for any $X_i \in D_i$ the points

$$B_i^-(X_i) = (x_i^-, y_i^-, q_i^-, p_i^-) \in O_i^-, \quad B_{i+1}^+(X_i) = (x_{i+1}^+, y_{i+1}^+, q_{i+1}^+, p_{i+1}^+) \in O_{i+1}^+,$$

given by the generating function $R_i(X_i)$ via

$$dR_i(X_i) = p_{i+1}^+ \, dq_{i+1}^+ + y_{i+1}^+ \, dx_{i+1}^+ + x_i^- \, dy_i^- + q_i^- \, dp_i^-,$$

are joined by a trajectory on Σ_0 close to σ_i.

Let $S_i^\pm$ be the generating functions of $W^\pm_{\mathrm{loc}}(V_i)$. As in (3.17), let

$$G_i(x_i, y_{i+1}, X_i) = S_i^-(x_i, y_i^-, p_i^-) - R_i(X_i) + S_{i+1}^+(x_{i+1}^+, y_{i+1}, q_{i+1}^+).$$

Let

$$\hat{D}_i = \{X_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in D_i : |p_i^-| = |q_i^+| = r\}.$$

By Proposition 3.7, the function $X_i \in \hat{D}_i \to G_i(x_i, y_{i+1}, X_i)$ has a nondegenerate critical value

$$\operatorname{Crit}_{X_i} G_i(x_i, y_{i+1}, X_i) = G_i(x_i, y_{i+1}, X_i(x_i, y_{i+1})) = S_{k_i}(x_i, y_{i+1}), \tag{3.18}$$

which is the generating function of the symplectic map f_{k_i}.

Let us define a formal discrete action functional

$$\mathcal{B}(\mathbf{z}, \mathbf{X}) = \sum_{i \in Z} (G_i(x_i, y_{i+1}, X_i) - \langle x_i, y_i \rangle), \quad \mathbf{z} = (z_i)_{i \in \mathbb{Z}}, \quad \mathbf{X} = (X_i)_{i \in \mathbb{Z}},$$

where

$$z_i = (x_i, y_i) \in V_i, \quad X_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in \hat{D}_i.$$

The functional is defined on $\mathcal{V} \times \mathcal{D}$, where

$$\mathcal{V} = \prod_{i \in \mathbb{Z}} V_i, \quad \mathcal{D} = \prod_{i \in \mathbb{Z}} \hat{D}_i.$$

The derivatives of $\mathcal{B}$ are well defined since each variable appears in a finite number of terms. Hence the gradient $\nabla \mathcal{B}$ is in l_∞, and it is a continuous function on $\mathcal{V} \times \mathcal{D}$. Note that $\mathcal{B}(\mathbf{z}, \mathbf{X})$ is the Maupertuis action of a broken trajectory on Σ_0 with discontinuities at the points $B_i^\pm$. At critical points, discontinuities disappear. Hence a critical point $(\mathbf{c}, \mathbf{C})$ of $\mathcal{B}$ defines an orbit $\mathbf{c}$ of the scattering map and the corresponding heteroclinic chain σ.

Proposition 3.8.

- *For any $\mathbf{z} \in \mathcal{V}$ close to $\mathbf{c}$ (with small $\|\mathbf{z} - \mathbf{c}\|_\infty$), the function $\mathbf{X} \in \mathcal{D} \to \mathcal{B}(\mathbf{z}, \mathbf{X})$ has a strongly nondegenerate critical point $\mathbf{X}(\mathbf{z})$.*

- *The (formal) critical value equals the action functional* (2.2)*:*

$$\mathcal{A}(\mathbf{z}) = \mathcal{B}(\mathbf{z}, \mathbf{X}(\mathbf{z})).$$

- *If $\mathbf{c}$ is a strongly nondegenerate critical point of $\mathcal{A}$, then $(\mathbf{c}, \mathbf{X}(\mathbf{c}))$ is a strongly nondegenerate critical point of $\mathcal{B}$ on $\mathcal{V} \times \mathcal{D}$.*

We call a critical point strongly nondegenerate if the Hessian has an inverse which is bounded in l_∞. The first item of Proposition 3.8 follows from the fact that for fixed $\mathbf{z}$ the functional split into a sum of independent functions of X_i, and these functions have nondegenerate critical points with the critical values (3.18). The rest follows easily.

Now we complete the proofs of Theorems 2.1 and 2.2. We will deal simultaneously with the real and complex case. In the real case the chain σ is positive, and we choose small neighborhood V_i of c_i and small neighborhoods P_i of $\hat{p}_i^-$ and Q_i of $\hat{q}_i^+$ so that $s_+(z,q)s_-(z,p) > \delta > 0$ for $z \in V_i$, $q \in Q_i$, $p \in P_i$. In the complex case we take small neighborhoods P_i and Q_i of $\hat{p}_i^-$ and Q_i of $\hat{q}_i^+$ so that $\psi_{z_i}(P_i) \subset W^-_{\mathrm{loc}}(z_i) \setminus W^-_{\mathrm{strong}}(z_i)$ and $\psi_{z_i}(Q_i) \subset W^+_{\mathrm{loc}}(z_i) \setminus W^+_{\mathrm{strong}}(z_i)$ for all $z_i \in V_i$. In the complex case we also choose an integer sequence $0 \le n_i \le m_0$.

Take small $\mu_0 > 0$ and let $\mu \in (0,\mu_0]$ in the real case and $\mu \in [-\mu_0, 0) \cup (0,\mu_0]$ in the complex case. Let

$$F_i^\mu(z_i, Z_i), \quad z_i \in V_i, \quad Z_i = (x_i^+, y_i^-, q_i^+, p_i^-) \in V_i \times Q_i \times P_i,$$

be the generating function in Corollary 3.1, and $R_i^\mu(X_i)$, $X_i = (y_i^-, p_i^-, x_{i+1}^+, q_{i+1}^+) \in D_i$, the generating function in Corollary 3.2. Set

$$\mathcal{A}_\mu(\mathbf{z},\mathbf{X}) = \sum_{i\in\mathbb{Z}} (F_i^\mu(z_i, Z_i) + R_i^\mu(X_i)), \quad \mathbf{z} = (z_i)_{i\in\mathbb{Z}} \in \mathcal{V}, \quad \mathbf{X} = (X_i)_{i\in\mathbb{Z}} \in \mathcal{D}.$$

This formal functional is the Maupertuis action of a concatenation of trajectories on Σ_μ with breaks at the constraints, i.e., at the points $B_i^\pm = B_i^\pm(X_i,\mu)$ in Corollary 3.2. If $(\mathbf{z},\mathbf{X})$ is a critical point of $\mathcal{A}_\mu$ on $\mathcal{V}\times\mathcal{D}$, then the points $B_i^\pm = B_i^\pm(X_i,\mu)$ lie on a smooth trajectory $\gamma : \mathbb{R} \to \Sigma_\mu$ shadowing the collision chain.

By Corollaries 3.1 and 3.2,

$$\|\nabla\mathcal{A}_\mu - \nabla\mathcal{B}\|_\infty \le C\mu|\ln\mu|.$$

By Proposition 3.8, the functional $\mathcal{B}$ has a strongly nondegenerate critical point $(\mathbf{c},\mathbf{X}(\mathbf{c}))$. Now the proof is completed by the implicit function theorem in l_∞.

4 Slowly time dependent systems

Let ϕ_τ^t be the flow of the frozen system and

$$W^\pm(\tau) = \{x : \phi_\tau^t(x) \to z_0(\tau) \quad \text{as } t \to \pm\infty\}$$

the stable and unstable manifolds of the equilibrium $z_0(\tau)$. If $\gamma_\tau : \mathbb{R} \to W^-(\tau) \cap W^+(\tau)$ is a transverse homoclinic orbit of the frozen system:

$$T_{\gamma_\tau(t)}W^+(\tau) \cap T_{\gamma_\tau(t)}W^-(\tau) = \mathbb{R}\,\dot\gamma_\tau(t),$$

then it smoothly depends on τ. The Maupertuis action

$$P(\tau) = A(\gamma_\tau) = \oint_\gamma p\,dq, \quad \omega = dp \wedge dq, \tag{4.1}$$

is called the Poincaré potential, and

$$\int_{-\infty}^{+\infty} \partial_\tau H(\gamma_\tau(t),\tau)\,dt = -P'(\tau) \tag{4.2}$$

is called Melnikov's function. If ω is nonexact, the Poincaré potential may be multivalued, but $P'(\tau)$ is always correctly defined.

Let us find the scattering map for the extended system (1.4). The stable and unstable manifolds $W^\pm(\tau,h_0)$ of a point $(z_0(\tau),\tau,h_0) \in \hat{N}$ are

$$W^\pm(\tau,h_0) = \Big\{(x,\tau,h) : x \in W^-(\tau),\ h = h_0 + \int_0^{\pm\infty} \partial_\tau H(\phi_\tau^t(x),\tau)\,dt\Big\}. \tag{4.3}$$

A homoclinic γ_τ of the frozen system defines a family of heteroclinics

$$\hat{\gamma}_{\tau,h} : \mathbb{R} \to W^+(\tau,h_+) \cap W^-(\tau,h_-)$$

of the extended system, where by (4.3),

$$h_+ - h_- = -\int_{-\infty}^{+\infty} \partial_\tau H(\gamma_\tau(t),\tau)\,dt = P'(\tau).$$

If γ_τ is transverse, then the heteroclinics $\hat{\gamma}_{\tau,h}$ are transverse.

The branch f of the scattering map $\mathcal{F} : \hat{N} \to \hat{N}$ associated to γ_τ is defined by $f(\hat{\gamma}_{\tau,h}(-\infty)) = \hat{\gamma}_{\tau,h}(+\infty)$. This is a symplectic map represented by the generating function $S(\tau_+,h_-) = \tau_+ h_- + P(\tau_+)$:

$$f(\tau_-,h_-) = (\tau_+,h_+) \ \Leftrightarrow\ \tau_- = \partial_{h_-} S = \tau_+, \quad h_+ = \partial_{\tau_+} S = h_- + P'(\tau_+).$$

The Hamiltonian (1.5) is unbounded as $h \to 0$. Thus we fix small $\delta > 0$ and study the system for $h \in [\delta,\delta^{-1}]$. Then we can try to apply the results of section 2 to the symplectic critical manifold

$$\tilde{N}_\delta = \hat{N} \cap \{\delta \le h \le \delta^{-1}\}.$$

Unfortunately $\tilde{N}_\delta$ is not invariant under the scattering map: the trajectory

$$f^n(\tau,h) = (\tau, h + nP'(\tau))$$

will exit $\tilde{N}_\delta$ after a finite number of steps. Also, since τ is conserved by the scattering map, all heteroclinic chains are degenerate. Hence we can't use Theorems 2.1 and 2.2. But weaker Theorems 2.3 and 2.4 do apply.

Let $\{\gamma_\tau^k\}_{k\in K}$ be a finite collection of families of transverse homoclinics, and $P_k(\tau) = A(\gamma_\tau^k)$ the corresponding Poincaré potentials. Then there are several branches f_k of the scattering map with generating functions

$$S_k(\tau_+, h_-) = \tau_+ h_- + P_k(\tau).$$

As in Section 2, we assume that the leading eigenvalues of the equilibrium $z_0(\tau)$ are simple. In the real case they are $\pm\alpha(\tau)$, and in the complex case $\pm\alpha(\tau) \pm i\beta(\tau)$.

First consider the real case. Define the functions $s_\pm$ on $W^\pm_{\mathrm{loc}}(\tau)$ as in (2.5) and define $\rho_\pm(\gamma_\tau^k)$ as in (2.6). We call a sequence (code) $k_i \in K$ negative if

$$\rho_+(\gamma_\tau^{k_i})\rho_-(\gamma_\tau^{k_{i+1}}) < 0 \quad \text{for all } \ i.$$

We construct trajectories with negative energy corresponding to a negative code. For positive codes and positive energy there is a similar result. Theorem 2.3 with $\mu = -\varepsilon < 0$ implies the following:

Theorem 4.1. *Let $h_0 > 0$ and $\tau_0 \in \mathbb{R}$. Let $(k_i)_{i=0}^{n-1}$ be a negative code such that*

$$h_0 + \sum_{j=0}^{i} P'_{k_j}(0) > 0, \quad i = 0, \cdots, n-1.$$

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ there exists a sequence $(\tau_i)_{i=0}^n$ and a trajectory $\gamma(t) \in M$, $\tau = \tau_0 + \varepsilon t$, such that:

- *$E(0) = -\varepsilon h_0$ and $E(t) < 0$ for $0 \le t \le T = (\tau_n - \tau_0)/\varepsilon$.*
- *The trajectory γ shadows the homoclinic chain $(\gamma_{\tau_i}^{k_i})_{i=0}^{n-1}$:*

$$d(\gamma(t), \gamma_{\tau_i}^{k_i}(\mathbb{R})) \le C\sqrt{\varepsilon}, \quad t_i \le t \le t_{i+1}, \quad t_i = (\tau_i - \tau_0)/\varepsilon.$$

- *The sequences τ_i and $h_i = -E(t_i)/\varepsilon$ satisfy*

$$\Delta\tau_i = \tau_{i+1} - \tau_i = -\frac{\varepsilon \ln \varepsilon}{\alpha(\tau_i)} + \mathcal{O}(\varepsilon), \tag{4.4a}$$

$$\Delta h_i = h_{i+1} - h_i = P'_{k_i}(\tau_i) + \mathcal{O}(\varepsilon \ln \varepsilon). \tag{4.4b}$$

The sequence (τ_i, h_i) shadows a trajectory of a composition of the symplectic maps $f_\varepsilon^{k_i} : (\tau_i, h_i) \to (\tau_{i+1}, h_{i+1})$ with the generating functions

$$S_\varepsilon^{k_i}(\tau_{i+1}, h_i) = \tau_{i+1} h_i + \frac{\varepsilon \ln \varepsilon}{\alpha(\tau_{i+1})} h_i + P_{k_i}(\tau_{i+1}).$$

Thus (τ_i, h_i) shadows a trajectory of the scattering map $\mathcal{F}$.

Theorem 4.1 was proved in [4] by a different method. The complex case was not considered in [4]. Then we use Theorem 2.4. The code is now a pair of sequences (k_i, n_i), where the sequence $k_i \in K$ is arbitrary and $0 \le n_i \le m_0$. The assertion is the same except that (4.4a) is replaced by

$$\Delta\tau_i = \frac{\varepsilon|\ln\varepsilon|}{\alpha(\tau_i)} - \frac{2\pi\varepsilon n_i}{\beta(\tau_i)} + \mathcal{O}(\varepsilon),$$

where $\mathcal{O}(\varepsilon)$ is bounded independent of n_i.

Appendix

We have seen that for $H = o(\varepsilon)$ the reduction (1.5) does not work. Let us discuss this case briefly. The frozen system has a compact normally hyperbolic invariant manifold $N_0 = \{(z_0(\tau), \tau) : \tau \in \mathbb{T}\}$ in $M \times \mathbb{T}$. Hence for small ε there is a normally hyperbolic compact invariant manifold

$$N_\varepsilon = \{(z_\varepsilon(\tau), \tau) : \tau \in \mathbb{T}\}, \quad z_\varepsilon(\tau) = z_0(\tau) + \mathcal{O}(\varepsilon).$$

By (1.3) we have $H|_{N_\varepsilon} = H(z_\varepsilon(\tau), \tau) = \mathcal{O}(\varepsilon^2)$. Let us describe multibump trajectories coming exponentially close to N_ε.

Let $\{\gamma_\tau^k\}_{k\in K}$ be a finite collection of families of transverse homoclinics of the frozen system, and $P_k(\tau) = A(\gamma_\tau^k)$ the Poincaré potentials. Take a discrete periodic set $\mathcal{T}_k \subset \mathbb{R}$ of nondegenerate critical points of P_k. Fix any

$$0 < \delta < \min\{|t - s| : t, s \in \cup_{k\in K}\mathcal{T}_k,\ t \neq s\}.$$

A trajectory will correspond to a code which is a strictly increasing sequence $(\tau_i \in \mathcal{T}_{k_i})_{i\in\mathbb{Z}}$.

Theorem A.1. *Suppose that $\varepsilon > 0$ is sufficiently small. Then for any code $(\tau_i)_{i\in\mathbb{Z}}$ there exists a unique trajectory $\gamma(t) \in M$, $\tau = \tau_0 + \varepsilon t$, such that for all $i \in \mathbb{Z}$,*

- $d(\gamma(t), \gamma_{\tau_i}^{k_i}(\mathbb{R})) \le C\varepsilon$ *for* $|\tau - \tau_i| \le \delta$.
- $d((\gamma(t), \tau), N_\varepsilon) \le e^{-C/\varepsilon}$ *for* $\min_i |\tau - \tau_i| \ge \delta$.

Here C is a constant independent of ε and the code. Theorem A.1 was proved in [5] by global variational methods for natural Hamiltonian systems under the assumption that the homoclinics γ_τ^k are action minimizing but may be nontransverse. Under the transversality assumption the proof is easier, see [9].

The multibump trajectory $\gamma : \mathbb{R} \to M$ shadows the infinite homoclinic chain $(\gamma_{\tau_i}^{k_i})_{i\in\mathbb{Z}}$ and comes exponentially close to the manifold N_ε. Most of the time the energy $E(t) = \mathcal{O}(\varepsilon^2)$. At the critical points $\tau = \tau_i$ the energy has spikes of order ε. The spikes are narrow and rare: $\Delta t_i = \varepsilon^{-1}(\tau_{i+1} - \tau_i) \sim \varepsilon^{-1}$. There is no substantial chainge of energy as in Theorem 4.1.

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