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Index Iteration Theory for Brake Orbit Type Solutions and Applications

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper, we give a survey on the index iteration theory of an index theory for brake orbit type solutions and its applications in the study of brake orbit problems including the Seifert conjecture and the minimal period solution problems in brake orbit cases.

Key Words: Hamiltonian system, Index theory, Iteration theory, Lagrangian boundary problem, Seifert conjecture, Brake orbits.

AMS Subject Classifications: 58E05, 70H05, 34C25, 37J45

1 Introduction

Since 1990, the iteration theory of the Maslov-type index theory for symplectic paths has been systematically developed by Long and his research group [54–56]. It has become a powerful tool in the study of various problems on periodic solutions (or orbits) of nonlinear Hamiltonian systems including: existence, multiplicity, and stability of periodic solution orbits [57, 59, 67, 68] and closed geodesic [5, 13, 14], stability problems of periodic orbits of n-body problems [35–37], Rabinowitz's minimal periodic solution conjecture [12,50–52], Conley's conjecture on sub-harmonic periodic orbits for second as well as first order Hamiltonian systems [27,32]. Recently, Long, Duan, and Zhu published a survey paper [15] on this topic. Interested readers are referred to this paper and references therein.

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Since the paper of [57] of Long, Zhang, and Zhu in 2006, motivated by the study of the existence, multiplicity and stability of brake orbit type periodic solutions of Hamiltonian systems, in order to get more precise information, the index theory and its iteration theory of the Maslov-type index for symplectic paths under brake orbit boundary conditions have been systematically developed. This index and its iteration theory can be used to study brake orbits problems of reversible Hamiltonian systems [47,48,57,73,74] as well as other related problems including Conley conjecture and minimal periodic solution problems in brake orbit case [42,72].

In this survey, we shall give an introduction to this Maslov-type index theory and its relationship with the Maslov index theory in Section 1. In Section 2, we shall describe the main results in the iteration theory of such an index theory. For applications, in Section 3, we shall introduce recent developments on brake orbit problem for compact convex reversible hypersurfaces in $\mathbf{R}^{2n}$, which yields some partial answer to the Seifert conjecture on the multiplicity of brake orbits proposed by Seifert in 1948 [64]. In Section 4, we shall briefly summarize the study of the minimal period solution problems of reversible Hamiltonian systems in brake orbit case.

In this paper, let $\mathbf{N}$, $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{C}$ denote the sets of natural integers, integers, rational numbers, real numbers and complex numbers respectively. Let $\mathbf{U}$ be the unit circle of the complex plane $\mathbf{C}$, i.e., $\mathbf{U} = \{z \in \mathbf{C} |\, |z| = 1\}$.

2 A review on the Maslov-type index theory i_L for symplectic paths under Lagrangian boundary condition

2.1 The i_ω index theory for symplectic paths

We firstly give the definition of i_ω index for symplectic paths which was first introduced by Long in [54] of 1999, all the materials here with historical notes can be found in [56] of 2002 and the recent survey paper [15]. Let $(\mathbf{R}^{2n}, \omega_0)$ be the standard symplectic vector space with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Let

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

be the standard symplectic matrix, where I_n is the identity matrix in $\mathbf{R}^n$. The real symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) | M^T J M = J\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$, where M^T is the transpose of matrix M. The set of symplectic paths in $\mathrm{Sp}(2n)$ starting from the identity matrix is defined by

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) | \gamma(0) = I_{2n}\}, \quad \forall \tau > 0,$$

whose topology is induced from $\mathrm{Sp}(2n)$. Denote the set of all $2n \times 2n$ real matrices by $\mathcal{L}(\mathbf{R}^{2n})$ and its subset of symmetric ones by $\mathcal{L}_s(\mathbf{R}^{2n})$. For any $M \in \mathcal{L}_s(\mathbf{R}^{2n})$, we denote the dimensions of the maximal positive definite subspace, negative definite subspace, and kernel of M by $m^+(M), m^-(M)$ and $m^0(M)$, respectively.

For $\omega \in \mathbf{U}$ and $M \in Sp(2n)$, the function

$$D_\omega(M) = (-1)^{n-1}\omega^{-n} det(M - \omega I)$$

was defined and proved to be real by Long in [53] of 1999. Following [53] and [54], for any $\omega \in \mathbf{U}$, we define

$$\begin{aligned}
&\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) | D_\omega(M) = 0\},\\
&\mathrm{Sp}(2n)^*_\omega = \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^0_\omega,\\
&\mathcal{P}^*_{\tau,\omega}(2n) = \{\gamma \in \mathcal{P}_\tau(2n) | \gamma(\tau) \in \mathrm{Sp}(2n)^*_\omega\},\\
&\mathcal{P}^0_{\tau,\omega}(2n) = \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_{\tau,\omega}(2n).
\end{aligned}$$

For any two continuous paths ξ and $\eta : [0,\tau] \to Sp(2n)$ with $\xi(\tau) = \eta(0)$, we define their joint path by

$$\eta * \xi(t) = \begin{cases} \xi(2t), & 0 \le t \le \dfrac{\tau}{2},\\ \eta(2t-\tau), & \dfrac{\tau}{2} \le t \le \tau. \end{cases}$$

For any two $2k_i \times 2k_i$ matrices of square block form

$$M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$$

with $i = 1, 2$ as in [56], the $\diamond$-sum of M_1 and M_2 is defined by the following $2(k_1 + k_2) \times 2(k_1 + k_2)$ matrix

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

We denote by $M^{\diamond k}$ the k-time self $\diamond$-sum of M for any $k \in \mathbf{N}$.

Define a special path $\xi_n \in \mathcal{P}_\tau(2n)$ by

$$\xi_n(t) = \begin{pmatrix} 2 - \dfrac{t}{\tau} & 0 \\ 0 & \left(2 - \dfrac{t}{\tau}\right)^{-1} \end{pmatrix}^{\diamond n}, \quad \forall t \in [0,\tau].$$

Definition 2.1 ([54,56]). *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define*

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker(M - \omega I_{2n}).$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, let

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).$$

If $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, then we define

$$i_\omega(\gamma) = [\mathrm{Sp}(2n)^0_\omega : \gamma * \xi_n], \tag{2.1}$$

where the right-hand side of (2.1) is the usual homotopy intersection number and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed endpoints.

If $\omega = 1$, simply denote by $i(\gamma)$ instead of $i_1(\gamma)$. If $\gamma(\tau) \in \mathcal{P}^0_{\tau,\omega}(2n)$, let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{i_\omega(\beta) | \beta \in U \cap \mathcal{P}^*_{\tau,\omega}(2n)\}.$$

Then $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \cdots, 2n\}$ is called the index function of γ at ω.

For any $M \in \mathrm{Sp}(2n)$, following [54] we define

$$\Omega(M) = \{P \in \mathrm{Sp}(2n) | \sigma(P) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U},\ \nu_\lambda(P) = \nu_\lambda(M),\ \forall \lambda \in \sigma(M) \cap \mathbf{U}\}.$$

Denote by $\Omega^0(M)$ the path-connected component of $\Omega(M)$ containing M, and call it the homotopy component of $M \in \mathrm{Sp}(2n)$.

Definition 2.2 ([54,56]). *For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, we define the splitting number of M by*

$$S^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma)$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ with $\gamma(\tau) = M$.

2.2 The Maslov type i_L-index theory associated with a Lagrangian subspace for symplectic paths

In this section, we give a brief introduction to the Maslov type i_L-index theory. We refer to the papers [40,41,47,48,57] for more details.

A Lagrangian subspace L of the standard symplectic space $(\mathbf{R}^{2n}, \omega_0)$ with

$$\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$$

is an n dimensional subspace satisfying $\omega_0|_L = 0$. The set of all Lagrangian subspaces in $(\mathbf{R}^{2n}, \omega_0)$ is denoted by $\Lambda(n)$.

When $\tau = 1$, we write simply $\mathcal{P}(2n)$ instead of $\mathcal{P}_1(2n)$. For a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$, we write it in the following form

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix}, \tag{2.2}$$

where $S(t), T(t), V(t), U(t)$ are $n \times n$ matrices. The n vectors consisting of columns of the matrix $\begin{pmatrix} V(t) \\ U(t) \end{pmatrix}$ are linearly independent and they span a Lagrangian subspace path of $(\mathbf{R}^{2n}, \omega_0)$. For $L_0 = \{0\} \times \mathbf{R}^n \in \Lambda(n)$, we define the following two subsets of $\mathrm{Sp}(2n)$ by

$$\begin{aligned} \mathrm{Sp}(2n)^*_{L_0} &= \{M \in \mathrm{Sp}(2n) \,|\, \det V \neq 0\}, \\ \mathrm{Sp}(2n)^0_{L_0} &= \{M \in \mathrm{Sp}(2n) \,|\, \det V = 0\}, \end{aligned}$$

for $M = \begin{pmatrix} S & V \\ T & U \end{pmatrix}$.

Since the space $\mathrm{Sp}(2n)$ is path connected, and the set of $n \times n$ non-degenerate matrices has two path connected components consisting of matrices with positive and negative determinants respectively, we denote by

$$\begin{aligned} \mathrm{Sp}(2n)^{\pm}_{L_0} &= \{M \in \mathrm{Sp}(2n) \,|\, \pm \det V > 0\}, \\ \mathcal{P}_\tau(2n)^*_{L_0} &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(1) \in \mathrm{Sp}(2n)^*_{L_0}\}, \\ \mathcal{P}_\tau(2n)^0_{L_0} &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(1) \in \mathrm{Sp}(2n)^0_{L_0}\}. \end{aligned}$$

Definition 2.3 ([40,57]). *We define the L_0-nullity of any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ by*

$$\nu_{L_0}(\gamma) = \dim \ker V(\tau)$$

with the $n \times n$ matrix function $V(t)$ defined in (2.2).

Note that the complex matrix $U(t) \pm \sqrt{-1}V(t)$ is invertible. We define a complex matrix function by

$$\mathcal{Q}(t) = [U(t) - \sqrt{-1}V(t)][U(t) + \sqrt{-1}V(t)]^{-1}.$$

The matrix $\mathcal{Q}(t)$ is unitary for any $t \in [0, \tau]$. We denote by

$$M_+ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n = \mathrm{diag}(-1, 1, \cdots, 1).$$

It is clear that $M_\pm \in \mathrm{Sp}(2n)^{\pm}_{L_0}$.

For a path $\gamma \in \mathcal{P}_\tau(2n)^*_{L_0}$, we define a symplectic path by

$$\tilde{\gamma}(t) = \begin{cases} I \cos \dfrac{(\tau - 2t)\pi}{2\tau} + J \sin \dfrac{(\tau - 2t)\pi}{2\tau}, & t \in [0, \tau/2], \\ \gamma(2t - \tau), & t \in [\tau/2, \tau], \end{cases}$$

and choose a symplectic path $\beta(t)$ in $\mathrm{Sp}(2n)^*_{L_0}$ starting from $\gamma(\tau)$ and ending at M_+ or M_- according to $\gamma(\tau)\in\mathrm{Sp}(2n)^+_{L_0}$ or $\gamma(\tau)\in\mathrm{Sp}(2n)^-_{L_0}$ respectively. We now define a joint path by

$$\bar{\gamma}(t)=\beta*\tilde{\gamma}:=\begin{cases}\tilde{\gamma}(2t), & t\in[0,\tau/2],\\ \beta(2t-\tau), & t\in[\tau/2,\tau].\end{cases}$$

By the definition, we see that the symplectic path $\bar{\gamma}$ starts from $-M_+$ and ends at either M_+ or M_-. As above, we define

$$\bar{\mathcal{Q}}(t)=[\bar{U}(t)-\sqrt{-1}\bar{V}(t)][\bar{U}(t)+\sqrt{-1}\bar{V}(t)]^{-1}$$

for

$$\bar{\gamma}(t)=\begin{pmatrix}\bar{S}(t) & \bar{V}(t)\\ \bar{T}(t) & \bar{U}(t)\end{pmatrix}.$$

We can choose a continuous function $\bar{\Delta}(t)$ on $[0,\tau]$ such that

$$\det\bar{\mathcal{Q}}(t)=e^{2\sqrt{-1}\bar{\Delta}(t)}.$$

By the above arguments, we see that the number $\frac{1}{\pi}(\bar{\Delta}(\tau)-\bar{\Delta}(0))\in\mathbf{Z}$ and it does not depend on the choices of the path β and the function $\bar{\Delta}(t)$ by a proof similar to that in [56].

Definition 2.4 ([40]). *For a symplectic path $\gamma\in\mathcal{P}_\tau(2n)^*_{L_0}$, we define the L_0-index of γ by*

$$i_{L_0}(\gamma)=\frac{1}{\pi}(\bar{\Delta}(\tau)-\bar{\Delta}(0)).$$

Definition 2.5 ([40]). *For a symplectic path $\gamma\in\mathcal{P}_\tau(2n)^0_{L_0}$, let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, we define the L_0-index of γ by*

$$i_{L_0}(\gamma)=\sup_{U\in\mathcal{F}(\gamma)}\inf\{i_{L_0}(\gamma^*)\,|\,\gamma^*\in U\cap\mathcal{P}_\tau(2n)^*_{L_0}\}.$$

In the general situation, let $L\in\Lambda(n)$. It is well known that $\Lambda(n)=U(n)/O(n)$, this means that for any linear subspace $L\in\Lambda(n)$, there is an orthogonal symplectic matrix

$$P=\begin{pmatrix}A & -B\\ B & A\end{pmatrix}$$

with $A\pm\sqrt{-1}B\in U(n)$ such that $PL_0=L$. We define the conjugated symplectic path $\gamma_c\in\mathcal{P}_\tau(2n)$ of γ by $\gamma_c(t)=P^{-1}\gamma(t)P$.

Definition 2.6 ([40]). *We define the L-nullity of any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ by*

$$\nu_L(\gamma) = \dim\ker V_c(\tau),$$

the $n\times n$ matrix function $V_c(t)$ is defined in (2.2) *with the symplectic path γ replaced by γ_c, i.e.,*

$$\gamma_c(t) = \begin{pmatrix} S_c(t) & V_c(t) \\ T_c(t) & U_c(t) \end{pmatrix}.$$

Definition 2.7 ([40]). *For a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$, we define the L-index of γ by*

$$i_L(\gamma) = i_{L_0}(\gamma_c).$$

Remark 2.1. (1) The Definitions 2.6 and 2.7 do not depend on the special choice of P. (2) In [57] of 2006, Long, Zhang and Zhu studied the multiple solutions of the brake orbit problem on a convex hypersurface, they introduced indices $(\mu_1(\gamma),\nu_1(\gamma))$ and $(\mu_2(\gamma),\nu_2(\gamma))$ for any symplectic path γ. The indices $\mu_1(\gamma)$ and $\mu_2(\gamma)$ are special cases of the L-index $i_L(\gamma)$ for Lagrangian subspaces $L_0 = \{0\}\times\mathbf{R}^n$ and $L_1 = \mathbf{R}^n\times\{0\}$ respectively up to a constant n. In [63] of 1993, Robbin and Salamon defined a half integer valued index for symplectic paths with Lagrange boundary conditions.

The i_L index can also be defined by Maslov indices of the corresponding Lagrangian subspace pair paths. There is a good introduction on Maslov indices of the corresponding real Lagrangian subspace pair paths in [11] of 1994 by Cappel, Lee and Miller, which can be extended to complex Lagrangian subspace pair paths [58]. Recently the first author of this paper defined also an i_L-index theory for general symplectic curves in the monograph [43], such an index theory satisfies an axioms characterization in terms of the affine scale invariance, homotopy invariance, path additivity, symplectic additivity, symplectic invariance and normalization.

We denote by

$$F = \mathbf{R}^{2n}\oplus\mathbf{R}^{2n}$$

equipped with the standard inner product $(\cdot,\cdot)$ and define the symplectic structure of F by

$$\{v,w\} = (\mathcal{J}v,w), \quad \forall v,w\in F, \quad \text{where } \mathcal{J} = (-J)\oplus J = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

We denote by $\mathrm{Lag}(F)$ the set of Lagrangian subspaces of F, and equip it with the topology as a subspace of the Grassmannian of all $2n$-dimensional subspaces of F.

It is easy to check that, for any $M\in\mathrm{Sp}(2n)$ its graph

$$\mathrm{Gr}(M) \equiv \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \middle|\, x\in\mathbf{R}^{2n} \right\}$$

is a Lagrangian subspace of F.

Let

$$V_1 = L_0 \times L_0 = \{0\} \times \mathbf{R}^n \times \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{4n},$$
$$V_2 = L_1 \times L_1 = \mathbf{R}^n \times \{0\} \times \mathbf{R}^n \times \{0\} \subset \mathbf{R}^{4n}.$$

Proposition 2.1 ([47,59]). *For any continuous path $\gamma \in \mathcal{P}_\tau(2n)$, there hold*

$$\begin{aligned} &i_{L_0}(\gamma) = \mu_F^{CLM}(V_1, \mathrm{Gr}(\gamma), [0,\tau]) - n, \\ &i_{L_1}(\gamma) = \mu_F^{CLM}(V_2, \mathrm{Gr}(\gamma), [0,\tau]) - n, \\ &\nu_{L_j}(\gamma) = \dim(\gamma(\tau)L_j \cap L_j), \qquad j = 0,1, \end{aligned}$$

where we denote by $i_F^{CLM}(V, W, [a,b])$ the Maslov index for Lagrangian subspace path pair (V, W) in F on $[a,b]$ defined by Cappell, Lee, and Miller in [11]. For any $M \in \mathrm{Sp}(2n)$ and $j = 0,1$, we also denote by $\nu_{L_j}(M) = \dim(ML_j \cap L_j)$.

In [16], Duistermaat introduced the Hörmander index which expresses the difference of Maslov index of the same symplectic path under different lagrangian boundary conditions as signature of corresponding quadratic form in nondegenerate case. In [75] of 2018, Zhou, Wu, and Zhu extended the Hörmander index into degenerate case. By the computation of the Hörmander index, in [57] of 2006, Long, Zhang and Zhu proved

Theorem 2.1 ([57]). *For any continuous path $\gamma \in \mathcal{P}_\tau(2n)$, there hold*

$$|i_{L_0}(\gamma) - i_{L_1}(\gamma)| \le n.$$

More precisely, for any $P \in \mathrm{Sp}(2n)$ and $\varepsilon \in \mathbf{R}$, set

$$M_\varepsilon(P) = P^T \begin{pmatrix} \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} P + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix}.$$

In [72], Zhang proved

Theorem 2.2 ([72]). *For $\gamma \in \mathcal{P}_\tau(2n)$ with $\tau > 0$, we have*

$$i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2}\mathrm{sgn} M_\varepsilon(\gamma(\tau)),$$

where $\mathrm{sgn} M_\varepsilon(\gamma(\tau))$ is the signature of the symmetric matrix $M_\varepsilon(\gamma(\tau))$ and $\varepsilon > 0$ is sufficiently small.

We also have,

$$(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) = \frac{1}{2}\mathrm{sgn} M_\varepsilon(\gamma(\tau)),$$

where $\varepsilon < 0$ and $|\varepsilon|$ is sufficiently small.

The key ingredients in the proof of Theorem 3.2 below are some ideas from [47, 57] and the following estimate, where the iteration path γ^2 will be defined in Section 2 below.

Theorem 2.3 ([48]). *For $\gamma \in \mathcal{P}_\tau(2n)$, let $P = \gamma(\tau)$. If $i_{L_0}(\gamma) \geq 0$, $i_{L_1}(\gamma) \geq 0$, $i(\gamma) \geq n$, $\gamma^2(t) = \gamma(t-\tau)\gamma(\tau)$ for all $t \in [\tau, 2\tau]$, then*

$$i_{L_1}(\gamma) + S^+_{P^2}(1) - \nu_{L_0}(\gamma) \geq 0.$$

3 The iteration theory for i_{L_0} and i_{L_1} index

In many problems related to nonlinear Hamiltonian systems, it is necessary to study iterations of periodic solutions. In order to distinguish two geometrically distinct periodic solutions, one way is to study the Maslov-type indices of the iteration paths of the fundamental solutions of the corresponding linearized Hamiltonian systems. For $\gamma \in \mathcal{P}_\tau(2n)$, we define $\tilde{\gamma}(t) = \gamma(t-j)\gamma(1)^j$, $j \leq t \leq j+1$, $j \in \{0\} \cup \mathbf{N}$, and the k-times iteration path of γ by $\gamma^k = \tilde{\gamma}|_{[0,k]}$ for any $k \in \mathbf{N}$. In the paper [54] of Long in 1999, the following result was proved

$$i(\gamma^k) = \sum_{\omega^k=1} i_\omega(\gamma), \quad \nu(\gamma^k) = \sum_{\omega^k=1} \nu_\omega(\gamma).$$

In [55] of 2000, Long established the precise index iteration formula for the Maslov-type indices. From these results, various iteration index formulas were obtained and were used to study the multiplicity and stability problems of periodic solutions related to the nonlinear Hamiltonian systems. We refer to the book [56] of Long in 2002 and the references therein for these topics.

In order to study the brake orbit problem, it is necessary to study iterations of the brake orbits. In order to do this, one way is to study the L_0-index of the iteration path γ^k of the fundamental solution γ of the corresponding linear system for any $k \in \mathbf{N}$. In this case, the L_0-iteration path γ^k of γ is different from that of the general periodic case mentioned above. Its definition is given below.

In 1956, Bott in [10] established the famous iteration formula of the Morse index for closed geodesics on Riemannian manifolds. For convex Hamiltonian systems, Ekeland developed the similar Bott-type iteration index formulas for the Ekeland index theory [17] of 1990. In [54] of 1999, Long established the Bott-type iteration formulas for the Maslov-type index theory. Motivated by the above results, in [47] of Liu and Zhang in 2014, the following Bott-type iteration formulas for the L_0-index was established. In [70] of 2018, Wu and Zhu extended the iteration formula to weak symplectic Hilbert spaces.

Define the involution matrix of $(\mathbf{R}^{2n}, \omega_0)$ by

$$N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

It is anti-symplectic, i.e., $NJ = -JN$. The fixed point set of N and $-N$ are the Lagrangian subspaces $L_0 = \{0\} \times \mathbf{R}^n$ and $L_1 = \mathbf{R}^n \times \{0\}$ of $(\mathbf{R}^{2n}, \omega_0)$, respectively.

For simplicity, we suppose $\gamma \in \mathcal{P}_1(2n)$, i.e., we take $\tau = 1$. For $j \in \mathbf{N}$, we define the j-times iteration path $\gamma^j : [0,j] \to \mathrm{Sp}(2n)$ of γ by

$$\gamma^1(t) = \gamma(t), \quad t \in [0,1],$$
$$\gamma^2(t) = \begin{cases} \gamma(t), & t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1,2], \end{cases}$$

and in general, for $k \in \mathbf{N}$, we define $\gamma(2) = N\gamma(1)^{-1}N\gamma(1)$ and

$$\gamma^{2k-1}(t) = \begin{cases} \gamma(t), & t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1,2], \\ \cdots & \\ N\gamma(2k-2-t)N\gamma(2)^{k-1}, & t \in [2k-3,2k-2], \\ \gamma(t-2k+2)\gamma(2)^{k-1}, & t \in [2k-2,2k-1], \end{cases}$$
$$\gamma^{2k}(t) = \begin{cases} \gamma(t), & t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1,2], \\ \cdots & \\ \gamma(t-2k+2)\gamma(2)^{k-1}, & t \in [2k-2,2k-1], \\ N\gamma(2k-t)N\gamma(2)^{k}, & t \in [2k-1,2k]. \end{cases}$$

For $\gamma \in \mathcal{P}_\tau(2n)$, we define

$$\gamma^k(t) = \tilde{\gamma}^k\left(\frac{t}{\tau}\right) \quad \text{with } \tilde{\gamma}(t) = \gamma(\tau t). \tag{3.1}$$

Theorem 3.1 ([47] of Liu and Zhang in 2014)**.** *Suppose $\gamma \in \mathcal{P}_\tau(2n)$, for the iteration symplectic paths γ^k, when k is odd, there hold*

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} i_{\omega_k^{2i}}(\gamma^2), \quad \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} \nu_{\omega_k^{2i}}(\gamma^2),$$

when k is even, there hold

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} i_{\omega_k^{2i}}(\gamma^2),$$
$$\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \nu^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} \nu_{\omega_k^{2i}}(\gamma^2),$$

where $\omega_k = e^{\pi\sqrt{-1}/k}$, and $(i^{L_0}_\omega(\gamma^1), \nu^{L_0}_\omega(\gamma^1))$ is the (L_0,ω)-index pair first defined in [47] for $\omega \in \mathbf{U}$ (see also [72] and Definition 3.1 *below in a different way).*

From the Bott-type formulas in Theorem 3.1, the abstract precise iteration index formula of i_{L_0} was given in [47]. We recall the definition of $E(a) = \min\{k \in \mathbf{Z} | k \geq a\}$ for $a \in \mathbf{R}$.

Theorem 3.2 ([47]). *Let $\gamma \in \mathcal{P}_\tau(2n)$, γ^k is defined by as above, and $M = \gamma^2(2\tau)$. Then for every $k \in 2\mathbf{N}-1$, there holds*

$$\begin{aligned} i_{L_0}(\gamma^k) =& i_{L_0}(\gamma^1) + \frac{k-1}{2}(i(\gamma^2) + S_M^+(1) - C(M)) \\ &+ \sum_{\theta\in(0,2\pi)} E\left(\frac{k\theta}{2\pi}\right) S_M^-(e^{\sqrt{-1}\theta}) - C(M), \end{aligned}$$

where $C(M)$ is defined by

$$C(M) = \sum_{\theta\in(0,2\pi)} S_M^-(e^{\sqrt{-1}\theta})$$

and $S_M^\pm(\omega)$ is the splitting number of the symplectic matrix M at ω for $\omega \in \mathbf{U}$ defined in Subsection 1.1. For every $k \in 2\mathbf{N}$, there holds

$$\begin{aligned} i_{L_0}(\gamma^k) =& i_{L_0}(\gamma^2) + \left(\frac{k}{2} - 1\right) \left(i(\gamma^2) + S_M^+(1) - C(M)\right) \\ &- C(M) - \sum_{\theta\in(\pi,2\pi)} S_M^-(e^{\sqrt{-1}\theta}) + \sum_{\theta\in(0,2\pi)} E\left(\frac{k\theta}{2\pi}\right) S_M^-(e^{\sqrt{-1}\theta}). \end{aligned}$$

In [72] of Zhang in 2015, the minimal period problems for symmetric brake orbits was studied, the Maslov-type index of symmetric brake orbits was defined as follows.

Definition 3.1 ([72]). *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega = e^{\sqrt{-1}\theta}$ with $\theta \in (0,\pi)$, let $V_\omega = L_0 \times (e^{\theta J} L_0)$, we define*

$$\begin{aligned} i_\omega^{L_0}(\gamma) &= \mu_F^{CLM}(V_\omega, \mathrm{Gr}(\gamma), [0,\tau]), \\ \nu_\omega^{L_0}(\gamma) &= \dim\left(\gamma(\tau) L_0 \cap e^{\sqrt{-1}\theta J} L_0\right). \end{aligned}$$

In order to estimate the minimal period for symmetric brake orbits, we need the iteration formula of the Maslov-type index of $(i^{L_0}_{\sqrt{-1}}, \nu^{L_0}_{\sqrt{-1}})$ for symplectic paths starting at the identity. Precisely the following Bott-type iteration formula was proved.

Theorem 3.3 ([72]). *Let $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega_k = e^{\pi\sqrt{-1}/k}$. For odd k we have*

$$\begin{aligned} i^{L_0}_{\sqrt{-1}}(\gamma^k) &= i^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{(k-1)/2} i_{\omega_k^{2i-1}}(\gamma^2), \\ \nu^{L_0}_{\sqrt{-1}}(\gamma^k) &= \nu^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{(k-1)/2} \nu_{\omega_k^{2i-1}}(\gamma^2), \end{aligned}$$

and for even k, we have

$$i^{L_0}_{\sqrt{-1}}(\gamma^k)=\sum_{i=1}^{k/2} i_{\omega_k^{2i-1}}(\gamma^2),\quad \nu^{L_0}_{\sqrt{-1}}(\gamma^k)=\sum_{i=1}^{k/2}\nu_{\omega_k^{2i-1}}(\gamma^2). \tag{3.2}$$

For any given compact strictly convex (or star-shaped) C^2 hypersurface Σ in $\mathbf{R}^{2n}$, a closed characteristic (τ,y) on Σ is a solution of the problem

$$\begin{cases} \dot{y}=Jn_\Sigma(y),\\ y(\tau)=y(0),\end{cases} \tag{3.3}$$

where $n_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $n_\Sigma(y)\cdot y=1$. Here $a\cdot b$ denotes the standard inner product of $a,b\in R^{2n}$.

In the study of closed characteristics problems and closed geodesics, the common index jump theorem [59] of Long and Zhu in 2002 and the enhanced common index jump theorem [13] of Duan, Long and Wang in 2016 for a finite collection of symplectic paths starting from identity with positive mean indices $i_1(\gamma_j)$ play important roles. The common index jump theorem of the i_{L_0}-index for a finite collection of symplectic paths starting from identity with positive mean i_{L_0}-indices was established in [47]. In the following of this paper, we write $(i_{L_0}(\gamma,k),\nu_{L_0}(\gamma,k))=(i_{L_0}(\gamma^k),\nu_{L_0}(\gamma^k))$ for any symplectic path $\gamma\in\mathcal{P}_\tau(2n)$ and $k\in\mathbf{N}$. From Theorems 3.1 and 3.2, we know that the mean indices

$$\hat{i}_{L_0}(\gamma):=\lim_{k\to\infty}\frac{i_{L_0}(\gamma,k)}{k}\quad\text{and}\quad \hat{i}(\gamma):=\lim_{k\to\infty}\frac{i(\gamma,k)}{k}$$

are well defined and we have

$$\hat{i}_{L_0}(\gamma)=\hat{i}_{L_1}(\gamma)=\hat{i}(\gamma)$$

for any symplectic path $\gamma\in\mathcal{P}_\tau(2n)$.

Theorem 3.4 (The common index jump theorem for brake orbit boundary condition [47, 59])**.** *Let $\gamma_j\in\mathcal{P}_{\tau_j}(2n)$ and $M_j=\gamma_j^2(2\tau_j)=N\gamma_j(\tau_j)^{-1}N\gamma_j(\tau_j)$ for $j=1,\cdots,q$. Suppose*

$$\hat{i}_{L_0}(\gamma_j)>0,\quad \forall j=1,\cdots,q.$$

Then there exist infinitely many $(R,m_1,m_2,\cdots,m_q)\in\mathbf{N}^{q+1}$ such that

(i) $\nu_{L_0}(\gamma_j,2m_j\pm1)=\nu_{L_0}(\gamma_j)$,

(ii) $i_{L_0}(\gamma_j,2m_j-1)+\nu_{L_0}(\gamma_j,2m_j-1)=R-(i_{L_1}(\gamma_j)+n+S^+_{M_j}(1)-\nu_{L_0}(\gamma_j))$,

(iii) $i_{L_0}(\gamma_j,2m_j+1)=R+i_{L_0}(\gamma_j)$,

(iv) $\nu(\gamma_j^2,2m_j\pm1)=\nu(\gamma_j^2)$,

(v) $i(\gamma_j^2, 2m_j - 1) + \nu(\gamma_j^2, 2m_j - 1) = 2R - (i(\gamma_j^2) + 2S^+_{M_j}(1) - \nu(\gamma_j^2))$,

(vi) $i(\gamma_j^2, 2m_j + 1) = 2R + i(\gamma_j^2)$,

where we have set

$$i(\gamma_j^2, n_j) = i(\gamma_j^{2n_j}, [0, 2n_j\tau_j]), \quad \nu(\gamma_j^2, n_j) = \nu(\gamma_j^{2n_j}, [0, 2n_j\tau_j])$$

for $n_j \in \mathbf{N}$.

4 Applications, brake orbits on given reversible hypersurfaces in $\mathbf{R}^{2n}$

The i_L index theory is suitable for studying the Lagrangian boundary value problems (L-solution, for short) related to nonlinear Hamiltonian systems. A typical application is to study the Seifert conjecture for the multiplicity of brake orbits.

4.1 The Seifert conjecture

Let us recall the famous conjecture proposed by H. Seifert in his pioneer work [64] of 1948 concerning the multiplicity of brake orbits in certain Hamiltonian systems in $\mathbf{R}^{2n}$.

We assume that $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$ possesses the following form

$$H(p,q) = \frac{1}{2}A(q)p \cdot p + V(q), \tag{4.1}$$

where $p, q \in \mathbf{R}^n$, $A(q)$ is a C^2 positive definite $n \times n$ symmetric matrix in $q \in \mathbf{R}^n$ and $V \in C^2(\mathbf{R}^n, \mathbf{R})$ is the potential energy. The solution of the following Hamiltonian system

$$\dot{x} = JH'(x), \quad x = (p,q), \tag{4.2a}$$

$$p(0) = p\Big(\frac{\tau}{2}\Big) = 0, \tag{4.2b}$$

is called a brake orbit. Moreover, if h is the total energy of a brake orbit (p,q), i.e., $H(p(t), q(t)) = h$ and $V(q(0)) = V(q(\tau)) = h$. Then $q(t) \in \bar{\Omega} \equiv \{q \in \mathbf{R}^n | V(q) \le h\}$ for all $t \in \mathbf{R}$.

In [64] of 1948, Seifert studied the existence of brake orbits for system (4.2a)-(4.2b) with the Hamiltonian function H being in the form of (4.1) and proved that the set $\mathcal{J}_b(\Sigma)$ of brake orbits on the energy surface $\Sigma = H^{-1}(h)$ is not empty, i.e., $\mathcal{J}_b(\Sigma) \neq \emptyset$ provided $V' \neq 0$ on $\partial\Omega$, V is analytic and $\bar{\Omega}$ is bounded and homeomorphic to the unit ball $B_1^n(0)$ in $\mathbf{R}^n$. Denoted by $\tilde{\mathcal{J}}_b(\Sigma)$ the set of geometrically distinct brake orbits on the energy surface Σ. The precise sense of the sets $\mathcal{J}_b(\Sigma)$ and $\tilde{\mathcal{J}}_b(\Sigma)$ are explained in the next subsection. Then in the same paper he proposed the following conjecture $^{\#}\tilde{\mathcal{J}}_b(\Sigma) \ge n$ for Σ described as above.

We note that for the Hamiltonian function

$$H(p,q)=\frac{1}{2}|p|^2+\sum_{j=1}^{n}a_j^2q_j^2,\quad q,p\in\mathbf{R}^n,$$

where $a_i/a_j\in\mathbf{R}\setminus\mathbf{Q}$ for all $i\neq j$ and $q=(q_1,q_2,\cdots,q_n)$, there are exactly n geometrically distinct brake orbits on the energy hypersurface $\Sigma=H^{-1}(h)$.

4.2 The generalized Seifert conjecture for reversible hypersurfaces

In general, we suppose that $H\in C^2(\mathbf{R}^{2n}\setminus\{0\},\mathbf{R})\cap C^1(\mathbf{R}^{2n},\mathbf{R})$ satisfies the following reversible condition

$$H(Nx)=H(x),\quad \forall x\in\mathbf{R}^{2n}. \tag{4.3}$$

For given $h>0$, we consider the following fixed energy problem of nonlinear Hamiltonian system with Lagrangian boundary conditions

$$\dot{x}(t)=JH'(x(t)), \tag{4.4a}$$
$$H(x(t))=h, \tag{4.4b}$$
$$x(0)\in L_0,\quad x(\tau/2)\in L_0. \tag{4.4c}$$

It is clear that a solution (τ,x) of (4.4a)-(4.4c) is a characteristic chord on the contact submanifold $\Sigma:=H^{-1}(h)=\{y\in\mathbf{R}^{2n}\mid H(y)=h\}$ of $(\mathbf{R}^{2n},\omega_0)$ and satisfies

$$x(-t)=Nx(t), \tag{4.5a}$$
$$x(\tau+t)=x(t). \tag{4.5b}$$

In general, this kind of τ-periodic characteristic (τ,x) is called a brake orbit on the hypersurface Σ. We note that the problem (4.2a)-(4.2b) with the Hamiltonian function H defined in (4.1) is a special case of the problem (4.4a)-(4.4c). We denote by $\mathcal{J}_b(\Sigma,H)$ the set of all brake orbits on Σ. Two brake orbits $(\tau_i,x_i)\in\mathcal{J}_b(\Sigma,H)$ with $i=1,2$, are equivalent, if the two brake orbits are geometrically the same, i.e., $x_1(\mathbf{R})=x_2(\mathbf{R})$. We denote by $[(\tau,x)]$ the equivalence class of $(\tau,x)\in\mathcal{J}_b(\Sigma,H)$ in this equivalence relation and by $\tilde{\mathcal{J}}_b(\Sigma,\ H)$ the set of $[(\tau,x)]$ for all $(\tau,x)\in\mathcal{J}_b(\Sigma,H)$. In fact $\tilde{\mathcal{J}}_b(\Sigma,\ H)$ is the set of geometrically distinct brake orbits on Σ, which is independent of the choice of H. So from now on we simply denote it by $\tilde{\mathcal{J}}_b(\Sigma)$ and in the notation $[(\tau,x)]$ we always mean that x has the minimal period τ. We also denote by $\tilde{\mathcal{J}}(\Sigma)$ the set of all geometrically distinct closed characteristics on Σ. The number of elements in a set S is denoted by $^{\#}S$. It is well known that $^{\#}\tilde{\mathcal{J}}_b(\Sigma)$ (and also $^{\#}\tilde{\mathcal{J}}(\Sigma)$) depends only on Σ, that is to say, for simplicity we take $h=1$, if H and G are two C^2 functions satisfying (4.3) and $\Sigma_H:=H^{-1}(1)=\Sigma_G:=G^{-1}(1)$, then $^{\#}\mathcal{J}_b(\Sigma_H)={}^{\#}\ \mathcal{J}_b(\Sigma_G)$. So we can consider the brake orbit problem in a more general setting. Let Σ be a C^2 compact hypersurface in $\mathbf{R}^{2n}$ bounding a compact set C with nonempty interior. Suppose Σ has non-vanishing Gaussian curvature and satisfies the reversible condition $N(\Sigma - x_0) = \Sigma - x_0 := \{x - x_0 | x \in \Sigma\}$ for some $x_0 \in C$. Without loss of generality, we may assume $x_0 = 0$. We denote the set of all such hypersurfaces in $\mathbf{R}^{2n}$ by $\mathcal{H}_b(2n)$. For $x \in \Sigma$, let $n_\Sigma(x)$ be the outward unit normal vector at $x \in \Sigma$ as in (3.3). Note that here by the reversible condition there holds $n_\Sigma(Nx) = Nn_\Sigma(x)$. We consider the dynamics problem of finding $\tau > 0$ and a C^1 smooth curve $x : [0, \tau] \to \mathbf{R}^{2n}$ such that

$$\dot{x}(t) = Jn_\Sigma(x(t)), \qquad x(t) \in \Sigma, \tag{4.6a}$$

$$x(-t) = Nx(t), \qquad x(\tau + t) = x(t) \quad \text{for all } \ t \in \mathbf{R}. \tag{4.6b}$$

A solution (τ, x) of the problem (4.6a)-(4.6b) determines a brake orbit on Σ. Now the generalized Seifert conjecture can be represented as

The Generalized Seifert Conjecture: For any $\Sigma \in \mathcal{H}_b(2n)$, there holds

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq n.$$

We can view the above estimate as a result on the number of Lagrangian intersection (more precisely the Legendre intersection, since $\Sigma \cap L_0$ is a Legendre submanifold of the contact manifold Σ) of "reversible" Hamiltonian map φ_b:

$$^{\#}\{\Sigma \cap L_0 \cap \varphi_b(L_0)\} \geq n.$$

The famous Arnold conjecture is related to the Lagrangian boundary problem, it says that the number of Lagrangian intersection of a Hamiltonian map on a closed symplectic manifold M can be estimated from below by the Betti number of M in the non-degenerate case and by the cuplength of M [4] of 1965. In this direction readers are referred to [22, 34, 39] of 1989, 1998, and 2005 respectively.

4.3 Some related results since 1948

As a special case, letting $A(q) = I$ in (4.1), the problem (4.2a)-(4.2b) corresponds to the following classical fixed energy problem of the second order autonomous Hamiltonian system

$$\ddot{q}(t) + V'(q(t)) = 0 \qquad \text{for } \ q(t) \in \Omega, \tag{4.7a}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \qquad \forall t \in \mathbf{R}, \tag{4.7b}$$

$$\dot{q}(0) = \dot{q}\left(\frac{\tau}{2}\right) = 0, \tag{4.7c}$$

where $V \in C^2(\mathbf{R}^n, \mathbf{R})$ and h is a constant such that $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$ is nonempty, bounded and connected.

A solution (τ, q) of (4.7a)-(4.7c) is still called a brake orbit in $\bar{\Omega}$. Two brake orbits q_1 and $q_2 : \mathbf{R} \to \mathbf{R}^n$ are geometrically distinct if $q_1(\mathbf{R}) \neq q_2(\mathbf{R})$. We denote by $\mathcal{O}(\Omega, V)$ and $\tilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in $\bar{\Omega}$ respectively.

Remark 4.1. It is well known that via

$$H(p,q)=\frac{1}{2}|p|^2+V(q),$$

$x=(p,q)$ and $p=\dot{q}$, the elements in $\mathcal{O}(\Omega,V)$ and the solutions of (4.4a)-(4.4c) are one to one correspondent.

After 1948, various studies have been carried out for the brake orbit problem. Bolotin proved in [8] of 1978 the existence of brake orbits in general setting. Hayashi in [31] of 1983, Gluck and Ziller in [28] of 1983, and Benci in [6] of 1984 proved $^{\#}\tilde{\mathcal{O}}(\Omega)\geq 1$, if V is C^1, $\bar{\Omega}=\{V\leq h\}$ is compact, and $V'(q)\neq 0$ for all $q\in\partial\Omega$. Rabinowitz in [62] of 1978 proved that if H satisfies (4.3), $\Sigma\equiv H^{-1}(h)$ is star-shaped, and $x\cdot H'(x)\neq 0$ for all $x\in\Sigma$, then $^{\#}\tilde{\mathcal{J}}_b(\Sigma)\geq 1$. Benci and Giannoni gave a different proof of the existence of one brake orbit in [7] of 1989. In 2005, it was pointed out in [24] of Giambò, Giannoni and Piccione that the problem of finding brake orbits is equivalent to find orthogonal geodesic chords on manifold with concave boundary. Giambò, Giannoni and Piccione in [25] of 2010 proved the existence of an orthogonal geodesic chord on a Riemannian manifold homeomorphic to a closed disk and with concave boundary. For the multiplicity of the brake orbit problems, Weinstein in [69] of 1973 proved a local result: Assume H satisfies (4.3). For any h sufficiently close to $H(z_0)$ with z_0 being a nondegenerate local minimum of H, there exist at least n geometrically distinct brake orbits on the energy surface $H^{-1}(h)$. In [9] of 1978 of Bolotin and Kozlov and in [28] of 1983 of Gluck and Ziller, the existence of at least n brake orbits was proved under assumptions of Seifert in [64] and an additional assumption on the energy integral such that different minimax critical levels correspond to geometrically distinct brake orbits. Szulkin in [65] of 1989 proved that $^{\#}\tilde{\mathcal{J}}_b(H^{-1}(h))\geq n$, if H satisfies conditions in [62] of 1978 of Rabinowitz and the energy hypersurface $H^{-1}(h)$ is $\sqrt{2}$-pinched. Groesen in [29] of 1985 and Ambrosetti, Benci and Long in [1] of 1993 also proved $^{\#}\tilde{\mathcal{O}}(\Omega)\geq n$ under different pinching conditions.

4.4 Recent progress on the Seifert conjecture

Definition 4.1. *We denote by*

$$\begin{aligned}&\mathcal{H}_b^c(2n)=\{\Sigma\in\mathcal{H}_b(2n)|\ \Sigma \text{ is strictly convex }\},\\&\mathcal{H}_b^{s,c}(2n)=\{\Sigma\in\mathcal{H}_b^c(2n)|\ -\Sigma=\Sigma\}.\end{aligned}$$

Without any pinching conditions, Long, Zhang and Zhu in [57] of 2006 established a Maslov-type index theory for brake orbits and proved

Theorem 4.1 ([57]). *For any $\Sigma\in\mathcal{H}_b^{s,c}(2n)$, there holds*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma)\geq 2.$$

Liu and Zhang established the iteration theory for i_{L_0} index in [47] and proved that $^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq \left[\frac{n}{2}\right]+1$ for $\Sigma \in \mathcal{H}_b^{s,c}(2n)$. Moreover it was proved that if all brake orbits on Σ are nondegenerate, then $^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq n+\mathfrak{A}(\Sigma)$, where $2\mathfrak{A}(\Sigma)$ is the number of geometrically distinct asymmetric brake orbits on Σ. In [73] of 2014, Liu and Zhang improved the above result to that $^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq \left[\frac{n+1}{2}\right]+1$ for $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ with $n \geq 3$. In this case, the Seifert conjecture is true for $n \leq 3$. In [74] of 2013 Liu and Zhang proved that $^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq \left[\frac{n+1}{2}\right]+2$ for $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ with $n \geq 4$. In this case, the Seifert conjecture is true for $n \leq 5$.

For any integer n, the following result was proved by Liu and Zhang in 2014.

Theorem 4.2 ([48]). *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, there holds*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq n.$$

In order to show the role of iteration theory of i_L index, we give a sketch of the proof of Theorem 4.2 below by applying Theorem 3.4 and the estimate in Theorem 2.3. More details can be found in [48].

For any x_τ being a τ-periodic brake orbit solution, let γ_{x_τ} be the symplectic path associated to x_τ. We define

$$i_{L_0}(x_\tau)=i_{L_0}(\gamma_{x_\tau}|_{[0,\frac{\tau}{2}]}), \qquad \nu_{L_0}(x_\tau)=\nu_{L_0}\Big(\gamma_{x_\tau}\Big(\frac{\tau}{2}\Big)\Big).$$

Definition 4.2. *For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, a brake orbit (τ,x) on Σ is called symmetric if $x(\mathbf{R})=-x(\mathbf{R})$. Similarly, for a C^2 convex symmetric bounded domain $\Omega \subset \mathbf{R}^n$, a brake orbit $(\tau,q)\in \mathcal{O}(\Omega,V)$ is called symmetric if $q(\mathbf{R})=-q(\mathbf{R})$.*

A sketch of the proof of Theorem 4.2. Suppose that there are p symmetric and $2q$ asymmetric geometrically distinct brake orbits on Σ. Denote them by $\{(\tau_j,x_j)|j=1,2,\cdots,p\}$ and $\{(\tau_k,x_k),(\tau_k,-x_k)k=p+1,p+2,\cdots,p+q\}$ respectively, where τ_j is the minimal period of x_j for $j=1,2,\cdots,p+q$. Then the proof is completed in three steps.

Step 1 Applying Theorem 3.4 to the associated symplectic paths of

$$(\tau_1,x_1),(\tau_2,x_2),\cdots,(\tau_{p+q},x_{p+q}),(2\tau_{p+1},x_{p+1}^2),\cdots,(2\tau_{p+q},x_{p+q}^2)$$

we obtain a sufficient large integer R and the iteration times $m_1,m_2,\cdots,m_{p+q},m_{p+q},m_{p+q+1},\cdots,m_{p+2q}$ satisfying (i)-(vi) of Theorem 3.4.

By the proofs in [57] of Long, Zhang and Zhu in 2006 and the fact that x and $-x$ have the same i_{L_0} and ν_{L_0} index, we can define a map ϕ with

$$\phi(R-s+1)=((\tau_{k(s)},x_{k(s)}),m(s)) \quad \text{for } s=1,2,\cdots,n,$$

such that $1\leq k(s)\leq p+q$ and

$$i_{L_0}(x_{k(s)},m(s)) \leq R-s \leq i_{L_0}(x_{k(s)},m(s))+\nu_{L_0}(x_{k(s)},m(s))-1, \tag{4.8a}$$

$$m(j)>m(l), \quad \forall j<l \quad \text{with } k(j)=k(l). \tag{4.8b}$$

Step 2 Let

$$S_1 = \{s \in \{1,2,\cdots,n\} | \, k(s) \le p\}, \quad S_2 = \{1,2,\cdots,n\} \setminus S_1.$$

We shall prove that

$$m(s) = 2m_{k(s)}, \quad s \in S_1. \tag{4.9}$$

In fact, by the strict convexity of Σ, we have $i_{L_0}(x_{k(s)}) \ge 0$. Hence by (4.8a) and (iii) of Theorem 3.4, for every $s = 1,2,\cdots,n$, there holds

$$i_{L_0}(x_{k(s)}, m(s)) < R \le i_{L_0}(x_{k(s)}, 2m_{k(s)} + 1). \tag{4.10}$$

Note that γ_{x_k} satisfies conditions of Theorem 2.3 with $\tau = \frac{\tau_k}{2}$. So by Theorem 2.3, we have

$$i_{L_1}(x_k) + S^+_{M_k}(1) - \nu_{L_0}(x_k) \ge 0, \quad \forall k = 1,\cdots,p. \tag{4.11}$$

(4.11) and (ii) of Theorem 3.4 would yield

$$\begin{aligned} & i_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) \\ < & i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)). \end{aligned} \tag{4.12}$$

Note that the convex assumption of the hypersurface Σ implies that the $i_{L_0}(x,j)$ and $i_{L_0}(x,j) + \nu_{L_0}(x,j)$ are both strictly increasing in the iteration time j for any brake orbit x on Σ.

Hence by (4.10), (4.12), we obtain

$$2m_{k(s)} - 1 < m(s) < 2m_{k(s)} + 1.$$

Hence (4.9) holds.

Step 3 (4.8b) and (4.9) imply ${}^\# S_1 \le p$. By similar arguments, we obtain

$$2m_{k(s)} - 2 < m(s) < 2m_{k(s)} + 1,$$

which yields ${}^\# S_2 \le 2q$. So we have

$$p + 2q \ge^\# S_1 +^\# S_2 = n.$$

The proof of Theorem 4.2 is completed. □

Remark 4.2. Theorem 4.2 is a kind of multiplicity result related to the Arnold chord conjecture. The Arnold chord conjecture is an existence result which was proved by Mohnke in [60] of 2001. Another kind of multiplicity result related to the Arnold chord conjecture was proved by Guo and Liu in [30] of 2008.

A typical example of $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ is the ellipsoid $\mathcal{E}_n(r)$ defined as follows. Let $r = (r_1, \cdots, r_n)$ with $r_j > 0$ for $1 \le j \le n$. Define

$$\mathcal{E}_n(r) = \left\{ x = (x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbf{R}^{2n} \;\middle|\; \sum_{k=1}^{n} \frac{x_k^2 + y_k^2}{r_k^2} = 1 \right\}.$$

If $r_j/r_k \in \mathbf{R} \setminus \mathbf{Q}$ whenever $j \neq k$, one can easily see that there are precisely n geometrically distinct symmetric brake orbits on $\mathcal{E}_n(r)$ and all of them are nondegenerate.

The following two important results are direct consequences of Theorem 4.2.

Corollary 4.1 ([48]). *If $H(p,q)$ defined by* (4.1) *is even and convex, then Seifert conjecture holds.*

Corollary 4.2 ([48]). *Suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$ and $V''(q)$ is positive definite for all $q \in \mathbf{R}^n \setminus \{0\}$. Then for any given $h > 0$ and $\Omega \equiv \{q \in \mathbf{R}^n | V(q) < h\}$, there holds*

$$^{\#}\tilde{\mathcal{O}}(\Omega) \geq n.$$

In 2015, by non-smooth Lyusternik-Schnirelmann theory, Giambò, Giannoni and Piccione proved the following result which means Seifert conjecture holds in the case $n = 2$ while it is still open for $n \geq 3$.

Theorem 4.3 ([26]). *Under Seifert's condition with A and V weakened to C^2, for $n \geq 2$, there holds*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq 2.$$

We call a compact star-shaped reversible hypersurface $\Sigma \in \mathbf{R}^{2n}$ being dynamically convex, if for any brake orbit (τ, x) on Σ, there holds

$$i_{L_0}(x) \geq 0, \quad i_{L_1}(x) \geq 0.$$

We call Σ is L_0-nondegenerate if every brake orbits on it is nondegenerate.

In a preprint of 2018, using the equivariant wrapped Floer homology theory and the Common Index Jump Theorem 3.4, Kim, Kim and Kwon proved,

Theorem 4.4 ([38]). *For any L_0-nondegenerate compact dynamically convex reversible hypersurface $\Sigma \in \mathbf{R}^{2n}$, there holds*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq n.$$

The conclusion of Theorem 4.2 can be extended to dynamically convex case.

Theorem 4.5 ([49] of Z. Liu and Zhang). *For any compact symmetric dynamically convex reversible hypersurface $\Sigma \in \mathbf{R}^{2n}$, there holds*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq n.$$

In 2006, Zhang considered the multiplicity of symmetric brake orbits and proved,

Theorem 4.6 ([71]). *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, there exist at least two geometrically symmetric brake orbits on Σ.*

For stability of brake orbits, there are only few results, since information obtained from variational methods is not easy to be used. In 2017, Fan and Zhang proved,

Theorem 4.7 ([19]). *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, if there are exactly n geometrically distinct brake orbits on Σ, then there are at least $n-2$ among them possessing irrational mean i_{L_0} indices.*

4.5 Related results

It is interesting to ask the following question: whether all closed characteristics on any hypersurfaces $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ are symmetric brake orbits after suitable time translation provided ${}^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$? In this direction, the following result was proved in [48] by Liu and Zhang.

Theorem 4.8 ([48]). *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, suppose*

$$
{}^{\#}\tilde{\mathcal{J}}(\Sigma) = n.
$$

Then all of the n closed characteristics on Σ are symmetric brake orbits after suitable time translation.

Note that for the hypersurface

$$
\Sigma = \Big\{ (x_1, x_2, y_1, y_2) \in \mathbf{R}^4 \Big| \, x_1^2 + y_1^2 + \frac{x_2^2 + y_2^2}{4} = 1 \Big\},
$$

we have

$$
{}^{\#}\tilde{\mathcal{J}}_b(\Sigma) = +\infty \quad \text{and} \quad {}^{\#}\tilde{\mathcal{J}}_b^s(\Sigma) = 2,
$$

where we have denoted by $\tilde{\mathcal{J}}_b^s(\Sigma)$ the set of all symmetric brake orbits on Σ. We also note that on the hypersurface $\Sigma = \{x \in \mathbf{R}^{2n} | \, |x| = 1\}$ there are some non-brake closed characteristics.

For $n = 2$, it was proved in [33] of Hofer, Wysocki and Zehnder that ${}^{\#}\tilde{\mathcal{J}}(\Sigma)$ is either 2 or $+\infty$ for any C^2 compact convex hypersurface Σ in $\mathbf{R}^4$. In the brake orbit case, in [23] of 2016, Frauenfelder and Kang proved that ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma)$ is either 2 or $+\infty$ for any C^2 compact dynamically convex reversible hypersurface Σ in $\mathbf{R}^4$. So it is natural to propose the following conjecture:

Conjeture: ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma)$ is either n or $+\infty$ for any C^2 compact convex reversible hypersurface Σ in $\mathbf{R}^{2n}$.

The multiplicity of closed characteristics on a fixed energy hypersurface in $\mathbf{R}^{2n}$ is a very important problem in nonlinear Hamiltonian systems. Similar to Seifert conjecture, there is a long standing (more than 100 years) conjecture that on every compact convex hypersurface in $\mathbf{R}^{2n}$, there exist at least n closed characteristics. An important progress was made by Long and Zhu in [59] of 2002, by using the iteration theory of Maslov-type index of symplectic paths and establishing the common index jump theorem. They proved that

$$^{\#}\tilde{\mathcal{J}}(\Sigma) \geq \left[\frac{n}{2}\right] + 1.$$

Then in 2002, Liu, Long and Zhu in [45] proved that

$$^{\#}\tilde{\mathcal{J}}(\Sigma) \geq n$$

provided Σ is a convex compact symmetric hypersurface in $\mathbf{R}^{2n}$.

In [68] of 2007, Wang, Hu and Long proved that the conjecture holds in the case $n = 3$. In [66] of 2016, Wang proved that $^{\#}\tilde{\mathcal{J}}(\Sigma) \geq \left[\frac{n+1}{2}\right] + 1$ when $n \geq 2$. In [67] of 2016, Wang proved that the conjecture holds in the case $n = 4$.

It is also interesting to ask the following question: Can we prove the above conjecture on closed characteristics if the hypersurface is additionally reversible?

5 Applications, minimal period solution problems in the brake orbit case

For the nonlinear Hamiltonian system:

$$\dot{x} = JH'(x), \quad x \in \mathbf{R}^{2n}, \tag{5.1}$$

in his pioneering paper [61] of 1978, Rabinowitz proved the following famous result via a variational method. Suppose H satisfies the following conditions:

(H1′) $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$.

(H2) There exist constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu H(x) \leq H'(x) \cdot x, \quad \forall |x| \geq r_0.$$

(H3) $H(x) = o(|x|^2)$ at $x = 0$.

(H4) $H(x) \geq 0$ for all $x \in \mathbf{R}^{2n}$.

Then for any $T > 0$, the system (5.1) possesses a non-constant T-periodic solution.

Because a T/k periodic function is also a T-periodic function, in [61] Rabinowitz proposed the conjecture that under conditions (H1′) and (H2)-(H4), there is a periodic solution of (5.1) with T as its minimal period for any $T > 0$. Since 1978, this conjecture has been deeply studied by many mathematicians. A significant progress was made by Ekeland and Hofer in their celebrated paper [18] of 1985, where they confirmed Rabinowitz's conjecture for the strictly convex Hamiltonian systems. For Hamiltonian systems with convex or weakly convex assumptions, we refer to [2,3,12,17,44,56] and references therein for more details. For the seconder order case without any convex condition we refer to [50–52], and Chapter 13 of [56] as well as references therein. In [12] of 1997, Dong and Long gave the first index theoretical proof of the Ekeland-Hofer theorem, and discovered the deep connection between the minimal period and the index theory.

It is natural to consider the minimal period solution problem of brake orbits in reversible first order nonlinear Hamiltonian systems. The key tool is the iteration theory of i_L index. Motivated by [12, 20, 21, 50, 51] we consider the case that the systems are semipositive, i.e., the Hessian $H''(x)$ has no negative eigenvalues at any $x \in \mathbf{R}^{2n}$, which guarantees the existence of a lower bound of i_L indices of brake orbits of the system.

It is natural to suppose H satisfies the following reversible and semipositive conditions:

(H5) $H(Nx) = H(x)$ for all $x \in \mathbf{R}^{2n}$.

(H6) $H''(x)$ is semipositive definite for all $x \in \mathbf{R}^{2n}$.

We also suppose the following smooth condition:

(H1) $H \in C^2(\mathbf{R}^{2n}, \mathbf{R})$.

In order to introduce our results more conveniently, we define the following (**B1**) condition. Since the Hamiltonian systems considered here are reversible, this condition is natural.

(B1) Condition. For any $\tau > 0$ and $B \in C([0,\tau], \mathcal{L}_s(\mathbf{R}^{2n})$ of the $n \times n$ matrix square block form

$$B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$$

satisfying $B_{12}(0) = B_{21}(0) = 0 = B_{12}(\tau) = B_{21}(\tau)$, we call B satisfying the condition **(B1)**.

For any $B \in C([0,\tau], \mathcal{L}_s(\mathbf{R}^{2n}))$, denote by γ_B the fundamental solution of the following problem

$$\begin{aligned} &\dot{\gamma}_B(t) = JB(t)\gamma_B(t), \\ &\gamma_B(0) = I_{2n}. \end{aligned}$$

Then $\gamma_B \in \mathcal{P}_\tau(2n)$. We call γ_B the symplectic path associated to B.

If H is even and x_τ is a τ-periodic symmetric brake orbit solution of (5.1), let $B(t) = H''(x(t))$, we define $\gamma_{x_\tau} = \gamma_B|_{[0,\frac{\tau}{4}]}$ and call it the symplectic path associated to x_τ. We define

$$i^{L_0}_{\sqrt{-1}}(x_\tau) = i^{L_0}_{\sqrt{-1}}(\gamma_{x_\tau}), \quad \nu^{L_0}_{\sqrt{-1}}(x_\tau) = i^{L_0}_{\sqrt{-1}}(\gamma_{x_\tau}).$$

Theorem 5.1 ([42] of Liu in 2010)**.** *Suppose that H satisfies conditions* (H1)-(H6)*. Then for any $\tau > 0$, the system* (5.1) *possesses a τ periodic brake orbit x_τ. Furthermore if x_τ satisfies*

(H6′) $\displaystyle\int_0^{\frac{\tau}{2}} H''(x_\tau(t))\, dt > 0.$

Then the minimal period of x_τ belongs to $\{\tau, \frac{\tau}{2}\}$.

Theorem 5.2 ([72] of Zhang in 2015)**.** *Suppose that H satisfies conditions* (H1)-(H6)*. Then for any $\tau > 0$, the system* (5.1) *possesses a τ periodic brake orbit x_τ with minimal period not less than $\frac{\tau}{2n+2}$. Moreover, for $x = (x_1, x_2)$ with $x_1, x_2 \in \mathbf{R}^n$, denote by $H''_{22}(x)$ the second order differential of H with respect to x_2, if*

$$\int_0^{\frac{\tau}{2}} H''_{22}(x_\tau(t))\, dt > 0, \tag{5.2}$$

then the minimal period of x_τ belongs to $\{\tau, \frac{\tau}{2}\}$.

For readers' convenience we give the idea of Theorem 5.2 below as an example to show how to estimate the minimal period by the index iteration theory, more details can be found in [72] of Zhang in 2015.

Idea of the proof of Theorem 5.2. For any $\tau > 0$, by the Galerkin approximation and the variational argument, we obtain a τ-periodic brake orbit solution x_τ of (5.1) such that $i_{L_0}(x_\tau) \le 1$. Suppose its minimal period is T, then $\tau/T \equiv k$ is a positive integer and $x_\tau = x_T^k$ with $x_T = x_\tau|_{[0,T]}$. In the variational argument we also have $i_{L_1}(x_T) + \nu_{L_1}(x_T) \ge 1$. By the semipositive assumption we have

$$i^{L_1}_{\sqrt{-1}}(\gamma_{x_T}) \ge 0 \tag{5.3}$$

and $i_{L_0}(\gamma_{x_T}) + \nu_{L_0}(\gamma_{x_T}) \ge 0$ which implies

$$i_{L_0}(\gamma_{x_T}) \ge -n, \tag{5.4}$$

where γ_{x_T} is the associated symplectic path of the brake orbit x_T. Hence $i_1(\gamma^2_{x_T}) + \nu_1(\gamma^2_{x_T}) \ge n+1$. Thus by the above Theorem 3.1 and Proposition 4.1 of [44] of Liu and Long in 2000 Theorem 10.1.1 of [56] of Long in 2002, we have

$$1 \ge i_{L_0}(x_\tau) = i_{L_0}(\gamma^k_{x_T}) \ge \begin{cases} i_{L_0}(\gamma_{x_T}) + (k-1)/2, & k \text{ odd}, \\ i_{L_0}(\gamma_{x_T}) + i^{L_1}_{\sqrt{-1}}(\gamma_{x_T}) + k/2 - 1, & k \text{ even}. \end{cases} \tag{5.5}$$

By (5.3)-(5.5) we have $k \le 2n+4$. By Theorem 2.2 and the estimate of Hörmander index we have $k \ne 2n+3,\ 2n+4$. So we have $k \le 2n+2$ which means that the minimal period of x_τ is not less than $\frac{\tau}{2n+2}$.

Moreover, if (5.2) holds, then we have $i_{L_0}(\gamma_{x_T}) \ge 0$, by the same argument as above we have $k \le 2$ which means that the minimal period of $x_\tau$belongs to $\{\tau, \frac{\tau}{2}\}$. □

Next we consider the minimal period problem for

$$H(x) = \frac{1}{2} B_0 x \cdot x + \hat{H}(x),$$

where $B_0 \in \mathcal{L}_s(\mathbf{R}^{2n})$.

Theorem 5.3 ([72]). *Let* $B_0 = \mathrm{diag}(B_{11}, B_{22})$ *be a* $2n\times 2n$ *real semipositive matrix with* B_{11} *and* B_{22} *being* $n\times n$ *matrices. Assume*

$$H(x) = \frac{1}{2}B_0x\cdot x + \hat{H}(x) \quad \textit{for all} \quad x\in \mathbf{R}^{2n},$$

and $\hat{H}$ *satisfies conditions* (H1)-(H6).

Then for any $\tau > 0$, *the system* (5.1) *possesses a* τ*-periodic brake orbit* x_τ *with its minimal period not less than* $\frac{\tau}{2i_{L_0}(B_0)+2\nu_{L_0}(B_0)+2n+2}$, *where* B_0 *is viewed as an element in* $C([0,\tau/2],\mathcal{L}_s(\mathbf{R}^{2n}))$ *which satisfies the condition (B1).*

Remark 5.1. In [72], it was proved that if B_0 is semipositive, then $i_{L_0}(B_0)+\nu_{L_0}(B_0)\geq 0$.

Theorem 5.4 ([72]). *Suppose that* H *satisfies the conditions* (H1)-(H6) *and*
(H7) $H(-x) = H(x)$ *for all* $x\in\mathbf{R}^{2n}$.

Then for any $\tau > 0$, *the system* (5.1) *possesses a symmetric brake orbit with minimal period belonging to* $\{\tau, \tau/3\}$.

Theorem 5.5 ([72]). *Let* $B_0 = \mathrm{diag}(B_{11}, B_{22})$ *be a* $2n\times 2n$ *real semipositive matrix with* B_{11} *and* B_{22} *being* $n\times n$ *matrix. Assume* $H(x) = \frac{1}{2}B_0x\cdot x+\hat{H}(x)$ *for all* $x\in\mathbf{R}^{2n}$, *and* $\hat{H}$ *satisfies the conditions* (H1)-(H7). *Then for any* $\tau>0$, *the system* (5.1) *possesses a symmetric brake orbit* x_τ *with minimal period not less than* $\frac{\tau}{4(i^{L_0}_{\sqrt{-1}}(B_0)+\nu^{L_0}_{\sqrt{-1}}(B_0))+7}$. *Moreover, if* $i^{L_0}_{\sqrt{-1}}(B_0)+\nu^{L_0}_{\sqrt{-1}}(B_0)$ *is even, then the minimal period of* x_τ *is not less than* $\frac{\tau}{4(i^{L_0}_{\sqrt{-1}}(B_0)+\nu^{L_0}_{\sqrt{-1}}(B_0))+3}$, *where* B_0 *is viewed as an element in* $C([0,\tau/4],\mathcal{L}_s(\mathbf{R}^{2n}))$ *satisfying the condition* (B1).

Remark 5.2. In [72], it was proved that, if B_0 is semipositive, then $i^{L_0}_{\sqrt{-1}}(B_0)\geq 0$, hence

$$i^{L_0}_{\sqrt{-1}}(B_0)+\nu^{L_0}_{\sqrt{-1}}(B_0)\geq 0.$$

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