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Deformation Argument under PSP Condition and Applications

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper we introduce a new deformation argument, in which C^0-group action and a new type of Palais-Smale condition PSP play important roles. This type of deformation results are studied in [17, 21] and has many different applications [10, 11, 17, 21] et al. Typically it can be applied to nonlinear scalar field equations. We give a survey in an abstract functional setting. We also present another application to nonlinear elliptic problems in strip-like domains. Under conditions related to [5,6], we show the existence of infinitely many solutions. This extends the results in [8].

Key Words: Deformation theory, nonlinear elliptic equations, radially symmetric solutions, strip-like domains, Pohozaev functional.
AMS Subject Classifications: 35B38, 35J20, 58E05

1 Introduction

We study nonlinear differential equations with scaling properties via variational methods. A typical example is the following nonlinear scalar field equations:

$$-\Delta u = g(u) \quad \text{in } \mathbf{R}^N, \tag{1.1a}$$
$$u \in H^1(\mathbf{R}^N), \tag{1.1b}$$

where $N \geq 2$ and we consider the existence of radially symmetric solutions. This type of problem appears in many models in mathematical physics and is well-studied by many authors. Especially Berestycki and Lions [5, 6] and Berestycki, Gallouët and Kavian [7] obtained almost necessary and sufficient conditions for the existence of non-trivial solutions. More precisely they consider (1.1) under the following conditions

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(g0) $g(\xi)\in C(\mathbf{R},\mathbf{R})$ and $g(\xi)$ is odd.

(g1) For $N\geq 3$,

$$\limsup_{\xi\to\infty}\frac{g(\xi)}{\xi^{(N+2)/(N-2)}}\leq 0.$$

For $N=2$,

$$\limsup_{\xi\to\infty}\frac{g(\xi)}{e^{\alpha\xi^2}}\leq 0 \quad \text{for any} \quad \alpha>0.$$

(g2) $-\infty<\liminf_{\xi\to 0}\dfrac{g(\xi)}{\xi}\leq\limsup_{\xi\to 0}\dfrac{g(\xi)}{\xi}<0.$

(g3) There exists a $\zeta_0>0$ such that $G(\zeta_0)>0$, where $G(\xi)=\int_0^\xi g(\tau)\,d\tau$.

The [5,6] (for $N\geq 3$) and [7] (for $N=2$) showed the existence of a positive solution of (1.1) and infinitely many possibly sign-changing radially symmetric solutions. We note that in [5–7] solutions are found as critical points of constraint functional

$$u\mapsto\int_{\mathbf{R}^N}|\nabla u|^2\,dx;\ \Big\{u\in H_r^1(\mathbf{R}^N);\ \int_{\mathbf{R}^N}G(u)\,dx=1\Big\}\to\mathbf{R} \qquad \text{for } N\geq 3, \tag{1.2a}$$

$$u\mapsto\int_{\mathbf{R}^2}|\nabla u|^2\,dx;\ \Big\{u\in H_r^1(\mathbf{R}^2);\ \int_{\mathbf{R}^2}G(u)\,dx=0\Big\}\to\mathbf{R} \qquad \text{for } N=2, \tag{1.2b}$$

after a suitable scaling and solutions satisfy Pohozaev identity. See Coleman, Glazer and Martin [12] for related argument. We also note that a positive solution is obtained as a minimizer after scaling and it is a least energy solution.

Remark 1.1. When $N=2$, in [7] the existence of solution is obtained under slightly stronger conditions (g0), (g1), (g3) and

(g2′) $\lim_{\xi\to 0}\dfrac{g(\xi)}{\xi}<0$ exists.

In [16], Hirata, Ikoma and the second author introduced a new approach to (1.1), in which we try to apply minimax argument to the natural functional associated to (1.1):

$$I(u)=\frac{1}{2}\int_{\mathbf{R}^N}|\nabla u|^2\,dx-\int_{\mathbf{R}^N}G(u)\,dx:\ H_r^1(\mathbf{R}^N)\to\mathbf{R}. \tag{1.3}$$

We note that it is difficult to verify so-called Palais-Smale condition ((PS) in short) for $I(u)$ and we cannot apply the standard deformation argument directly to $I(u)$. We also remark that the constraint functional (1.2a) and (1.2b) satisfy (PS) condition.

To avoid lack of (PS) condition, we make use of a special scaling property of the functional $I(u)$ and we introduce following Pohozaev functional:

$$P(u)=\frac{N-2}{2}\int_{\mathbf{R}^N}|\nabla u|^2\,dx-N\int_{\mathbf{R}^N}G(u)\,dx:\ H_r^1(\mathbf{R}^N)\to\mathbf{R}.$$

Using the scaling $u_\lambda(x)=u(x/\lambda)$, we have formally

$$\begin{aligned} P(u)=&-I'(u)(x\cdot\nabla u)=I'(u)\left(\frac{d}{d\lambda}\Big|_{\lambda=1}u_\lambda\right)\\ =&\frac{d}{d\lambda}\Big|_{\lambda=1}I(u_\lambda). \end{aligned}$$

In [16], Hirata, Ikoma and the second author found a (PS) sequence with an extra property at minimax level $b=\inf_{\gamma\in\Gamma}\max_{\xi\in D}I(\gamma(\xi))$. That is, there exists a sequence $(u_j)_{j=1}^\infty\subset H_r^1(\mathbf{R}^N)$ such that as $j\to\infty$

$$I(u_j)\to b, \tag{1.4a}$$

$$I'(u_j)\to 0 \quad \text{in} \ \ (H_r^1(\mathbf{R}^N))^*, \tag{1.4b}$$

$$P(u_j)\to 0. \tag{1.4c}$$

For example, at a mountain pass level for $I(u)$, they find a sequence $(u_j)_{j=1}^\infty$ with (1.4a)–(1.4c) and under the condition (g0)–(g3) and moreover they show that $(u_j)_{j=1}^\infty$ has a strongly convergent subsequence whose limit is a solution of (1.1). See Remark 4.1 in Section 4.

The condition (1.4c) means $(u_j)_{j=1}^\infty$ satisfies Pohozaev identity $P(u)=0$ asymptotically and we call such sequence $(u_j)_{j=1}^\infty$ as (PSP) sequences.

Existence of such (PSP) sequences was firstly found by the second author. Jeanjean [22] used the second author's approach for L^2 normalized solutions for L^2 super critical problems and in [16], we studied nonlinear scalar field equation (1.1) through (PSP) sequence and showed the existence of positive radially symmetric solutions via mountain pass method.

Such a strategy turned out to be useful for various problems with suitable scaling properties. See [23] for an application for nonlinear Choquard equations, [2, 18–20] for fractional scalar field equations, [9] for FitzHugh-Nagumo elliptic systems, [8] for nonlinear elliptic equations in strip-like domains, [3] for nonlinear Schrödiger-Maxwell systems, [4] for nonlinear eigenvalue problems.

For even functionals, it is natural to ask the existence of infinitely many solutions. We note that our argument in [16] does not provide a deformation theory for $I(u)$ and we cannot apply genus theory directly to $I(u)$. So to find infinitely many solutions, we need to use some comparison argument to ensure the existence of unbounded sequence of minimax values. See [2, 9, 16].

In this paper we give a survey of some deformation theorems contained in previous papers [10, 11, 17, 21] and a new application to semilinear elliptic equations in strip-like domains. Our deformation result works for $I(u)$ under (g0)–(g3) and enables us to apply genus theory directly to $I(u)$. It also shows that critical points with Pohozaev identity are essential in the deformation argument (see Corollary 3.1 and Remark 3.1 in Section 3). A special scaling property and a new type of Palais-Smale condition, which we call (PSP)

condition and which claims any (PSP) sequence has a strongly convergent subsequence, play important roles in our argument. We give our deformation result in a general setting in Sections 2–3. We also give an existence result for (PSP) sequence at a minimax level in Section 4.

We note that such a deformation argument is firstly given in [17] for L^2 normalized problem for nonlinear scalar field equations. We also refer to [21] for L^2 super critical problems and [10, 11] for our recent works on L^2 normalized solutions for nonlinear Choquard equations and fractional nonlinear scalar field equations.

In Section 5, we give a new application to a nonlinear elliptic problem in a strip-like domain:

$$
\begin{aligned}
&-\Delta u = f(u) \quad \text{in } \mathbf{R}^k \times D,\\
&u \in H^1_0(\mathbf{R}^k \times D),
\end{aligned}
$$

where $k \geq 2$. Under conditions (f0)–(f3) in Section 5, which are related to (g0)–(g3), we show the existence of infinitely many solutions.

2 Deformation argument under (PSP)

We give our deformation argument in an abstract framework. Let $(E, \|\cdot\|_E)$ be a Hilbert space and $\Phi: \mathbf{R} \to L(E); \theta \mapsto \Phi_\theta$ be a continuous group action of $\mathbf{R}$. For $I \in C^1(E, \mathbf{R})$ we assume

Assumption 2.1.

(i) Φ_θ *is a* C^0*-group action, that is,*

$$
\begin{aligned}
&\Phi_0 = id,\\
&\Phi_{\theta+\theta'} = \Phi_\theta \circ \Phi_{\theta'} \quad \textit{for all } \theta, \theta' \in \mathbf{R},\\
&\theta \mapsto \Phi_\theta u;\ \mathbf{R} \to E \textit{ is continuous for all } u \in E.
\end{aligned}
$$

(ii) Let $M = \mathbf{R} \times E$ *and we regard* M *as a Hilbert manifold and we introduce a metric by*

$$
\|(\kappa, v)\|_{(\theta,u)} = \big(|\kappa|^2 + \|\Phi_\theta u\|_E^2\big)^{1/2}
$$

for all $(\kappa, v) \in \mathbf{R} \times E = T_{(\theta,u)}M$ *and* $(\theta, u) \in M$*. We assume* $\|\cdot\|_{(\theta,u)}$ *is a metric of class* C^2.

(iii) Let

$$
J(\theta, u) = I(\Phi_\theta u): M \to \mathbf{R}.
$$

Then we assume that $J(\theta, u) \in C^1(M, \mathbf{R})$.

Under the Assumption 2.1, we introduce

$$P(u)=\frac{\partial J}{\partial\theta}(0,u):\ E\to\mathbf{R}.$$

For $b\in\mathbf{R}$ we request the following Palais-Smale type condition $(PSP)_b$ for $I(u)$.

$(PSP)_b$ Assume that $(u_j)_{j=1}^\infty\subset E$ satisfies

$$I(u_j)\to b, \tag{2.1a}$$

$$I'(u_j)\to 0 \quad \text{in } \ E^*, \tag{2.1b}$$

$$P(u_j)\to 0. \tag{2.1c}$$

Then $(u_j)_{j=1}^\infty$ has a strongly convergent subsequence.

Under the above assumptions, we have the following deformation result in which we use notation:

$$\begin{aligned}&K_b=\{u\in E;\, I(u)=b,\ I'(u)=0,\ P(u)=0\},\\ &[I\le c]=\{u\in E;\, I(u)\le c\} \quad \text{for } \ c\in\mathbf{R}.\end{aligned}$$

We note that K_b is different from the usual critical set at level b and it requests $P(u)=0$.

Theorem 2.1. *Assumption* 2.1 *and for* $b\in\mathbf{R}$ $(PSP)_b$ *holds. Then*

(i) K_b *is compact in* E.

(ii) For any neighborhood O *of* K_b *and for any* $\bar\varepsilon>0$ *there exists* $\varepsilon\in(0,\bar\varepsilon)$ *and a continuous map* $\eta(t,u):\ [0,1]\times E\to E$ *such that*

(1) $\eta(0,u)=u$ *for all* $u\in E$.

(2) $\eta(t,u)=u$ *for all* $t\in[0,1]$ *and* $u\in[I\le b-\bar\varepsilon]$.

(3) $I(\eta(t,u))\le I(u)$ *for all* $(t,u)\in\mathbf{R}\times E$.

(4) $\eta(1,[I\le b+\varepsilon]\setminus O)\subset[I\le b-\varepsilon]$, $\eta(1,[I\le b+\varepsilon])\subset[I\le b-\varepsilon]\cup O$.

(5) If $K_b=\emptyset$, *then* $\eta(1,[I\le b+\varepsilon])\subset[I\le b-\varepsilon]$.

(6) If $I(u)$ *is an even functional, then* $\eta(t,-u)=-\eta(t,u)$ *for all* $(t,u)\in\mathbf{R}\times E$.

A typical situation, where the Assumption 2.1 is satisfied, is given in the following example.

Example 2.1. Let $E=H^1_r(\mathbf{R}^N)$ and

$$\|u\|_E=\Big(\int_{\mathbf{R}^N}|\nabla u|^2+|u|^2\,dx\Big)^{1/2}.$$

We consider the following action $\Phi:\ \mathbf{R}\to L(H^1_r(\mathbf{R}^N))$ defined by

$$(\Phi_\theta u)(x)=u(x/e^\theta).$$

Then we have

$$\begin{aligned}&\|(\kappa,v)\|^2_{(\theta,u)}=|\kappa|^2+\|u(x/e^\theta)\|^2_{H^1}\\ =&|\kappa|^2+e^{(N-2)\theta}\|\nabla u\|^2_{L^2}+e^{N\theta}\|u\|^2_{L^2}.\end{aligned}$$

For $I(u)\in C^1(E,\mathbf{R})$ given in (1.3), we have

$$\begin{aligned}&J(\theta,u)=I(u(x/e^\theta))\\ =&\frac{1}{2}e^{(N-2)\theta}\|\nabla u\|^2_{L^2}-e^{N\theta}\int_{\mathbf{R}^N}G(u)\,dx\in C^1(\mathbf{R}\times E,\mathbf{R}).\end{aligned}$$

We note that $\theta\mapsto\Phi_\theta u$ is not of class C^1 in general but $J(\theta,u)$ is of class C^1. That is, E, Φ_θ satisfy the Assumption 2.1. We also note that

$$P(u)=\partial_\theta J(0,u)=\frac{N-2}{2}\int_{\mathbf{R}^N}|\nabla u|^2\,dx-N\int_{\mathbf{R}^N}G(u)\,dx.$$

We also note that under (g0)–(g3), $I(u)$ satisfies $(PSP)_b$ condition for all $b\in\mathbf{R}$ (see Proposition 6.1 in [17]) and we can find infinitely many solutions via symmetric mountain pass theorem.

The statement (i) of Theorem 2.1 is a direct consequence from $(PSP)_b$. In Section 3, we give an outline of the proof of the statement (ii).

Remark 2.1. Theorem 2.1 can be generalized to the setting, where a functional $I(u)$ is defined on a submanifold in E. See [21], where applications to L^2 normalized solutions are also given.

3 Outline of the proof of Theorem 2.1

Proof of Theorem 2.1 is given using a deformation flow for $J(\theta,u):\ M\to\mathbf{R}$. We begin with some notation. First we define the standard distance dist_M on M by

$$\begin{aligned}&\mathrm{dist}_M((\theta_0,u_0),(\theta_1,u_1))\\ =&\inf\Big\{\int_0^1\Big\|\frac{d\sigma}{dt}(t)\Big\|_{\sigma(t)}dt;\ \sigma(t)\in C^1([0,1],M),\ \sigma(i)=(\theta_i,u_i)\text{ for }i=0,1\Big\}.\end{aligned}$$

We have easily that

$$\mathrm{dist}_M((\theta_0+\alpha,u_0),(\theta_1+\alpha,u_1))=\mathrm{dist}_M((\theta_0,\Phi_\alpha u_0),(\theta_1,\Phi_\alpha u_1)).\tag{3.1}$$

For $F \in T^*_{(\theta,u)}M$, we define its norm by

$$\|F\|_{(\theta,u),*} = \sup\{F(\kappa,v);\ (\kappa,v) \in \mathbf{R}\times E,\ \|(\kappa,v)\|_{(\theta,u)} \le 1\}.$$

Writing $D = (\partial_\theta, \partial_u)$, we have

$$DJ(\theta,u)(\kappa,v) = \partial_\theta J(\theta,u)\kappa + \partial_u J(\theta,u)v.$$

By the definition of $J(\theta,u)$, we have for $(\theta,u) \in M$

$$J(\theta,u) = J(0,\Phi_\theta u) = I(\Phi_\theta u), \tag{3.2a}$$

$$J(\theta+\alpha,u) = J(\alpha,\Phi_\theta u) \quad \text{for } \alpha \in \mathbf{R}, \tag{3.2b}$$

$$\partial_\theta J(\theta,u) = \partial_\theta J(0,\Phi_\theta u) = P(\Phi_\theta u), \tag{3.2c}$$

$$\partial_u J(\theta,u)v = \partial_u J(0,\Phi_\theta u)\Phi_\theta v = I'(\Phi_\theta u)\Phi_\theta v. \tag{3.2d}$$

In particular,

$$\begin{aligned} &\|DJ(\theta,u)\|_{(\theta,u),*} = \|DJ(0,\Phi_\theta u)\|_{(0,\Phi_\theta u),*} \\ &= \left(|P(\Phi_\theta u)|^2 + \|I'(\Phi_\theta u)\|^2_{E^*}\right)^{1/2}. \end{aligned} \tag{3.3}$$

Finally for $b \in \mathbf{R}$, we set

$$\widetilde{K}_b = \{(\theta,u) \in M;\ J(\theta,u) = b,\ DJ(\theta,u) = 0\}.$$

We note that

$$\begin{aligned} &(\theta,u) \in \widetilde{K}_b \quad \text{if and only if } \Phi_\theta u \in K_b, \\ &\widetilde{K}_b = \{(\alpha,\Phi_{-\alpha}u);\ u \in K_b,\ \alpha \in \mathbf{R}\}, \\ &\mathrm{dist}_M((\theta,u),\widetilde{K}_b) = \mathrm{dist}_M((0,\Phi_\theta u),\widetilde{K}_b) \le \mathrm{dist}_E(\Phi_\theta u, K_b). \end{aligned}$$

From the above properties we have

Lemma 3.1. *Suppose Assumption* 2.1 *holds and for* $b \in \mathbf{R}$ *assume* $(PSP)_b$. *Then*
(i) Let $(\theta_j,u_j) \subset M$ *satisfies*

$$J(\theta_j,u_j) \to b, \quad \|DJ(\theta_j,u_j)\|_{(\theta_j,u_j),*} \to 0 \quad \text{as } j \to \infty.$$

Then $(\Phi_{\theta_j}u_j)_{j=1}^\infty$ *has a strongly convergent subsequence in E. Moreover we have*

$$\mathrm{dist}_M((\theta_j,u_j),\widetilde{K}_b) \to 0 \quad \text{as } j \to \infty.$$

(ii) Suppose $K_b \ne \emptyset$*, i.e.,* $\widetilde{K}_b \ne \emptyset$*. Then for any* $\rho > 0$ *there exists* $\delta_\rho > 0$ *such that*

$$\|DJ(\theta,u)\|_{(\theta,u),*} \ge \delta_\rho \quad \text{if } J(\theta,u) \in [b-\delta_\rho, b+\delta_\rho] \quad \text{and} \quad (\theta,u) \notin \widetilde{N}_\rho(\widetilde{K}_b).$$

Here

$$\widetilde{N}_\rho(\widetilde{K}_b) = \{(\theta,u);\ \mathrm{dist}_M((\theta,u),\widetilde{K}_b) < \rho\}.$$

(iii) If $K_b = \emptyset$, i.e., $\widetilde{K}_b = \emptyset$, there exists $\delta_0 > 0$ such that

$$\|DJ(\theta,u)\|_{(\theta,u),*} \geq \delta_0 \quad \textit{for} \ \ (\theta,u) \in M \quad \textit{with} \ \ J(\theta,u) \in [b-\delta_0, b+\delta_0]. \tag{3.4}$$

Proof. By (3.3) and $(PSP)_b$, (i) follows. (ii) and (iii) follow from (i) easily. □

We use notation

$$[J \leq c]_M = \{(\theta,u) \in M;\ J(\theta,u) \leq c\} \quad \text{for } \ c \in \mathbf{R}.$$

By (ii), (iii) of Lemma 3.1, for any $\rho > 0$ there exists $\delta'_\rho > 0$ such that

$$\|DJ(\theta,u)\|_{(\theta,u),*} \geq \delta'_\rho$$

for $(\theta,u) \in ([J \leq b+\delta'_\rho]_M \setminus [J \leq b-\delta'_\rho]_M) \setminus \widetilde{N}_{\frac{1}{3}\rho}(\widetilde{K}_b)$.

Taking a pseudo-gradient vector field

$$\mathcal{V}(\theta,u):\ ([J \leq b+\delta'_\rho]_M \setminus [J \leq b-\delta'_\rho]_M) \setminus \widetilde{N}_{\frac{1}{3}\rho}(\widetilde{K}_b) \to \mathbf{R} \times E$$

corresponding to DJ and choosing suitable cut-off functions $\varphi, \psi:\ E \to [0,1]$ such that

$$\varphi(\theta,u) = \begin{cases} 1 & \text{for } \ (\theta,u) \in M \setminus \widetilde{N}_{\frac{2}{3}\rho}(\widetilde{K}_b), \\ 0 & \text{for } \ (\theta,u) \in \widetilde{N}_{\frac{1}{3}\rho}(\widetilde{K}_b), \end{cases}$$

$$\psi(\theta,u) = \begin{cases} 1 & \text{if } \ J(\theta,u) \in \Big[b-\frac{1}{2}\delta'_\rho, b+\frac{1}{2}\delta'_\rho\Big], \\ 0 & \text{if } \ J(\theta,u) \notin [b-\delta'_\rho, b+\delta'_\rho], \end{cases}$$

we consider the following ODE in M:

$$\frac{d\widetilde{\eta}}{dt} = -\varphi(\widetilde{\eta})\psi(\widetilde{\eta})\frac{\mathcal{V}(\widetilde{\eta})}{\|\mathcal{V}(\widetilde{\eta})\|_{\widetilde{\eta}}},$$
$$\widetilde{\eta}(0,\theta,u) = (\theta,u).$$

In a standard way, we have the following

Proposition 3.1. *For any $\bar{\varepsilon} > 0$ and $\rho > 0$, there exist $\varepsilon \in (0,\bar{\varepsilon})$ and $\widetilde{\eta} \in C([0,1] \times M, M)$ such that*

(i) $\widetilde{\eta}(0,\theta,u) = (\theta,u)$ *for* $(\theta,u) \in M$.

(ii) $\widetilde{\eta}(t,\theta,u) = (\theta,u)$ *for* $t \in [0,1]$ *if* $(\theta,u) \in [J \leq b-\bar{\varepsilon}]_M$.

(iii) $t \mapsto J(\widetilde{\eta}(t,\theta,u))$ *is non-increasing for* $(\theta,u) \in M$.

(iv) $\widetilde{\eta}(1,[J \le b+\varepsilon]_M) \setminus \widetilde{N}_\rho(\widetilde{K}_b)) \subset [J \le b-\varepsilon]_M$, $\widetilde{\eta}(1,[J \le b+\varepsilon]_M) \subset [J \le b-\varepsilon]_M \cup \widetilde{N}_\rho(\widetilde{K}_b)$.

(v) *If* $K_b = \emptyset$, *i.e.,* $\widetilde{K}_b = \emptyset$, *we have* $\widetilde{\eta}(1,[J \le b+\varepsilon]_M) \subset [J \le b-\varepsilon]_M$.

(vi) *If* $I(u)$ *is even in* u, $\widetilde{\eta}(t,\theta,u) = (\widetilde{\eta}_1(t,\theta,u), \widetilde{\eta}_2(t,\theta,u))$ *satisfies*

$$\widetilde{\eta}_1(t,\theta,-u) = \widetilde{\eta}_1(t,\theta,u), \quad \widetilde{\eta}_2(t,\theta,-u) = -\widetilde{\eta}_2(t,\theta,u).$$

Our Theorem 2.1 can be derived from Proposition 3.1. We need the following operator:

$$\pi : \ M = \mathbf{R} \times E \to E;\ (\theta,u) \mapsto \Phi_\theta u.$$

We need the following lemma.

Lemma 3.2. *For any* $\rho > 0$ *there exists a* $R(\rho) > 0$ *such that*

$$\pi(\widetilde{N}_\rho(\widetilde{K}_b)) \subset N_{R(\rho)}(K_b), \tag{3.5a}$$

$$R(\rho) \to 0 \quad \text{as} \ \ \rho \to 0. \tag{3.5b}$$

Here

$$N_r(K_b) = \{u \in E;\ \mathrm{dist}_E(u,K_b) < r\}.$$

Proof. Suppose that $(\theta,u) \in \widetilde{N}_\rho(\widetilde{K}_b)$. By (3.1), note that

$$\mathrm{dist}_M((0,\Phi_\theta u),\widetilde{K}_b) = \mathrm{dist}_M((\theta,u),\widetilde{K}_b) < \rho$$

and choose a $\sigma(t) \in C^1([0,1],M)$ such that $\sigma(0) = (0,\Phi_\theta u)$, $\sigma(1) \in \widetilde{K}_b$,

$$\int_0^1 \left\| \frac{d\sigma}{dt}(t) \right\|_{\sigma(t)} dt < \rho.$$

Writing $\sigma(t) = (\sigma_1(t),\sigma_2(t))$, we have

$$|\sigma_1(t)| \le \int_0^1 \left| \frac{d\sigma_1}{dt}(t) \right| dt \le \int_0^1 \left\| \frac{d\sigma}{dt}(t) \right\|_{\sigma(t)} dt < \rho.$$

We note that there exists $c_\rho > 0$ such that for some $\delta \in (0,1]$

$$\begin{aligned} &c_\rho \|v\|_E \le \|\Phi_\theta v\|_E && \text{for } \ |\theta| \le \rho \quad \text{and} \quad v \in E, \\ &\delta \le c_\rho \le 1 && \text{for } \ \rho \in (0,1]. \end{aligned}$$

Thus

$$\begin{aligned}\mathrm{dist}_E(\Phi_\theta u,\sigma_2(1)) &\le \int_0^1 \Big\|\frac{d\sigma_2}{dt}(t)\Big\|_E\,dt \le c_\rho^{-1}\int_0^1 \Big\|\Phi_{\sigma_1(t)}\frac{d\sigma_2}{dt}(t)\Big\|_E\,dt\\ &\le c_\rho^{-1}\int_0^1 \Big\|\frac{d\sigma}{dt}(t)\Big\|_{\sigma(t)}\,dt \le c_\rho^{-1}\rho.\end{aligned}$$

Therefore, noting $\Phi_{\sigma_1(1)}\sigma_2(1)\in K_b$,

$$\begin{aligned}\mathrm{dist}_E(\pi(\theta,u),K_b) &= \mathrm{dist}_E(\Phi_\theta u,K_b) \le \mathrm{dist}_E(\Phi_\theta u,\Phi_{\sigma_1(1)}\sigma_2(1))\\ &\le \mathrm{dist}_E(\Phi_\theta u,\sigma_2(1)) + \mathrm{dist}_E(\sigma_2(1),\Phi_{\sigma_1(1)}\sigma_2(1))\\ &\le c_\rho^{-1}\rho + \sup\{\mathrm{dist}_E(w,\Phi_\alpha w);\,|\alpha|\le\rho,\ w\in K_b\}.\end{aligned}$$

Since K_b is compact by $(PSP)_b$, we have

$$R(\rho) = c_\rho^{-1}\rho + \sup\{\mathrm{dist}_E(w,\Phi_\alpha w);\,|\alpha|\le\rho,\ w\in K_b\}\to 0 \quad \text{as } \rho\to 0$$

and (3.5a) and (3.5b) hold. □

Proof of Theorem 2.1(ii). For a given neighborhood O of K_b, we choose $\rho>0$ so small that $N_{R(\rho)}(K_b)\subset O$. By Lemma 3.2, we have

$$\pi(\widetilde{N}_\rho(\widetilde{K}_b))\subset N_{R(\rho)}(K_b).$$

For any $\bar\varepsilon>0$, by Proposition 3.1 there exists $\varepsilon\in(0,\bar\varepsilon)$ and $\widetilde{\eta}\in C([0,1]\times M)$ with the properties stated in Proposition 3.1. Then we define

$$\eta(t,u)=\pi(\widetilde{\eta}(t,0,u)):\ [0,1]\times E\to E.$$

Then we can see that η has the desired properties. □

As a corollary to Theorem 2.1, we have

Corollary 3.1. *Suppose that Assumption* 2.1 *and* $(PSP)_b$ *hold. Moreover suppose* $K_b=\emptyset$. *Then there exists* $\varepsilon>0$ *such that* $[I\le b+\varepsilon]$ *is deformable into* $[I\le b-\varepsilon]$.

Remark 3.1. From Corollary 3.1, if $(PSP)_b$ holds for $b\in\mathbf{R}$ and $K_b=\emptyset$, then even if the standard critical set $\widehat{K}_b=\{u\in E;\ I(u)=b,\ I'(u)=0\}$ is not empty, $[I\le b+\varepsilon]$ is deformable into $[I\le b-\varepsilon]$. Thus, critical points without Pohozaev identity $P(u)=0$ do not affect topology of level sets of I.

4 Generation of (PSP) sequences at minimax level

Under the Assumption 2.1 (but without assuming (PSP) condition), we have the following existence result for (PSP) sequence at minimax levels. It can be regarded as a refinement of Ekeland's principle under Assumption 2.1.

For the sake of simplicity, we state the result for Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1].

Theorem 4.1. *Suppose that Assumption* 2.1 *holds and $I(u)$ has a mountain pass geometry. That is, for $e \in E$ with $e \neq 0$, set*

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) = e\},$$
$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

and assume

$$b > \max\{I(0),\ I(e)\}.$$

Then there exists a (PSP) sequence $(u_j)_{j=1}^\infty$ at level b, that is, $(u_j)_{j=1}^\infty$ satisfies (2.1a)–(2.1c).

Proof. We argue indirectly and suppose that there does not exists sequences $(u_j)_{j=1}^\infty$ with (2.1a)–(2.1c). In particular, $K_b = \emptyset$. Then there exists $\delta_0 > 0$ such that

$$c_0 \equiv \inf\{(|P(u)|^2 + \|I'(u)\|_{E^*}^2)^{1/2};\ I(u) \in [b - \delta_0, b + \delta_0]\} > 0.$$

By (3.2a), (3.3), we have

$$\|DJ(\theta,u)\|_{(\theta,u),*} \geq c_0, \quad \text{if } J(\theta,u) \in [b-\delta_0, b+\delta_0].$$

Thus (3.4) holds. Repeating the argument in Proposition 3.1 for any $\bar{\varepsilon} > 0$ with $I(0)$, $I(e) < b - \bar{\varepsilon}$ there exists $\varepsilon \in (0, \bar{\varepsilon})$ and $\widetilde{\eta} \in C([0,1] \times M, M)$ satisfying (i)–(iii) and

$$\widetilde{\eta}(1, [J \leq b + \varepsilon]_M) \subset [J \leq b - \varepsilon]_M.$$

Thus $\eta(t,u) = \pi(\widetilde{\eta}(t,0,u))$ satisfies (1)–(3) and (5) in Theorem 2.1(ii). Now take a path $\gamma \in \Gamma$ with $\max_{t\in[0,1]} I(\gamma(t)) < b + \varepsilon$ and consider $\widetilde{\gamma}(t) = \eta(1, \gamma(t)) \in \Gamma$. Then $\widetilde{\gamma}(t)$ satisfies $\max_{t\in[0,1]} I(\widetilde{\gamma}(t)) < b - \varepsilon$, which is a contradiction. □

Remark 4.1. In [16], at mountain pass level b for $I(u)$, we find a (PS) sequence $(\theta_j, u_j) \subset \mathbf{R} \times E$ for $J(\theta,u)$ such that as $j \to \infty$

$$J(\theta_j, u_j) \to b,$$
$$\partial_u J(\theta_j, u_j) \to 0 \quad \text{in } E^*,$$
$$\partial_\theta J(\theta_j, u_j) \to 0,$$
$$\theta_j \to 0.$$

This sequence is obtained by applying Ekeland's principle to $J(\theta,u)$ in [16].

Clearly in this setting, $(\Phi_{\theta_j} u_j)$ is a $(PSP)_b$ sequence for $I(u)$. See also [23] for generation of (PSP) sequences.

5 An application to nonlinear elliptic problems in strip-like domains

Our abstract results can be applied to many problems. For example, nonlinear scalar fields equations, nonlinear Choquard equations, etc.

Here we give an application to a nonlinear elliptic problems in a strip-like domain:

$$-\Delta u = f(u) \quad \text{in } \mathbf{R}^k \times D, \tag{5.1a}$$

$$u \in H_0^1(\mathbf{R}^k \times D), \tag{5.1b}$$

where $k \geq 2$ and $D \subset \mathbf{R}^\ell$ ($\ell \geq 1$) is a bounded open domain with a smooth boundary ∂D.

In what follows we write an element in $\mathbf{R}^k \times D$ by (x,y) ($x = (x_1, \cdots, x_k) \in \mathbf{R}^k$, $y = (y_1, \cdots, y_\ell) \in \mathbf{R}^\ell$) and $\nabla_x = (\partial_{x_1}, \cdots, \partial_{x_k})$, $\nabla_y = (\partial_{y_1}, \cdots, \partial_{y_\ell})$, $\nabla_{x,y} = (\nabla_x, \nabla_y)$.

We define

$$I(u) = \int_{\mathbf{R}^k \times D} \left(\frac{1}{2} |\nabla_{x,y} u|^2 - F(u) \right) dxdy : \ H_0^1(\mathbf{R}^k \times D) \to \mathbf{R},$$

where $F(\xi) = \int_0^\xi f(\tau)\, d\tau$. Solutions of (5.1) are characterized as critical points of $I(u)$.

Defining

$$\mathcal{G}(v) = \int_D -\frac{1}{2} |\nabla_y v|^2 + F(v)\, dy : \ H_0^1(D) \to \mathbf{R},$$

formally we have

$$I(u) = \int_{\mathbf{R}^k} \frac{1}{2} \|\nabla_x u(x,y)\|^2_{L^2(D)} - \mathcal{G}(u(x,y))\, dx. \tag{5.2}$$

We set $N = k + \ell \geq 3$ and we assume

(f0) $f(\xi) \in C(\mathbf{R}, \mathbf{R})$ and $f(\xi)$ is odd.

(f1) $\lim_{\xi \to \infty} \frac{f(\xi)}{\xi^{\frac{N+2}{N-2}}} = 0$.

(f2) $m_0 \equiv \lim_{\xi \to 0} \frac{f(\xi)}{\xi} < \lambda_1(D)$, where $\lambda_1(D)$ is the first eigenvalue of $-\Delta$ under Dirichlet boundary condition.

(f3) There exists $\zeta_0(y) \in C_0^\infty(D)$ such that $\mathcal{G}(\zeta_0) > 0$.

To find critical points of $I(u)$, we restrict $I(u)$ to a space of axially symmetric functions in x:

$$\begin{aligned} H_{s,0}^1(\mathbf{R}^k \times D) = \{ & u \in H_0^1(\mathbf{R}^k \times D);\ u(x,y) = u(|x|,y) \text{ is axially symmetric} \\ & \text{with respect to } x \in \mathbf{R}^k \}. \end{aligned}$$

Our main result in this section is

Theorem 5.1. *Assume* (f0)–(f3). *Then* $I(u):\ H^1_{s,0}(\mathbf{R}^k\times D)\to\mathbf{R}$ *has a unbounded sequence of critical values. In particular,* (5.1) *has infinitely many solutions which are axially symmetric in* x, *that is* $u(x,y)=u(|x|,y)$.

Remark 5.1. Introducing a suitable truncation of $f(\xi)$, the condition (f1) can be relaxed to (see [8, Section 2])

(f1′) $\lim_{\xi\to\infty}\dfrac{f(\xi)}{\xi^{\frac{N+2}{N-2}}}\le 0.$

Remark 5.2. (i) Under condition (f0), (f1′), (f2), (f3) but without oddness of $f(\xi)$, existence of a least energy solution is shown in [8].

(ii) We refer to Esteban [13] and Grossinho [15] for earlier works. In [13,15], they study the case where $G(s)/s^{-\theta}$ is non-decreasing in $[0,\infty)$ for some $\theta>2$.

Remark 5.3. In view of (5.2), conditions (f0)–(f3) are analogies of (g0)–(g3) for elliptic problems in strip-like domains (5.1). It seems difficult to take an approach in spirit of [5, 6]: find critical points of

$$u\mapsto \frac{1}{2}\|\nabla_x u\|^2_{L^2(\mathbf{R}^k\times D)};\ \Big\{u\in H^1_0(\mathbf{R}^k\times D);\ \int_{\mathbf{R}^k\times D}-\frac{1}{2}|\nabla_y u|^2+F(u)\,dxdy=1\Big\}.$$

For a proof Theorem 5.1. We set

$$E=H^1_{s,0}(\mathbf{R}^k\times D)\quad\text{and}\quad \|u\|_E=(\|\nabla_{x,y}u\|^2_{L^2(\mathbf{R}^k\times D)}+\|u\|^2_{L^2(\mathbf{R}^k\times D)})^{1/2}.$$

We note that $E=H^1_{s,0}(\mathbf{R}^k\times D)$ is compactly embedded into $L^q(\mathbf{R}^k\times D)$ for $q\in(2,\frac{2N}{N-2})$. See [14].

We consider the following C^0-action $\Phi:\ \mathbf{R}\to L(E)$:

$$(\Phi_\theta u)(x,y)=u(x/e^\theta,y).$$

Then

$$\begin{aligned}
&J(\theta,u)=I(\Phi_\theta u)=I(u(x/e^\theta,y))\\
=&\frac{1}{2}e^{(k-2)\theta}\|\nabla_x u\|^2_{L^2(\mathbf{R}^k\times D)}+\frac{1}{2}e^{k\theta}\|\nabla_y u\|^2_{L^2(\mathbf{R}^k\times D)}-e^{k\theta}\int_{\mathbf{R}^k\times D}F(u)\,dxdy\\
=&\frac{1}{2}e^{(k-2)\theta}\|\nabla_x u\|^2_{L^2(\mathbf{R}^k\times D)}-e^{k\theta}\int_{\mathbf{R}^k}\mathcal{G}(u)\,dx\in C^1(\mathbf{R}\times E,\mathbf{R})
\end{aligned}$$

and E, Φ_θ satisfy the Assumption 2.1. We also define $P(u):\ E\to\mathbf{R}$ by setting

$$\begin{aligned}
&P(u)=\partial_\theta J(0,u)\\
=&\frac{k-2}{2}\|\nabla_x u\|^2_{L^2(\mathbf{R}^k\times D)}-k\left(-\frac{1}{2}\|\nabla_y u\|^2_{L^2(\mathbf{R}^k\times D)}+\int_{\mathbf{R}^k\times D}F(u)\,dxdy\right)\\
=&\frac{k-2}{2}\|\nabla_x u\|^2_{L^2(\mathbf{R}^k\times D)}-k\int_{\mathbf{R}^k}\mathcal{G}(u)\,dx.
\end{aligned}$$

We have the following

Proposition 5.1. *Assume* (f0)–(f3). *Then for any* $b \in \mathbf{R}$, $I(u)$ *satisfies* $(PSP)_b$.

This proposition is essentially shown in [8, Sections 5–6]. In fact, in [8], it is shown that if $(\theta_j, u_j) \in \mathbf{R} \times E$ satisfies

$$\theta_j \to 0, \quad J(\theta_j, u_j) \to b, \quad DJ(\theta_j, u_j) \to 0 \text{ strongly in } (\mathbf{R} \times E)^*,$$

then (θ_j, u_j) has a convergent subsequences. Consider a special case $\theta_j \equiv 0$. It is nothing but $(PSP)_b$ condition.

By Proposition 5.1 we can apply Theorem 2.1 to $I(u)$ and we have deformation flow for $I(u)$. We apply Symmetric Mountain Pass Theorem to $I(u)$.

First we note that by (f2) and (f1)

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^k \times D} \left(|\nabla_x u|^2 + |\nabla_y u|^2 - m_0 |u|^2 \right) dxdy + o(\|u\|_E^2) \quad \text{as } \|u\|_E \sim 0.$$

Since $m_0 < \lambda_1(D)$, we have for some $r_0, \rho_0 > 0$

$$I(u) \geq \rho_0 \quad \text{for } \|u\|_E = r_0. \tag{5.3}$$

To find symmetric mountain pass geometry, we need the following result due to [6]. To state it, we need some notation. For $n \geq 1$, we set

$$\pi_{n-1} = \Big\{ (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n; \sum_{i=1}^n |\xi_i| = 1 \Big\}.$$

Berestycki and Lions [6] showed the following

Proposition 5.2. *For* $R > 0$ *there exists a continuous map* $\tau_{n,R} : \pi_{n-1} \to Lip([0,\infty), \mathbf{R})$ *such that*

(1) supp $\tau_{n,R}(\xi) \subset [0, R]$ *for all* $\xi \in \pi_{n-1}$.

(2) $\tau_{n,R}(\xi)(r) \in [-1, 1]$ *for all* $\xi \in \pi_{n-1}$ *and* $r \in [0, \infty)$.

(3) $\tau_{n,R}(-\xi)(r) = -\tau_{n,R}(\xi)(r)$ *for all* $\xi \in \pi_{n-1}$ *and* $r \in [0, \infty)$.

(4) For each $\xi \in \pi_{n-1}$, $\tau_{n,R}(\xi)(r) \in \{+1, -1\}$ *on* $[0, R]$ *except in at most n intervals* $J_1, \cdots, J_p$ *of* $[0, R]$, *each of these intervals has length at most one. Moreover*

$$|\tau'_{n,R}(\xi)(r)| \leq 2 \quad \textit{for all } r \in [0, \infty).$$

This proposition is shown in Sections 9.2a, 9.2b in [6] (set $\zeta = 1$ in the argument in [6]). We note that $\tau_{n,R}(\xi)$ is explicitly given in [6]. Regarding

$$\pi_{n-2} = \Big\{ (\xi_1, \cdots, \xi_{n-1}, 0); \sum_{i=1}^{n-1} |\xi_i| = 1 \Big\} \subset \pi_{n-1},$$

we have

$$\tau_{n,R}(\xi) = \tau_{n-1,R}(\xi) \quad \text{for all } \xi \in \pi_{n-2}. \tag{5.4}$$

We also see that

$$\tau_{n,R}(\xi)(r) \text{ depends on continuously on } R, \tag{5.5a}$$

$$\tau_{n,R}(\xi)(r) \to 0 \text{ as } R \to 0^+ \text{ uniformly in } (\xi, r). \tag{5.5b}$$

For $\xi \in \mathbf{R}^k$, we write

$$|\xi|_1 = \sum_{i=1}^{n} |\xi_i|$$

and we define $\gamma_{0n} : \mathbf{R}^n \to E$ by

$$\gamma_{0n}(\xi)(x,y) = \begin{cases} \tau_{n,|\xi|_1}\Big(\dfrac{\xi}{|\xi|_1}\Big)(|x|)\zeta_0(y) & \text{for } \xi \neq 0, \\ 0 & \text{for } \xi = 0. \end{cases}$$

Here $\zeta_0(y)$ is given in (f3). We also note that by (5.4)

$$\gamma_{0n}(\xi_1, \cdots, \xi_{n-1}, 0) = \gamma_{0,n-1}(\xi_1, \cdots, \xi_{n-1}).$$

Lemma 5.1. *For each $n \in \mathbf{N}$ there exists $R_n > 0$ such that*

$$I(\gamma_{0n}(\xi)) < 0 \quad \textit{for } \xi \in \mathbf{R}^n \quad \textit{with } |\xi|_1 \geq R_n.$$

Proof. Let $u(x,y) = \gamma_{0n}(\xi)(x,y)$ for $|\xi|_1 = R$. We have by the definition of $\tau_{n,R}(\xi)$ and Proposition 5.2(1)–(4)

$$\begin{aligned} &\operatorname{supp} u(x,y) \subset B_R \times \overline{D}, \\ &\operatorname{meas}\{x \in \mathbf{R}^k;\, |x| \leq R,\, u(x,y) \neq \pm\zeta_0(y)\} \subset \operatorname{meas}(B_R \setminus B_{R-n}). \end{aligned}$$

Here we use notation: $B_R = \{x \in \mathbf{R}^k; |x| \leq R\}$. Since

$$|\nabla_x u(x,y)| = \Big|\nabla_x \tau_{n,R}\Big(\frac{\xi}{R}\Big)(|x|)\Big| |\zeta_0(y)|,$$

so

$$\begin{aligned} &\nabla_x u(x,y) = 0, && \text{if } u(x,y) = \pm\zeta_0(y), \\ &|\nabla_x u(x,y)| \leq 2|\zeta_0(y)|, && \text{otherwise.} \end{aligned}$$

Thus, setting $C_1 = 2\int_D |\zeta_0(y)|^2\, dy$,

$$\frac{1}{2}\int_{\mathbf{R}^k \times D} |\nabla_x u|^2\, dxdy \leq C_1 \operatorname{meas}(B_R \setminus B_{R-n}).$$

On the other hand, setting $C_2 = \max_{t\in[0,1]} |\mathcal{G}(t\zeta_0(y))|$, we have

$$\int_{\mathbf{R}^k} \mathcal{G}(u(x,y))\,dx \geq \mathcal{G}(\zeta_0)\,\mathrm{meas}(B_{R-n}) - C_2\,\mathrm{meas}(B_R \setminus B_{R-n}).$$

Thus we have

$$I(u) \leq -\mathcal{G}(\zeta_0)\,\mathrm{meas}(B_{R-n}) + (C_1 + C_2)\,\mathrm{meas}(B_R \setminus B_{R-n}).$$

Since $\mathcal{G}(\zeta_0) > 0$, we have $I(u) < 0$ for large R. Thus for large $R_n > 0$ we have the conclusion of Lemma 5.1. $\square$

Proof of Theorem 5.1. By Lemma 5.1, we choose $R_1, R_2, \cdots$, so that

$$R_1 < R_2 < \cdots < R_n < R_{n+1} < \cdots,$$

such that

$$I(\gamma_{0n}(\xi)) < 0 \quad \text{for } \xi \in \mathbf{R}^n \quad \text{with } |\xi|_1 = R_n. \tag{5.6}$$

We may also assume

$$\|\gamma_{0n}(\xi)\|_E \geq r_0 \quad \text{for } \xi \in \mathbf{R}^n \quad \text{with } |\xi|_1 \geq R_1, \tag{5.7}$$

where $r_0 > 0$ is given in (5.3).

Now we define a sequence of minimax values. We set for $n \in \mathbf{N}$

$$\begin{aligned} &D_n = \{\xi \in \mathbf{R}^n;\ |\xi|_1 \leq R_n\}, \\ &\Gamma_n = \{\gamma \in C(D_n, E);\ \gamma(\xi) = \gamma_{0n}(\xi) \ \text{ for } \xi \in \partial D_n\}, \end{aligned}$$

and we define

$$b_n = \inf_{\gamma\in\Gamma_n} \max_{\xi\in D_n} I(\gamma(\xi)).$$

We note that by (5.7)

$$\gamma(D_n) \cap \{u \in E;\ \|u\|_E = r_0\} \neq \emptyset.$$

Thus, $b_n \geq \rho_0$ for each $n \in \mathbf{N}$ and we can see that b_n is a critical value of $I(u)$ for each $n \in \mathbf{N}$. To show the multiplicity of critical points, we need another set of minimax values. Modifying the definition in [24], we set

$$\begin{aligned} \Lambda_n = \{&\gamma(\overline{D_{n+p} \setminus Y});\ p \geq 0,\ \gamma \in \Gamma_{n+p},\ Y \subset D_{n+p} \setminus \{0\} \text{ is closed, symmetric} \\ &\text{with respect to 0 and genus}(Y) \leq p\}, \\ c_n = {}&\inf_{A\in\Lambda_n} \max_{u\in A} I(u). \end{aligned}$$

Here genus(Y) is the genus of Y. Clearly we have

$$b_n \geq c_n \quad \text{for } n \in \mathbf{N}.$$

By our deformation result (Theorem 5.1), we can apply the argument in [24] to c_n. In particular, since $I(u)$ satisfies $(PSP)_b$ for any $b \in \mathbf{R}$, we have

$$c_n \to \infty \quad \text{as } n \to \infty.$$

Thus $I(u)$ has unbounded sequence of critical values. $\square$

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