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Gaussian BV Functions and Gaussian BV Capacity on Stratified Groups

Jizheng Huang[1], Pengtao Li[2] and Yu Liu[3]*

[1] *School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China*
[2] *School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong 266071, China*
[3] *School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China*

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. Let G be a stratified Lie group and let $\{X_1,\cdots,X_{n_1}\}$ be a basis of the first layer of the Lie algebra of G. The sub-Laplacian Δ_G is defined by

$$\Delta_G=-\sum_{j=1}^{n_1}X_j^2.$$

The operator defined by

$$\Delta_G-\sum_{j=1}^{n_1}\frac{X_jp}{p}X_j$$

is called the Ornstein-Uhlenbeck operator on G, where p is a heat kernel at time 1 on G. In this paper, we investigate Gaussian BV functions and Gaussian BV capacities associated with the Ornstein-Uhlenbeck operator on the stratified Lie group.

Key Words: Gaussian p bounded variation, capacity, perimeter, stratified Lie group.

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1 Introduction

A function of bounded variation, simply a BV-function, is a real-valued function whose total variation is finite. In recent decades, many scholars have been paying attention to

*Corresponding author. *Email addresses:* `hjzheng@163.com` (J. Huang), `ptli@qdu.edu.cn` (P. Li), `liuyu75@pku.org.cn` (Y. Liu)

the BV function due to its application to calculus of variation and image processing. In the multi-variable setting, a function defined on an open subset $\Omega \subseteq \mathbb{R}^d, d \geq 2$, is said to have bounded variation provided that its distributional derivative is a vector-valued finite Radon measure over the subset Ω (cf. [2] or [13]). Precisely,

Definition 1.1. *A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation $\|Du\|$ in Ω is called a function of bounded variation, where*

$$\|Du\| := \sup\Big\{ \int_\Omega u \, div \, \nu dx : \ \nu = (\nu_1, \cdots, \nu_d) \in C_0^\infty(\Omega, \mathbb{R}^d), \ |\nu(x)| \leq 1, \ x \in \Omega \Big\} < \infty.$$

The class of all such functions will be denoted by $BV(\Omega)$. The norm of $BV(\Omega)$ is defined as

$$\|u\|_{BV} := \|u\|_{L^1(\Omega)} + \|Du\|.$$

In the study of the pointwise behavior of a Sobolev function, the notion of capacity plays a crucial role. In recent years, the capacity related to bounded variation functions attracts the attentions of many researchers. Please refer to [13] for the classical BV-capacity in $\mathbb{R}^d$. After that, many scholars generalize the BV- capacity to other settings. Hakkarainen and Kinnunen [5] studied basic properties of the BV-capacity and the Sobolev capacity in a complete metric space equipped with a doubling measure and supporting a weak Poincaré inequality. In [11], J. Xiao introduced the BV-type capacity on Gaussian spaces $\mathbb{G}^d$, and as an application, the Gaussian BV-capacity was used to study the trace inequalities of Gaussian BV-space. Recently, the authors in this paper investigate the capacity and perimeters derived from α-Hermite Bounded Variation in [6]. The author in [9] investigates two analogues of the Ornstein-Uhlenbeck semi-group in the setting of stratified groups G, which can be regarded as the generalization of Gauss spaces to the case of Lie group. Motivated by the previous works, we will investigate the Gaussian BV function and the Gaussian BV capacity associated with Ornstein-Uhlenbeck operators on stratified Lie groups.

To state our results, we recall some basic facts on the stratified Lie group, which can be easily found in Folland and Stein's book [3]. Let G be a stratified group of dimension n with the Lie algebra $\mathfrak{g}$. This means that $\mathfrak{g}$ is equipped with a family of dilations $\{\alpha_r : r > 0\}$ and $\mathfrak{g}$ is a direct sum $\bigoplus_{j=1}^m \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\mathfrak{g}_1$ generates $\mathfrak{g}$, and $\alpha_r(X) = r^j X$ for $X \in \mathfrak{g}_j$. $Q = \sum_{j=1}^m j \, n_j$ is called the homogeneous dimension of G, where $n_j = \dim \mathfrak{g}_j$. G is topologically identified with $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \mapsto G$. α_r is also viewed as an automorphism of G and if $x \in G, r > 0$, we write

$$\alpha_r x = (r^{d_1} x_1, \cdots, r^{d_n} x_n),$$

where $1 \leq d_1 \leq \cdots \leq d_n$. We fix a homogeneous norm of G, which satisfies the generalized triangle inequalities

$$|xy| \leq \gamma(|x| + |y|) \qquad \text{for all } \ x, y \in G,$$

$$\big||xy| - |x|\big| \leq \gamma|y| \qquad \text{for all } \ x, y \in G \quad \text{with } \ |y| \leq \frac{|x|}{2},$$

where $\gamma \geq 1$ is a constant. The homogeneous norm induces a quasi-metric d which is defined by $d(x,y) := |x^{-1}y|$. In particularly,

$$d(e,x) = |x| \quad \text{and} \quad d(x,y) = d(e,x^{-1}y).$$

The ball of radius r centered at x is written by

$$B(x,r) = \{y \in G : \ d(x,y) \ < r\}.$$

The Haar measure on G is simply the Lebesgue measure on $\mathbb{R}^n$ under the identification of G with $\mathfrak{g}$ and the identification of $\mathfrak{g}$ with $\mathbb{R}^n$, where $n = \sum_{j=1}^m n_j$. The measure of $B(x,r)$ is

$$\big|B(x,r)\big| = b\, r^Q,$$

where b is a constant.

We identify $\mathfrak{g}$ with $\mathfrak{g}_L$, the Lie algebra of left-invariant vector fields on G. Let $\{X_j : j = 1, \cdots, n_1\}$ be a basis of $\mathfrak{g}_1$. The sub-Laplacian Δ_G is defined by

$$\Delta_G = \sum_{j=1}^{n_1} X_j^2,$$

and the horizontal gradient operator ∇_G is denoted by $\nabla_G = (X_1, \cdots, X_{n_1})$. Moreover, if $\phi = (\phi_1, \cdots, \phi_{n_1})$ is a vector-valued function such that $X_j\phi_j \in L^1_{\rm loc}(G)$ for $j = 1, \cdots, n_1$, we define the divergence $\mathrm{div}_G\phi$ as the real valued function

$$\mathrm{div}_G(\phi) := -\sum_{j=1}^{n_1} X_j^* \phi_j = \sum_{j=1}^{n_1} X_j\phi_j.$$

If f and g are measurable functions on G, their convolution $f * g$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy = \int_G f(xy^{-1})g(y)dy.$$

The [3] implies that for any $k = 1, 2, \cdots, n_1$,

$$X_k(f * g) = f * (X_k g).$$

Let $\{T_s : \ s > 0\} = \big\{e^{-s(-\Delta_G)} : \ s > 0\big\}$ be the heat semigroup with the convolution kernel $p_s(x)$. [7] and [12] imply that the heat kernel $p_s(x)$ satisfies the following estimates

$$0 < p_s(x) \leq Cs^{-\frac{Q}{2}} e^{-A_1 s^{-1}|x|^2}, \tag{1.1a}$$

$$|X_i p_s(x)| \leq C\, s^{-\frac{Q+1}{2}} e^{-A_2 s^{-1}|x|^2}, \tag{1.1b}$$

where A_1 and A_2 are positive constants, $i = 1, 2, \cdots, n_1$. Moreover, we know from [3, Proposition 1.68] that $p_s(x^{-1}) = p_s(x)$ for any $x \in G$. The Gauss p-divergence of $\phi \in C_c^1(O, \mathbb{R}^{n_1})$ is determined by

$$\mathrm{div}_p \phi = \mathrm{div}_G(\phi) + \sum_{j=1}^{n_1} \frac{X_j p}{p} \phi_j,$$

where p is a heat kernel at time 1 on G.

The paper is organized as follows. In Section 2, we introduce the Gaussian BV function on the stratified Lie group and study its properties. Section 3 is devoted to the proofs of some results for Gaussian BV capacity on the stratified Lie group.

2 Gaussian BV functions on the stratified Lie group

Let $\Omega \subseteq G$ be an open set. The p-total variation of $f \in L^1(\Omega)$ is defined by

$$\|D_{G,p} f\|(\Omega) = \sup_{\varphi \in \mathcal{F}(\Omega)} \Big\{ \int_\Omega f(x) \mathrm{div}_p \varphi(x) p(x) dx \Big\},$$

where $\mathcal{F}(\Omega)$ denotes the class of all functions

$$\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{n_1}) \in C_0^1(\Omega, \mathbb{R}^{n_1})$$

satisfying

$$\|\varphi\|_\infty = \sup_{x \in \Omega} (|\varphi_1(x)|^2 + \cdots + |\varphi_{n_1}(x)|^2)^{1/2} \leq 1.$$

It should be noted that when $p = 1$, $\|D_{G,p} f\|(\Omega)$ is the total variation on the stratified Lie group (cf. [1]) and when $G = \mathbb{R}^n$, $\|D_{G,p} f\|(\Omega)$ is the γ-total variation in [8].

An L^1 function f is said to have the bounded p-total variation on Ω if

$$\|D_{G,p} f\|(\Omega) < \infty,$$

and the collection of all such functions is denoted by $\mathcal{BV}_{G,p}(\Omega)$, which is a Banach space with the norm

$$\|f\|_{\mathcal{BV}_{G,p}(\Omega)} = \|f\|_{L^1} + \|D_{G,p} f\|(\Omega).$$

Please see Lemma 2.3. A function $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R})$ is said to be of locally p-total variation and we write $f \in \mathcal{BV}^{\mathrm{loc}}_{G,p}(\Omega)$ if

$$\|D_{G,p} f\|(U) < \infty$$

holds true for every open set $U \subset \Omega$.

Let $\Omega \subset G$ be an open and bounded set and $E \subset \Omega$ be a Borel set. Then using the Riesz representation theorem in [2, Theorem 1.38], it is easy to check that

$$\|D_{G,p} f\|(E) := \inf \Big\{ \|D_{G,p} f\|(U) :\ E \subset U,\ U \subset \Omega \text{ open} \Big\}$$

extends $\|D_{G,p}f\|(\cdot)$ to a Radon measure in Ω.

The Gaussian p perimeter of $E \subseteq \Omega$ can be defined as follows:

$$P_{G,p}(E,\Omega) = \|D_{G,p}1_E\|(\Omega) = \sup_{\varphi\in\mathcal{F}(\Omega)} \left\{ \int_E \mathrm{div}_p\varphi(x)p(x)dx \right\},$$

where 1_E denotes the characteristic function of E.

In what follows, we will collect some properties of the space $\mathcal{BV}_{G,p}(\Omega)$.

Lemma 2.1. *(i) If $f \in C_c^1(\Omega)$, then*

$$\|D_{G,p}f\|(\Omega) = \int_\Omega |\nabla_G f(x)|p(x)dx. \tag{2.1}$$

(ii) The Gaussian p variation has the following lower semicontinuity: if

$$f, f_k \in \mathcal{BV}_{G,p}(\Omega), \quad k\in\mathbb{N}, \quad \textit{be such that } \; f_k \to f \quad \textit{in } \; L^1_{\mathrm{loc}}(\Omega),$$

then

$$\liminf_{k\to\infty} \|D_{G,p}f_k\|(\Omega) \geq \|D_{G,p}f\|(\Omega). \tag{2.2}$$

Proof. (i) If $f \in C_c^1(\Omega)$, then we have $\nabla_G f \in L^1(\Omega)$. For every

$$\varphi \in C_c^1(\Omega,\mathbb{R}^{n_1}) \quad \text{with} \quad \|\varphi\|_{L^\infty(\Omega)} \leq 1,$$

we have

$$\begin{aligned}
&\left|\int_\Omega f(x)\mathrm{div}_p\varphi(x)p(x)dx\right| \\
=&\left|\int_\Omega \nabla_G(f(x)p(x))\cdot\varphi(x)dx - \int_\Omega f(x)\sum_{j=1}^{n_1} X_j p(x)\varphi_j(x)dx\right| \\
=&\left|\int_\Omega \nabla_G f(x)\cdot\varphi(x)p(x)dx\right| \leq \int_\Omega |\nabla_G f(x)|p(x)dx.
\end{aligned}$$

By taking the supremum over φ, we conclude that $f\in\mathcal{BV}_{G,p}(\Omega)$ and

$$\|D_{G,p}f\|(\Omega) \leq \int_\Omega |\nabla_G f(x)|p(x)dx. \tag{2.3}$$

Define $\varphi \in L^\infty(\Omega,\mathbb{R}^{n_1})$ as follows:

$$\varphi(x) := \begin{cases} \dfrac{\nabla_G f(x)}{|\nabla_G f(x)|}, & \text{if } x\in\Omega \text{ and } \nabla_G f(x)\neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\|\varphi\|_\infty \le 1$. We choose a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega,\mathbb{R}^{n_1})$ such that $\varphi_n \to \varphi$ as $n \to \infty$, with $\|\varphi_n\|_{L^\infty(\Omega)} \le 1$ for all $n \in \mathbb{N}$. Combining the definition of $\|D_{G,p}f\|(\Omega)$ with integration by parts derives that for every $n \ge 1$,

$$\begin{aligned}\|D_{G,p}f\|(\Omega) \geq &\sum_{i=1}^{n_1} \int_\Omega \big(X_i(f(x)p(x)) - f(x)X_i p(x)\big)\varphi_n^{(i)}(x)dx \\ = &\int_\Omega \nabla_G f(x)\cdot \varphi_n(x)p(x)dx.\end{aligned}$$

By the dominated convergence theorem and the definition of φ, we have

$$\|D_{G,p}f\|(\Omega) \geq \int_\Omega |D_{G,p}f(x)|dx$$

via letting $n \to \infty$, which is the opposite of the inequality (2.3).
(ii) Fix $\varphi \in C_c^\infty(\Omega,\mathbb{R}^{n_1})$ with $\|\varphi\|_{L^\infty(\Omega)} \le 1$. By the definition of $\|D_{G,p}f_k\|(\Omega)$, we have

$$\|D_{G,p}f_k\|(\Omega) \geq \int_\Omega f_k(x)\mathrm{div}_p\varphi(x)p(x)dx.$$

Since $\{f_k\}_{k\in\mathbb{N}}$ converges to f in $L^1_{\mathrm{loc}}(\Omega)$, so by Fatou's lemma, we get

$$\liminf_{k\to\infty} \|D_{G,p}f_k\|(\Omega) \geq \int_\Omega f(x)\mathrm{div}_p\varphi(x)p(x)dx.$$

Therefore, (ii) is proved. □

The following lemma gives the structure theorem for Gaussian BV functions on the stratified Lie group.

Lemma 2.2. *Let $\Omega \subset G$ be an open and bounded set. There exists a unique $\mathbb{R}^{n_1}$-valued finite Radon measure μ such that*

$$\int_\Omega u(x)\mathrm{div}_p\varphi(x)p(x)dx = \int_\Omega \varphi(x)\cdot d\mu(x)$$

for every $\varphi \in C_c^\infty(\Omega,\mathbb{R}^{n_1})$ and

$$\|D_{G,p}u\|(\Omega) = |\mu|(\Omega).$$

The lower semicontinuity and the standard procedure imply the follow lemma.

Lemma 2.3. *The space $(\mathcal{BV}_{G,p}(\Omega), \|\cdot\|_{\mathcal{BV}_{G,p}(\Omega)})$ is a Banach space.*

Next we will list the following approximation result for the Gaussian p variation.

Theorem 2.1. *If $u \in \mathcal{BV}_{G,p}(\Omega)$, there exists a sequence of functions $\{u_h\}_{h\in\mathbb{N}} \in C^\infty(\Omega) \cap \mathcal{BV}_{G,p}(\Omega)$ such that*

$$\lim_{h\to\infty} \|u_h - u\|_{L^1} = 0 \quad \textit{and} \quad \lim_{h\to\infty} \int_\Omega |\nabla_G u_h(x)| p(x) dx = \|D_{G,p} u\|(\Omega).$$

Proof. Inspired by the method in [2, Theorem 5.3] or [4, Theorem 1.14] and via the semi-continuity property of Lemma 2.1, we only need to verify that, for every $\varepsilon > 0$, there exists a function $u_\varepsilon \in C^\infty(\Omega)$ such that

$$\int_\Omega |u(x) - u_\varepsilon(x)| p(x) dx < \varepsilon \quad \text{and} \quad \|D_{G,p} u_\varepsilon\|(\Omega) < \|D_{G,p} u\|(\Omega) + \varepsilon. \tag{2.4}$$

Given a positive integer m, let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of open sets which are defined as follows

$$\Omega_j := \left\{ x \in \Omega \;\middle|\; \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+j} \right\} \bigcap B(0, k+m), \quad j \in \mathbb{N},$$

where $B(0, k+m)$ denotes the open ball of center 0 and radius $k+m$, and $\operatorname{dist}(x, \partial\Omega)$ represents the distance from x to $\partial\Omega$. Since $\|D_{G,p} u\|(\cdot)$ is a Radon measure, given $\varepsilon > 0$ we can choose $m \in \mathbb{N}$ so large that

$$\|D_{G,p} u\|(\Omega \setminus \Omega_0) < \varepsilon. \tag{2.5}$$

In fact, we find that the sequence of open sets $\{\Omega_j\}$ satisfy the following properties:

$$\Omega_j \subset \Omega_{j+1} \subset \Omega \quad \text{for any} \quad j \in \mathbb{N} \quad \text{and} \quad \bigcup_{j=0}^{\infty} \Omega_j = \Omega.$$

Set

$$U_0 := \Omega_0, \quad U_j := \Omega_{j+1} \setminus \overline{\Omega}_{j-1} \quad \text{for } j \geq 1.$$

The proof of Theorem 1.14 in [4] implies that there exists a partition of unity related to the covering $\{U_j\}_{j\in\mathbb{N}}$, which means that there exists $\{f_j\}_{j\in\mathbb{N}} \in C_c^\infty(U_j)$ such that $0 \leq f_j \leq 1$ for every $j \geq 0$ and $\sum_{j=0}^\infty f_j = 1$ on Ω. In particular, the following fact is valid:

$$\sum_{j=0}^{\infty} \nabla_G f_j = 0 \quad \text{on } \Omega. \tag{2.6}$$

Let $\eta \in C_c^\infty(G)$ be a radial nonnegative function with

$$\int_G \eta(x) dx = 1 \quad \text{and} \quad \operatorname{supp}(\eta) \subset B(0,1).$$

Given $\varepsilon > 0$ and $u \in L^1(\Omega, \mathbb{R})$, extended to zero out of Ω, define

$$u_\varepsilon(x) := \frac{1}{\varepsilon^Q} \int_G \eta\Big(\alpha_\varepsilon(xy^{-1})\Big) u(y) dy = \frac{1}{\varepsilon^Q} \int_{B(x,\varepsilon)} \eta\Big(\alpha_\varepsilon(xy^{-1})\Big) u(y) dy. \tag{2.7}$$

For every $j \geq 0$, there exists $0 < \varepsilon_j < \varepsilon$ such that

$$\begin{cases} \operatorname{supp}\big((f_j u)_{\varepsilon_j}\big) \subseteq U_j, \\ \int_\Omega |(f_j u)_{\varepsilon_j} - f_j u| dx < \varepsilon 2^{-(j+1)}, \\ \int_\Omega |(u \nabla_G f_j)_{\varepsilon_j} - u \nabla_G f_j| dx < \varepsilon 2^{-(j+1)}. \end{cases} \tag{2.8}$$

Define

$$\phi_\varepsilon := \sum_{j=0}^{\infty} (u f_j)_{\varepsilon_j}.$$

Since the sum is locally finite, then we conclude that

$$\phi_\varepsilon \in C^\infty(\Omega) \quad \text{and} \quad u = \sum_{j=0}^{\infty} u f_j$$

pointwise. Since η is radial,

$$\eta(xy^{-1}) = \eta(yx^{-1}).$$

Denote by

$$\eta_{\varepsilon_j}(x) = \frac{1}{\varepsilon_j^Q} \eta\Big(\alpha_{\varepsilon_j^{-1}}(x)\Big).$$

A direct computation implies that

$$\begin{aligned} &\int_\Omega \phi_\varepsilon(x) \operatorname{div}_p \varphi(x) dx \\ =& \sum_{j=0}^{\infty} \int_\Omega \big(\eta_{\varepsilon_j} * (u f_j)\big)(x) \operatorname{div}_p \varphi(x) dx \\ =& \sum_{j=0}^{\infty} \int_\Omega \int_\Omega \frac{1}{\varepsilon_j^Q} \eta\Big(\alpha_{\varepsilon_j^{-1}}(xy^{-1})\Big) u(y) f_j(y) \operatorname{div}_p \varphi(x) dy dx \\ =& \sum_{j=0}^{\infty} \int_\Omega \int_\Omega \frac{1}{\varepsilon_j^Q} \eta\Big(\alpha_{\varepsilon_j^{-1}}(xy^{-1})\Big) u(y) f_j(y) \Big[\operatorname{div}_G \varphi(x) + \sum_{k=1}^{n_1} \frac{X_k p(x)}{p(x)} \varphi_k(x)\Big] dy dx \\ =& : I + II, \end{aligned}$$

where

$$\begin{aligned} I =& \sum_{j=0}^{\infty} \int_\Omega \int_\Omega \frac{1}{\varepsilon_j^Q} \eta\Big(\alpha_{\varepsilon_j^{-1}}(xy^{-1})\Big) u(y) f_j(y) \Big[\sum_{k=1}^{n_1} \mathrm{X}_k \varphi_k(x)\Big] dy dx \\ =& \sum_{j=0}^{\infty} \int_\Omega u(y) \sum_{k=1}^{n_1} \int_\Omega \frac{1}{\varepsilon_j^Q} \eta\Big(\alpha_{\varepsilon_j^{-1}}(y^{-1}x)\Big) f_j(y) \Big[\mathrm{X}_k \varphi_k(x)\Big] dy dx \\ =& \sum_{j=0}^{\infty} \int_\Omega u(y) f_j(y) \sum_{k=1}^{n_1} \Big[\eta_{\varepsilon_j} * \mathrm{X}_k \varphi_k(y)\Big] dy \end{aligned}$$

$$= \sum_{j=0}^{\infty} \int_{\Omega} u(y) f_j(y) \sum_{k=1}^{n_1} \Big[X_k(\eta_{\varepsilon_j} * \varphi_k)(y)\Big] dydx,$$

$$II = \sum_{j=0}^{\infty} \int_{\Omega}\int_{\Omega} \frac{1}{\varepsilon_j^Q} \eta\Big(\alpha_{\varepsilon_j^{-1}}(xy^{-1})\Big) u(y) f_j(y) \Big[\sum_{k=1}^{n_1} \frac{X_k p(x)}{p(x)} \varphi_k(x)\Big] dydx.$$

As for I, we get $I = I_1 + I_2$, where

$$I_1 := \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big[\sum_{k=1}^{n_1} X_k\Big(f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y))\Big)\Big] dy,$$

$$\begin{aligned} I_2 &:= -\sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big[\sum_{k=1}^{n_1} X_k f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y))\Big] dy \\ &= -\sum_{j=0}^{\infty} \int_{\Omega} \Big[\sum_{k=1}^{n_1} \eta_{\varepsilon_j} * (uX_k f_j)(y)\varphi_k(y) - \sum_{k=1}^{n_1} (uX_k f_j)(y)\varphi_k(y)\Big] dy. \end{aligned}$$

When $\|\varphi\|_{L^\infty} \le 1$, it holds that

$$|(f_j(y))(\eta_{\varepsilon_j} * \varphi_k(y))| \le 1$$

for all $j \ge 0$ and $k = 1, \cdots, d$. Moreover, it follows from (2.8) that $|I_2| < \varepsilon$.

For II, a direct computation gives

$$\begin{aligned} II =& \sum_{j=0}^{\infty} \int_{\Omega}\int_{\Omega} \frac{1}{\varepsilon_j^d} \eta\Big(\alpha_{\varepsilon_j^{-1}}(xy^{-1})\Big) u(y) f_j(y) \Big[\sum_{k=1}^{n_1} \frac{X_k p(x)}{p(x)} \varphi_k(x)\Big] dydx \\ =& \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big(\sum_{k=1}^{d} \frac{X_k p(x)}{p(x)} f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y)) dy \\ =& \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big(\sum_{k=1}^{d} \frac{X_k p(y)}{p(y)} f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y)) dy \\ &+ \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big(\sum_{k=1}^{d} \Big(\frac{X_k p(x)}{p(x)} - \frac{X_k p(y)}{p(y)}\Big) f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y)) dy. \end{aligned}$$

Therefore, the above estimate for the term I_2 indicates that

$$\Big| \int_{\Omega} \phi_\varepsilon(x) \mathrm{div}_p \varphi(x) dx\Big| = |I_1 + I_2 + II| \le J_1 + J_2 + \varepsilon,$$

where

$$\begin{aligned} J_1 :=& \Big| \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big[\sum_{k=1}^{n_1} X_k\Big(f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y))\Big)\Big] dy \\ &+ \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big(\sum_{k=1}^{d} \frac{X_k p(y)}{p(y)} f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y)) dy\Big|, \end{aligned}$$

$$J_2 := \Big| \sum_{j=0}^{\infty} \int_{\Omega} u(y) \Big(\sum_{k=1}^{d} \Big(\frac{X_k p(x)}{p(x)} - \frac{X_k p(y)}{p(y)} \Big) f_j(y) (\eta_{\varepsilon_j} * \varphi_k(y)) dy \Big|.$$

Note that, by the construction of U_j, every point $x \in \Omega$ belongs to at most three of the sets U_j. Then we have

$$\begin{aligned} J_1 \leq & \Big| \Big\{ \int_{\Omega} u(y) \Big[\sum_{k=1}^{n_1} X_k \Big(f_0(y)(\eta_{\varepsilon_j} * \varphi_k(y)) \Big) \Big] dy \Big\} \\ & + \int_{\Omega} u(y) \Big[\Big(\sum_{k=1}^{d} \frac{X_k p(y)}{p(y)} f_0(y)(\eta_{\varepsilon_j} * \varphi_k(y)) \Big] dy \Big\} \Big| \\ & + \Big| \Big\{ \sum_{j=1}^{\infty} \int_{\Omega} u(y) \Big[\sum_{k=1}^{n_1} X_k \Big(f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y)) \Big) \Big] dy \Big\} \\ & + \Big\{ \sum_{j=1}^{\infty} \int_{\Omega} u(y) \Big[\sum_{k=1}^{n_1} \frac{X_k p(y)}{p(y)} f_j(y)(\eta_{\varepsilon_j} * \varphi_k(y)) \Big] dy \Big\} \Big| \\ \lesssim & c \Big(\|D_{G,p} u\|(\Omega) + \sum_{j=1}^{\infty} \|D_{G,p} u\|(U_j) \Big) \\ \lesssim & c \Big(\|D_{G,p} u\|(\Omega) + 3 \|D_{G,p} u\|(\Omega \backslash \Omega_0) \Big) \\ \lesssim & c \Big(\|D_{G,p} u\|(\Omega) + 3\varepsilon \Big) \leq c, \end{aligned}$$

where we have used (2.5) in the last inequality.

It follows the stratified mean value theorem in [3] that

$$\Big| \frac{X_k p(x)}{p(x)} - \frac{X_k p(y)}{p(y)} \Big| \leq \Big\| \nabla_G \Big(\frac{X_k p}{p} \Big) \Big\|_{L^\infty} |y^{-1} x|,$$

that is, $\frac{X_k p(y)}{p(y)}$ is Lipschitz continuous, $\|\varphi\| \leq 1$ and $\operatorname{supp}(\eta) \subseteq B_1(0)$, then we have

$$J_2 \leq c\varepsilon \Big\| \nabla_G \Big(\frac{X_k p}{p} \Big) \Big\|_{L^\infty} \int_G \eta(z) |z| dz \int_{\Omega} \sum_{j=1}^{\infty} |f_j(y)| |u(y)| dy \leq c\varepsilon.$$

By taking the supremum over φ and the arbitrariness of $\varepsilon > 0$, we conclude that (2.4) holds true. □

Moreover, we have the following max-min property of the Gaussian p variation.

Theorem 2.2. *Let $u, v \in L^1(\Omega)$. Then*

$$\|D_{G,p} \max\{u,v\}\|(\Omega) + \|D_{G,p} \min\{u,v\}\|(\Omega) \leq \|D_{G,p} u\|(\Omega) + \|D_{G,p} v\|(\Omega).$$

Proof. Without loss of generality, we may assume

$$\|D_{G,p}u\|(\Omega)+\|D_{G,p}v\|(\Omega)<\infty.$$

Take two functions $u_h, v_h \in C_c^\infty(\Omega)\cap\mathcal{BV}_{G,p}(\Omega)$, $h=1,2,\cdots$, such that

$$\begin{cases} u_h\to u, \quad v_h\to v \qquad \text{in} \quad L^1(\Omega),\\ \displaystyle\int_\Omega |D_{G,p}u_h(x)|dx \to \|D_{G,p}u\|(\Omega),\\ \displaystyle\int_\Omega |D_{G,p}v_h(x)|dx \to \|D_{G,p}v\|(\Omega). \end{cases}$$

Since

$$\max\{u_h,v_h\}\to\max\{u,v\}, \quad \min\{u_h,v_h\}\to\min\{u,v\} \quad \text{in} \ L^1(\Omega),$$

it follows that

$$\begin{aligned} &\|D_{G,p}\max\{u,v\}\|(\Omega)+\|D_{G,p}\min\{u,v\}\|(\Omega)\\ \le&\liminf_{h\to\infty}\int_\Omega |D_{G,p}\max\{u_h,v_h\}|dx+\liminf_{h\to\infty}\int_\Omega |D_{G,p}\min\{u_h,v_h\}|dx\\ \le&\liminf_{h\to\infty}\Big(\int_\Omega |D_{G,p}\max\{u_h,v_h\}|dx+\int_\Omega |D_{G,p}\min\{u_h,v_h\}|dx\Big)\\ \le&\lim_{h\to\infty}\int_\Omega |D_{G,p}u_h(x)|dx+\lim_{h\to\infty}\int_\Omega |D_{G,p}v_h(x)|dx\\ =&\|D_{G,p}u\|(\Omega)+\|D_{G,p}v\|(\Omega). \end{aligned}$$

Thus, we complete the proof. □

For any compact subsets E,F in Ω, via choosing $u=1_E$ and $u=1_F$, the following lemma can be deduced from Theorem 2.2 immediately.

Lemma 2.4. *For any subsets E,F in Ω, we have*

$$P_{G,p}\Big(E\bigcap F,\Omega\Big)+P_{G,p}\Big(E\bigcup F,\Omega\Big)\le P_{G,p}(E,\Omega)+P_{G,p}(F,\Omega).$$

In what follows, we establish the coarea formula for Gaussian BV functions on the stratified Lie group.

Theorem 2.3. *If $f\in\mathcal{BV}_{G,p}(\Omega)$, then*

$$\|D_{G,p}f\|(\Omega)=\int_{-\infty}^{\infty}\|D_{G,p}1_{E_t}\|(\Omega)dt. \tag{2.9}$$

Proof. Let $f:\ \Omega\to G$ and $t\in\mathbb{R}$. Denote by $E_t=\{x\in\Omega:\ f(x)>t\}$. The structure of the Gaussian p-divergence and [2, Section 5.5, Lemma 1] imply the following fact: if $f\in\mathcal{BV}_{G,p}(\Omega)$, the mapping $t\mapsto\|D_{G,p}1_{E_t}\|(\Omega)$ is Lebesgue measurable for $t\in\mathbb{R}$.

Let $\varphi \in C_c^1(\Omega, \mathbb{R}^{n_1})$ and $\|\varphi\|_{L^\infty} \leq 1$. Firstly, we claim that

$$\int_\Omega f(x)\mathrm{div}_p\varphi(x)p(x)dx = \int_{-\infty}^{\infty}\Big(\int_{E_t}\mathrm{div}_p\varphi p(x)dx\Big)dt.$$

The above claim can be proved by the following facts

$$\int_\Omega f(x)\frac{X_jp(x)}{p(x)}\varphi p(x)dx = \int_{-\infty}^{\infty}\Big(\int_{E_t}\frac{X_jp(x)}{p(x)}\varphi(x)p(x)dx\Big)dt$$

for $j = 1, 2, \cdots, n_1$ and

$$\int_\Omega f\mathrm{div}_G\varphi p(x)dx = \int_{-\infty}^{\infty}\Big(\int_{E_t}\mathrm{div}_G\varphi p(x)dx\Big)dt,$$

where the latter can be proved by [4, (5.2)]. Therefore, we conclude that for all φ as above

$$\int_\Omega f\mathrm{div}_p\varphi dx \leq \int_{-\infty}^{\infty}\|D_{G,p}1_{E_t}\|(\Omega)dt.$$

Furthermore,

$$\|D_{G,p}f\|(\Omega) \leq \int_{-\infty}^{\infty}\|D_{G,p}1_{E_t}\|(\Omega)dt.$$

Secondly, we claim that (2.9) holds true for all $f \in \mathcal{BV}_{G,p}(\Omega)\bigcap C^\infty(\Omega)$. Next we will prove the claim according to the idea of [10, Proposition 4.2]. Let

$$m(t) = \int_{\{x\in\Omega: f(x)\leq t\}}|\nabla_G f|p(x)dx. \tag{2.10}$$

Then it is obvious that

$$\int_{-\infty}^{\infty}m'(t)dt \leq \int_\Omega|\nabla_G f|p(x)dx. \tag{2.11}$$

Define a function g_h as follows:

$$g_h(s) = \begin{cases} 0, & \text{if } s \leq t, \\ h(t-s)+1, & \text{if } t \leq s \leq t+1/h, \\ 1, & \text{if } s \geq t+1/h, \end{cases}$$

where $t \in \mathbb{R}$. We define the sequence $v_h(x) := g_h(f(x))$. At this time, $v_h \to 1_{E_t}$ in $L^1(\Omega)$. In fact,

$$\begin{aligned}\int_\Omega|v_h(x) - 1_{E_t}(x)|dx &= \int_{\{x\in\Omega: t<f(x)\leq t+1/h\}}g_h(f(x))dx \\ &\leq \Big|\Big\{x\in\Omega : t < f(x) \leq t+1/h\Big\}\Big| \to 0,\end{aligned}$$

since $\{x\in\Omega : t<f(x)\le t+1/h\}\to\emptyset$ when $h\to\infty$. Then

$$\begin{aligned}\int_\Omega |\nabla_G v_h(x)|p(x)dx =&h\int_{\{x\in\Omega:t<f(x)\le t+1/h\}}|\nabla_G f(x)|p(x)dx\\ =&h\big(m(t+1/h)-m(t)\big).\end{aligned}$$

Taking the limit $h\to\infty$ and noting that Theorem 2.1, we have

$$\|D_{G,p}1_{E_t}\|(\Omega)\le\limsup_{h\to\infty}\|D_{G,p}v_h\|(\Omega)\le m'(t). \tag{2.12}$$

Integrating (2.12) and using (2.13), we obtain

$$\int_{-\infty}^{\infty}\|D_{G,p}1_{E_t}\|(\Omega)dt\le\int_\Omega|\nabla_G f|p(x)dx. \tag{2.13}$$

Finally, by approximation and using the lower semi-continuity of the Gaussian p perimeter, we conclude that (2.9) holds true for all $f\in\mathcal{BV}_{G,p}(\Omega)$ (see Evans and Gariepy [2] for details). □

3 Gaussian BV capacity on the stratified Lie group

In terms of the results on Gaussian BV spaces, we introduce the Gaussian BV capacity and investigate its properties on the stratified Lie group.

Definition 3.1. *For a set $E\subseteq G$, let $\mathcal{A}(E,\mathcal{BV}_{G,p}(G))$ be the class of admissible functions on G, i.e., $f\in\mathcal{BV}_{G,p}(G)$ satisfying $0\le f\le 1$ and $f=1$ in a neighborhood of E (an open set containing E). The Gaussian BV capacity of E is defined by*

$$\operatorname{cap}(E,\mathcal{BV}_{G,p}(G)):=\inf_{f\in\mathcal{A}(E,\mathcal{BV}_{G,p}(G))}\Big\{\|f\|_{L^1}+\|D_{G,p}f\|(G)\Big\}. \tag{3.1}$$

Via the co-area formula for Gaussian BV functions in Theorem 2.3, we obtain the following basic assertion.

Theorem 3.1. *A geometric description of the Gaussian BV capacity of a set in G is given as follows:*

(i) For any set $K\subseteq G$,

$$\operatorname{cap}(K,\mathcal{BV}_{G,p}(G))=\inf_A\Big\{|A|+P_{G,p}(A)\Big\},$$

where the infimum is taken over all sets $A\subseteq G$ such that $K\subseteq int(A)$.

(ii) For any compact set $K \subseteq G$,

$$\operatorname{cap}(K, \mathcal{BV}_{G,p}(G)) = \inf_A \Big\{ |A| + P_{G,p}(A) \Big\},$$

where the infimum is taken over all bounded open sets A with smooth boundary in G containing K.

Proof. (i) If $A \subseteq G$ with $K \subseteq int(A)$ and $|A| + P_{G,p}(A) < \infty$, $1_A \in \mathcal{A}(K, \mathcal{BV}_{G,p}(G))$ and hence,

$$\operatorname{cap}(K, \mathcal{BV}_{G,p}(G)) \leq |A| + P_{G,p}(A).$$

By taking the infimum over all such sets A, we obtain

$$\operatorname{cap}(K, \mathcal{BV}_{G,p}(G)) \leq \inf_A \Big\{ |A| + P_{G,p}(A) \Big\}.$$

In order to prove the reverse inequality, we may assume that $\operatorname{cap}(K, \mathcal{BV}_{G,p}(G)) < \infty$. Let $\varepsilon > 0$ and $f \in \mathcal{A}(K, \mathcal{BV}_{G,p}(\mathbb{R}^d))$ such that

$$\|f\|_{L^1} + \|D_{G,p} f\|(G) < \operatorname{cap}(K, \mathcal{BV}_{G,p}(G)) + \varepsilon.$$

Using the co-area formula (2.9) and the Cavalieri principle, we have

$$\begin{aligned} \int_G f(x)dx + \|D_{G,p} f\|(G) &= \int_0^1 \Big[\Big| \Big\{ x \in G : f(x) > t \Big\} \Big| + P_{G,p}(\{x \in G : f(x) > t\} \Big] dt \\ &\geq \inf_A \{ |A| + P_{G,p}(A) \}, \end{aligned}$$

where we have used the fact: $K \subseteq int\big\{x \in G : f(x) > t\big\}$ for $0 < t < 1$. Then

$$\inf_A \Big\{ |A| + P_{G,p}(A) \Big\} \leq \operatorname{cap}(K, \mathcal{BV}_{G,p}(G)) + \varepsilon.$$

The desired inequality now follows by letting $\varepsilon \to 0$.

(ii) Using the co-area formula (2.9) and the Cavalieri principle again, we can also prove (ii) similar to the proof of (i) and so we omit the details here. □

In what follows, we give the measure-theoretic nature of Gaussian BV capacity.

Theorem 3.2. *Assume A, B are subsets of G.*

(i)

$$\operatorname{cap}(\emptyset, \mathcal{BV}_{G,p}(G)) = 0.$$

(ii) If $A \subseteq B$, then

$$\operatorname{cap}(A, \mathcal{BV}_{G,p}(G)) \leq \operatorname{cap}(B, \mathcal{BV}_{G,p}(G)).$$

(iii)

$$\begin{aligned}&\mathrm{cap}\Big(A\bigcup B,\mathcal{BV}_{G,p}(G)\Big)+\mathrm{cap}\Big(A\bigcap B,\mathcal{BV}_{G,p}(G)\Big)\\ \le&\mathrm{cap}(A,\mathcal{BV}_{G,p}(G))+\mathrm{cap}(B,\mathcal{BV}_{G,p}(G)).\end{aligned}$$

(iv) If $A_k, k=1,2,\cdots$, are subsets in G, then

$$\mathrm{cap}\Big(\bigcup_{k=1}^{\infty}A_k,\mathcal{BV}_{G,p}(G)\Big)\le\sum_{k=1}^{\infty}\mathrm{cap}(A_k,\mathcal{BV}_{G,p}(G)).$$

(v) For any sequence $\{A_k\}_{k=1}^{\infty}$ of subsets of G with $A_1\subseteq A_2\subseteq A_3\subseteq\cdots$,

$$\lim_{k\to\infty}\mathrm{cap}(A_k,\mathcal{BV}_{G,p}(G))=\mathrm{cap}\Big(\bigcup_{k=1}^{\infty}A_k,\mathcal{BV}_{G,p}(G)\Big).$$

(vi) If $A_k, k=1,2,\cdots$, are compact sets in $\mathbb{R}^d$ and $A_1\supseteq A_2\supseteq A_3\supseteq\cdots$, then

$$\lim_{k\to\infty}\mathrm{cap}(A_k,\mathcal{BV}_{G,p}(G))=\mathrm{cap}\Big(\bigcap_{k=1}^{\infty}A_k,\mathcal{BV}_{G,p}(G)\Big).$$

Proof. (i)-(ii). Statements (i) and (ii) are the evident consequences of Definition 3.1.
(iii) Without loss of generality, we may assume

$$\mathrm{cap}(A,\mathcal{BV}_{G,p}(G))+\mathrm{cap}(B,\mathcal{BV}_{G,p}(G))<\infty.$$

For any $\varepsilon>0$, there are two functions $\phi\in\mathcal{A}(A,\mathcal{BV}_{G,p}(G))$ and $\psi\in\mathcal{A}(B,\mathcal{BV}_{G,p}(G))$, such that

$$\begin{cases}\|\phi\|_{L^1}+\|D_{G,p}\phi\|(G)<\mathrm{cap}(A,\mathcal{BV}_{G,p}(G))+\dfrac{\varepsilon}{2},\\ \|\psi\|_{L^1}+\|D_{G,p}\psi\|(G)<\mathrm{cap}(B,\mathcal{BV}_{G,p}(G))+\dfrac{\varepsilon}{2}.\end{cases}$$

Let

$$\varphi_1=\max\{\phi,\psi\},\quad \varphi_2=\min\{\phi,\psi\}.$$

It is easy to see that

$$\varphi_1\in\mathcal{A}\Big(A\bigcup B,\mathcal{BV}_{G,p}(G)\Big),\quad \varphi_2\in\mathcal{A}\Big(A\bigcap B,\mathcal{BV}_{G,p}(G)\Big).$$

Then by Theorem 2.2,

$$\begin{aligned}&\mathrm{cap}\Big(A\bigcup B,\mathcal{BV}_{G,p}(G)\Big)+\mathrm{cap}\Big(A\bigcap B,\mathcal{BV}_{G,p}(G)\Big)\\ \le&\int_G\varphi_1(x)dx+\int_G\varphi_2(x)dx+\|D_{G,p}\varphi_1\|(G)+\|D_{G,p}\varphi_2\|(G)\\ \le&\int_G\phi(x)dx+\int_G\psi(x)dx\|D_{G,p}\phi\|(G)+\|D_{G,p}\psi\|(G)\\ \le&\mathrm{cap}(A,\mathcal{BV}_{G,p}(G))+\mathrm{cap}(B,\mathcal{BV}_{G,p}(G))+\varepsilon.\end{aligned}$$

The assertion (iii) is proved.

(iv) Suppose

$$\sum_{k=1}^{\infty} \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) < \infty.$$

For any $\varepsilon > 0$ and $k = 1, 2, \cdots$, there is $f_k \in \mathcal{A}(A_k, \mathcal{BV}_{G,p}(G))$ such that

$$\|f_k\|_{L^1} + \|D_{G,p} f_k\|(G) < \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) + \frac{\varepsilon}{2^k}.$$

Setting $f = \sup_k f_k$ gives

$$\int_G f(x)dx \le \sum_{k=1}^{\infty} \int_G f_k(x)dx < \sum_{k=1}^{\infty} \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) + \frac{\varepsilon}{2^k} < \infty,$$

which implies $f \in L^1(G)$.

Via the lower semicontinuity (2.2) of the Gaussian p variation we get

$$\begin{aligned}
\int_G f(x)dx + \|D_{G,p} f\|(G) &\le \sum_{k=1}^{\infty} \int_G f_k(x)dx + \lim\inf_{k\to\infty} \|D_{G,p} \max\{f_1, \cdots, f_k\}\|(G) \\
&\le \sum_{k=1}^{\infty} \int_G f_k(x)dx + \sum_{k=1}^{\infty} \|D_{G,p} f_k\|(G) \\
&\le \sum_{k=1}^{\infty} \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) + \varepsilon.
\end{aligned}$$

Then we have $f \in \mathcal{A}(\cup_{k=1}^{\infty} A_k, \mathcal{BV}_{G,p}(G))$ and this completes the proof of (iv) via letting $\varepsilon \to 0$.

(v) It is obvious that

$$\lim_{k\to\infty} \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) \le \operatorname{cap}\Big(\bigcup_{k=1}^{\infty} A_k, \mathcal{BV}_{G,p}(G)\Big).$$

The equality holds if

$$\lim_{k\to\infty} \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) = \infty.$$

Let $\varepsilon > 0$ and assume

$$\lim_{k\to\infty} \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) < \infty.$$

For $k = 1, 2, \cdots$, there is

$$f_k \in \mathcal{A}(A_k, \mathcal{BV}_{G,p}(G)),$$

such that

$$\|f_k\|_{L^1} + \|D_{G,p} f_k\|(G) < \operatorname{cap}(A_k, \mathcal{BV}_{G,p}(G)) + \frac{\varepsilon}{2^k}.$$

Set

$$\begin{cases} \phi_k = \max\limits_{1\le i\le k} f_i = \max\{\phi_{k-1},\ f_k\}, \\ \phi_0 = 0, \quad A_0 = \emptyset, \\ \varphi_k = \min\{\phi_{k-1},\ f_k\}. \end{cases}$$

Then

$$\phi_k, \varphi_k \in \mathcal{BV}_{G,p}(G), \quad A_{k-1} \subseteq int\{x \in G : \varphi_k(x) = 1\}.$$

Since $\phi_k = \max\{\phi_{k-1}, \phi_k\}$, an application of Theorem 2.2 derives

$$\begin{aligned} &\|D_{G,p}\max\{\phi_{k-1}, \phi_k\}\|(G) + \|D_{G,p}\min\{\phi_{k-1}, \phi_k\}\|(G) \\ \le &\|D_{G,p}\phi_{k-1}\|(G) + \|D_{G,p}\phi_k\|(G), \end{aligned}$$

and then

$$\begin{aligned} &\|\phi_k\|_{L^1} + \|D_{G,p}\phi_k\|(G) + \mathrm{cap}(A_{k-1}, \mathcal{BV}_{G,p}(G)) \\ \le &\|\phi_k\|_{L^1} + \|D_{G,p}\phi_k\|(G) + \|\varphi_k\|_{L^1} + \|D_{G,p}\varphi_k\|(G) \\ \le &\|\phi_k\|_{L^1} + \|\phi_{k-1}\|_{L^1} + \|D_{G,p}\phi_k\|(G) + \|D_{G,p}\phi_{k-1}\|(G) \\ \le &\|\phi_{k-1}\|_{L^1} + \|D_{G,p}\phi_{k-1}\|(G) + \mathrm{cap}(A_k, \mathcal{BV}_{G,p}(G)) + \frac{\varepsilon}{2^k}, \end{aligned}$$

where we have used the fact that $A_{k-1} \subseteq A_k$. Therefore,

$$\begin{aligned} &\|\phi_k\|_{L^1} + \|D_{G,p}\phi_k\|(G) - \|\phi_{k-1}\|_{L^1} - \|D_{G,p}\phi_{k-1}\|(G) \\ \le &\mathrm{cap}(A_k, \mathcal{BV}_{G,p}(G)) - \mathrm{cap}(A_{k-1}, \mathcal{BV}_{G,p}(G)) + \frac{\varepsilon}{2^k}. \end{aligned}$$

By adding the above inequalities from $k = 1$ to $k = j$, we get

$$\|\phi_j\|_{L^1} + \|D_{G,p}\phi_j\|(G) \le \mathrm{cap}(A_j, \mathcal{BV}_{G,p}(G)) + \varepsilon.$$

Let $\tilde{\phi} = \lim_{j\to\infty} \phi_j$. Via the monotone convergence theorem, we obtain

$$\int_G \tilde{\phi}(x)dx = \lim_{j\to\infty} \int_G \phi_j(x)dx \le \lim_{j\to\infty} \mathrm{cap}(A_j, \mathcal{BV}_{G,p}(G)) + \varepsilon.$$

Then via the lower semicontinuity (2.2) of the Gaussian p variation, we have

$$\tilde{\phi} \in \mathcal{A}\Big(\bigcup_{j=1}^{\infty} A_j, \mathcal{BV}_{G,p}(G)\Big)$$

and

$$\begin{aligned} \mathrm{cap}\Big(\bigcup_{j=1}^{\infty} A_j, \mathcal{BV}_{G,p}(G)\Big) &\le \|\tilde{\phi}\|_{L^1} + \|D_{G,p}\tilde{\phi}\|(G) \\ &\le \liminf_{j\to\infty} \Big(\int_G \phi_j(x)dx + \|D_{G,p}\phi_j\|(G)\Big) \\ &\le \lim_{j\to\infty} \mathrm{cap}(A_j, \mathcal{BV}_{G,p}(G)) + \varepsilon. \end{aligned}$$

(vi) Let $A=\cap_{k=1}^{\infty}A_k$. By monotonicity,

$$\mathrm{cap}\Big(\bigcap_{k=1}^{\infty}A_k,\mathcal{BV}_{G,p}(G)\Big)\le\lim_{k\to\infty}\mathrm{cap}(A_k,\mathcal{BV}_{G,p}(G)).$$

Let U be an open set containing A. Then by the compactness of A, we know that $A_k\subseteq U$ for all sufficiently large k. Therefore,

$$\lim_{k\to\infty}\mathrm{cap}(A_k,\mathcal{BV}_{G,p}(G))\le\mathrm{cap}(U,\mathcal{BV}_{G,p}(G)).$$

Corollary 3.1 implies that $\mathrm{cap}(\cdot,\mathcal{BV}_{G,p}(G))$ is an outer capacity. Then we obtain the claim by taking infimum over all open sets U containing A. □

Corollary 3.1. *(i) If $E\subseteq G$, then*

$$\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))=\inf_{\text{open }O\supseteq E}\Big\{\mathrm{cap}(O,\mathcal{BV}_{G,p}(G))\Big\}.$$

(ii) If $E\subseteq G$ is a Borel set, then

$$\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))=\sup_{\text{compact }K\subseteq E}\Big\{\mathrm{cap}(K,\mathcal{BV}_{G,p}(G))\Big\}.$$

Proof. (i) The statement (ii) of Theorem 3.2 implies

$$\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))\le\inf_{\text{open }O\supseteq E}\Big\{\mathrm{cap}(O,\mathcal{BV}_{G,p}(G))\Big\}.$$

To prove the reverse inequality, we may assume

$$\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))<\infty.$$

Via Definition 3.1, for any $\varepsilon>0$, there is $f\in\mathcal{A}(E,\mathcal{BV}_{G,p}(G))$ such that

$$\|f\|_{L^1}+\|D_{G,p}f\|(G)<\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))+\varepsilon.$$

Hence, there exists an open set $O\supseteq E$ such that $f=1$ on O, which implies

$$\mathrm{cap}(O,\mathcal{BV}_{G,p}(G))\le\|f\|_{L^1}+\|D_{G,p}f\|(G)<\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))+\varepsilon.$$

Therefore,

$$\mathrm{cap}(E,\mathcal{BV}_{G,p}(G))\ge\inf_{\text{open }O\supseteq E}\Big\{\mathrm{cap}(O,\mathcal{BV}_{G,p}(G))\Big\}.$$

(ii) This follows from (v) and (vi) of Theorem 3.2. □

Remark 3.1. In this paper, we generalize the BV functions and BV capacity on the classical Gauss space to the case of the stratified Lie group. There are many other problems which are worth studying, such as isoperimetric inequality and Sobolev inequality etc. We will investigate them in the future study.

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