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Compactness of the Commutators of Fractional Hardy Operator with Rough Kernel

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. The more explicit decomposition of the operator and the kernel are utilized to investigate a characterization of the central $BMO(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$ space via the compactness of the commutators of fractional Hardy operator with rough kernel.

Key Words: Fractional Hardy operator, commutator, compactness.
AMS Subject Classifications: 47B47, 47G10, 42B25

1 Introduction

Problem of commutators draws recently more and more attention of Harmonic analysis, such as its application in the study of elliptic equations [1,7]. For example, Sun, Wang and Zhang simplify the proof of the famous Wu's theorem on Navier-Stokes equations greatly in [18] and the technique used is some estimates for commutators by Lu and Yan [13]. The commutator formed by an operator T and a suitable function b can be recalled as

$$[b,T]f := b(Tf) - T(bf).$$

We call a function $b \in L_{loc}(\mathbb{R}^n)$ is a central $BMO(\mathbb{R}^n)$ (the mean oscillation function space) function, denoted by $CBMO(\mathbb{R}^n)$ which was introduced by Lu and Yang [14], if

$$\|b\|_{CBMO(\mathbb{R}^n)} := \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} |b(x) - b_{B_r}|dx < \infty.$$

Here and in what follows, $B_r := B(0,r)$ is a ball centered at 0 with radius $r > 0$. $CBMO(\mathbb{R}^n)$ can be understood as a local version of $BMO(\mathbb{R}^n)$ at the origin, $BMO(\mathbb{R}^n) \subset CBMO(\mathbb{R}^n)$ and they have quite different properties since for $1 < p < \infty$,

$$\|b\|_{BMO(\mathbb{R}^n)} \approx \|b\|_{BMO^p(\mathbb{R}^n)} \quad \text{and} \quad \|b\|_{CBMO(\mathbb{R}^n)} \lesssim \|b\|_{CBMO^p(\mathbb{R}^n)}$$

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with

$$\|b\|_{BMO^p(\mathbb{R}^n)} = \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|}\int_B |b(x)-b_B|^p dx\right)^{\frac{1}{p}},$$
$$\|b\|_{CBMO^p(\mathbb{R}^n)} = \sup_{r>0} \left(\frac{1}{|B_r|}\int_{B_r} |b(x)-b_{B_r}|^p dx\right)^{\frac{1}{p}}.$$

Thus, the John-Nirenberg inequality is not true for $CBMO(\mathbb{R}^n)$. We follow the notation used in the existed work: $VMO(\mathbb{R}^n)$ denotes the $BMO(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$ (the space of all functions being infinite-times continuously differential in $\mathbb{R}^n$ with compact support), $CVMO(\mathbb{R}^n)$ stands for the $CBMO(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$.

This paper provides a characterization of the $CVMO(\mathbb{R}^n)$ space by the compactness of $[b,T]$, when T is the following fractional Hardy operator

$$H_{\Omega,\alpha}f(x) = \frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|} \Omega(x-y)f(y)dy,$$
$$H^*_{\Omega,\alpha}f(x) = \int_{|y|\geq|x|} \frac{\Omega(x-y)f(y)}{|y|^{n-\alpha}}dy, \quad 0<\alpha<n.$$

Here Ω satisfies

$$\Omega(tx) = \Omega(x), \qquad \forall t>0, \quad x\in\mathbb{R}^n, \tag{1.1a}$$
$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1b}$$
$$\Omega\in L^q(\mathbb{S}^{n-1}), \qquad \forall q\geq 1. \tag{1.1c}$$

The $L^{q\geq 1}$-Dini condition of Ω can be recalled as

$$\int_0^1 \frac{w_q(\delta)}{\delta} < \infty \quad \text{with } w_q(\delta) = \sup_{\|\tau\|\leq\delta} \left(\int_{\mathbb{S}^{n-1}} |\Omega(\tau x') - \Omega(x')|^q d\sigma(x')\right)^{\frac{1}{q}}$$

and τ is a rotation on $\mathbb{S}^{n-1}$ with

$$\|\tau\| = \sup_{x'\in\mathbb{S}^{n-1}} |\tau x' - x'|.$$

For a suitable function h, $H^*_{\Omega,\alpha}$ is said to be the dual operator of $H_{\Omega,\alpha}$ in the following sense

$$\int_{\mathbb{R}^n} h(x)H_{\Omega,\alpha}f(x)dx = \int_{\mathbb{R}^n} f(x)H^*_{\Omega,\alpha}h(x)dx.$$

Fu, Lu and Zhao considered the boundedness of $H_{\Omega,\alpha}$ and $[b,H_{\Omega,\alpha}]$ on homogeneous Herz spaces and Lebesgue spaces for $b\in BMO(\mathbb{R}^n)$ in [11]. For $\Omega=1$, see for example [9,16].

The pioneer work on the compactness of operators can be traced to Uchiyama [19], where a characterization of $VMO(\mathbb{R}^n)$ via the compactness of $[b,T]$ with T is the classical Calderón-Zygmund singular integral operator is obtained. To date, much work has been reported in these field. For example, the compactness of $[b,T]$ on Lebesgue space when b is in an appropriately BMO space and T is the multiplication operator [2]; a characterization of $VMO(\mathbb{R}^n)$ by the compactness of $[b,T]$ when T is the parabolic singular integral [4]; the compactness theory of $[b,T]$ when T is the generalized Toeplitz operators by Krantzl and Li [12]; the characterizations of $VMO(\mathbb{R}^n)$ via the compactness of $[b,T]$ when T is the Riesz potential [5] and T is the singular integral operator [6] on Morrey type space; the compactness of $[b,T]$ for bilinear operators on Morrey spaces [8]; the characterization of $CVMO(\mathbb{R}^n)$ by compactness of $[b,T]$ when T is the classical Hardy operator and the Hardy operator with homogeneous kernels [10,15].

The know results for the function characterizations highly depended on the smoothness of Ω and there have been many attempts to weak the condition of Ω have been undertaken, see e.g., [19] for $\Omega\in Lip_1(\mathbb{S}^{n-1})$ (Lipschitz functional space), [3,4] for Ω satisfies

$$|\Omega(x')-\Omega(y')|\le \frac{A}{\left(\log\frac{2}{|x'-y'|}\right)^{\gamma}} \quad \text{with } A>0,\quad \gamma>1 \quad \text{and} \quad x',y'\in\mathbb{S}^{n-1}. \tag{1.2}$$

It is obvious that (1.2) is weaker than the Lipschitz condition $Lip_{0<\gamma\le 1}(\mathbb{S}^{n-1})$ and is stronger than (1.1c). Furthermore, if Ω satisfies (1.2), then for $q\ge 1$,

$$\int_0^1 \frac{w_q(\delta)}{\delta}(1+|\log\delta|)d\delta<\infty. \tag{1.3}$$

The major goal of this paper is to give the following characterization of $CVMO(\mathbb{R}^n)$ via the compactness of $[b,H_{\Omega,\alpha}]$ and $[b,H^*_{\Omega,\alpha}]$.

Theorem 1.1. *Let $0<\alpha<n$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, Ω satisfy (1.1a), (1.1b), (1.2) and $b\in BMO(\mathbb{R}^n)$. Then $b\in CVMO(\mathbb{R}^n)$ $\iff$ Both $[b,H_{\Omega,\alpha}]$ and $[b,H^*_{\Omega,\alpha}]$ are compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Remark 1.1. The assumption $b\in BMO(\mathbb{R}^n)$ in Theorem 1.1 can not be weakened in the proof of the necessity part since the John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ function is used and it is not true for $CBMO(\mathbb{R}^n)$. Since $H_{\Omega,\alpha}$ is centrosymmetric, the method used to consider the Calderón-Zygmund singular integral [4] can not be applied to $H_{\Omega,\alpha}$ directly.

Section 2 devoted to the basic lemmas for the proof Theorem 1.1; in Section 3, we shall give the proof of Theorem 1.1 by more general case.

In what follows, the symbol C stands for a positive constant which may vary from line to line. $A\lesssim B$ means $A\le CB$ and $A\simeq B$ whenever $A\lesssim B$ and $B\lesssim A$. $\mathbb{Z}$ denotes the set of all integers. $B_k:=B_{2^k}$, $C_k:=B_k\setminus B_{k-1}$ and $\chi_k:=\chi_{C_k}$ with $k\in\mathbb{Z}$.

2 Preparation

Four lemmas will be described in this section which are useful for the analysis of Theorem 1.1. We first recall the John-Nirenberg type inequality of $BMO(\mathbb{R}^n)$ function and some properties of $CBMO(\mathbb{R}^n)$ type function from [15, Lemma 2.1].

Lemma 2.1. *(a) Let $b \in BMO(\mathbb{R}^n)$. Then for $C_2 > C_1 > 2$ and $\forall x_0 \in \mathbb{R}^n$, there exist positive constants C_3, C_4, C_5 (depending on C_1, C_2 and b), such that*

$$\begin{aligned}&|\{C_1 r < |x - x_0| \langle C_2 r : |b(x) - b_{B(x_0,r)}| \rangle \nu + C_3\}| \\ \leq & C_4 |B(x_0, r)| e^{-C_5 \nu} \quad with \;\; 0 < \nu < \infty.\end{aligned} \tag{2.1}$$

(b) Write

$$\Phi(b, B_r) := \inf_{c \in \mathbb{R}} \frac{1}{|B_r|} \int_{B_r} |b(y) - c| dy$$

and assume that $b \in CBMO(\mathbb{R}^n)$, then $b \in CVMO(\mathbb{R}^n)$ if and only if b satisfies the following two conditions:

$$\lim_{r \to 0} \sup_r \Phi(b, B_r) = 0, \tag{2.2a}$$

$$\lim_{r \to \infty} \sup_r \Phi(b, B_r) = 0. \tag{2.2b}$$

(c) $\|b\|_{CBMO(\mathbb{R}^n)} \simeq \sup_r \Phi(b, B_r)$.

Some estimates for Ω will be concluded in the next lemma, part of which can be deduced from [10, Lemma 2.1] directly.

Lemma 2.2. *Let Ω satisfy* (1.1a) *and* (1.2)*. Then*
(a) $|\Omega(x - y) - \Omega(x)| \leq \frac{C}{(\log(|x|/|y|))^\gamma}$ with $|x| \geq 4|y|$ and γ be given in (1.2).
(b) if furthermore Ω satisfies the $L^{q\geq 1}$-Dini condition, then there is a constant $C > 0$ such that for $0 < C < 1/2$, $r > 0$, $x \in \mathbb{R}^n$ with $|x| < Cr$, one has

$$\begin{cases} \left(\displaystyle\int_{r<|y|<2r} |\Omega(y - x) - \Omega(y)|^q dy \right)^{1/q} \leq C r^{\frac{n}{q}} \displaystyle\int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta, \\ \left(\displaystyle\int_{r<|y|<2r} \frac{|\Omega(y - x) - \Omega(y)|^q}{|y|^{(n-\alpha)q}} dy \right)^{1/q} \leq C r^{-\frac{n}{q'}+\alpha} \displaystyle\int_{|x|/2r}^{|x|/r} \frac{w_q(\delta)}{\delta} d\delta. \end{cases}$$

Proof. We only need to show the second part of (b) since (a) and the first part of (b) is just [10, Lemma 2.1]. This can be done by the fact that Ω satisfies the L^q-Dini condition.

In fact,

$$
\begin{aligned}
&\left(\int_{r<|y|<2r}\frac{|\Omega(y-x)-\Omega(y)|^q}{|y|^{(n-\alpha)q}}dy\right)^{1/q}\\
=&Cr^{-\frac{n}{q'}+\alpha}\left(\int_r^{2r}\int_{\mathbb{S}^{n-1}}|\Omega(y'-t^{-1}x')-\Omega(y')|^q d\delta(y')\frac{dt}{t}\right)^{1/q}\\
\leq&Cr^{-\frac{n}{q'}+\alpha}\int_{|x|/2r}^{|x|/r}\frac{w_q(\delta)}{\delta}d\delta.
\end{aligned}
$$

Thus, we complete the proof. □

The following known estimates from [17] and [20] will help us to complete the proof of Theorem 1.1.

Lemma 2.3. *(a) Let $g(x)$ be a measurable function,*

$$\lambda(\mu)=|\{x\in\mathbb{R}^n:|g(x)|>\mu>0\}|$$

and S be a measurable set. Define

$$g^*(t)=\inf\{\mu:\lambda(\mu)\leq t\}\quad \textit{for}\quad t>0,$$

then

$$\int_S|g(x)|^p dx\leq\int_0^{|S|}|g^*(t)|^p dt\quad \textit{with}\ 1\leq p<\infty.$$

(b) Let $0<\alpha<n$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and Ω satisfy (1.1a) *and* (1.1c)*. Then both $H_{\Omega,\alpha}$ and $H^*_{\Omega,\alpha}$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

In the end of this section, we give the boundedness for the truncated operators of $H_{\Omega,\alpha}$ and $H^*_{\Omega,\alpha}$, which can be seen as a fractional case of [10, Lemma 2.5].

Lemma 2.4. *Suppose that $0<\alpha<n$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and set*

$$
\begin{cases}
H^{\eta}_{\Omega,\alpha}f(x)=\dfrac{1}{|x|^{n-\alpha}}\displaystyle\int_{S_1}\Omega(x-z)f(z)dz & \textit{with}\ S_1=\{z:|z|<|x|,\ |x-z|>\eta\},\\
H^{*,\eta}_{\Omega,\alpha}f(x)=\displaystyle\int_{S_2}\frac{\Omega(x-z)f(z)}{|z|^{n-\alpha}}dy & \textit{with}\ S_2=\{z:|z|\geq|x|,\ |x-z|>\eta\}.
\end{cases}
$$

If Ω satisfies (1.1a) *and the L^q-Dini condition, then $H^{\eta}_{\Omega,\alpha}$ and $H^{*,\eta}_{\Omega,\alpha}$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. We only give the outline of the proof since the similarity, more details see [10, Lemma 2.5]. It is sufficient to show that for $f\in L^p(\mathbb{R}^n)$, there are constants $C>0$ satisfying

$$\|H^{\eta}_{\Omega,\alpha}f\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)}\quad\text{and}\quad\|H^{*,\eta}_{\Omega,\alpha}f\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

Let us first prove the boundedness of $H^{\eta}_{\Omega,\alpha}$ after the decomposition that

$$|H^{\eta}_{\Omega,\alpha}f(x)|=|H_{\Omega,\alpha}f(y)-H_{\Omega,\alpha}f_1(y)-H_{\Omega,\alpha}f_2(y)+H^{\eta}_{\Omega,\alpha}f(x)|,$$

where $f_1=f\chi_{4B}$, $f_2=f-f_1$ and $B=B(x,\eta/4)$. Therefore,

$$\begin{aligned}|H^{\eta}_{\Omega,\alpha}f(x)|\leq&\frac{1}{|B|}\int_B|H_{\Omega,\alpha}f(y)|dy+\frac{1}{|B|}\int_B|H_{\Omega,\alpha}f_1(y)|dy\\&+\frac{1}{|B|}\int_B|H_{\Omega,\alpha}f_2(y)-H^{\eta}_{\Omega,\alpha}f(x)|dy\\\leq&M(H_{\Omega,\alpha}f)(x)+If(x)+IIf(x).\end{aligned}$$

Combining the L^p-boundedness of the maximal operator M, the (L^p,L^q)-boundeness of the fractional maximal operator M_α and the (L^p,L^q)-boundeness of $H_{\Omega,\alpha}$ [20], we get

$$\begin{aligned}&\|M(H_{\Omega,\alpha}f)\|_{L^q(\mathbb{R}^n)}\leq\|H_{\Omega,\alpha}f\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)},\\&\|If\|_{L^q(\mathbb{R}^n)}\leq C\|M_\alpha(|f|^p)^{\frac{1}{p}}\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)},\\&\|IIf\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

as desired.

The task is now to show the boundedness of $H^{*,\eta}_{\Omega,\alpha}$. Analysis similar to that in the proof of $H^{\eta}_{\Omega,\alpha}$ shows that

$$\begin{aligned}&|H^{*,\eta}_{\Omega,\alpha}f(x)|\leq M(H^{*}_{\Omega,\alpha}f)(x)+J(f)(x)+JJ(f)(x),\\&\|M(H^{*}_{\Omega,\alpha}f)\|_{L^q(\mathbb{R}^n)}\leq\|H^{*}_{\Omega,\alpha}f\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)},\\&\|Jf\|_{L^q(\mathbb{R}^n)}\leq C\|M_\alpha(|f|^p)^{\frac{1}{p}}\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Set $\widetilde{S_2}=\{z:|z|\geq y,\ |x-z|>\eta\}$, we obtain that

$$|JJf(x)|\leq\frac{C}{|B_{\frac{\eta}{4}}|}\int_{B_{\frac{\eta}{4}}}\left|\int_{\widetilde{S_2}}\frac{|\Omega(x-y-z)-\Omega(x-z)||f(z)|}{|z|^{n-\alpha}}dz\right|dy.$$

Accordingly, we conclude from the Minkowski inequality, Lemma 2.2 and the fact $|x-z|\geq 3|z|/4$ for $|x|=2^{k_0-1}\eta$, $|z|>\eta$ and $|y|<\eta/4$ that

$$\|JJf\|_{L^q(\mathbb{R}^n)}\leq C\|f\|_{L^p(\mathbb{R}^n)},$$

whence reaching the required estimation. $\square$

3 Proof of Theorem 1.1

We begin with the proof of the necessity of Theorem 1.1 which is partly inspired by [10, Theorem 4.1]. If $[b, H_{\Omega,\alpha}]$ and $[b, H^*_{\Omega,\alpha}]$ are both compact operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then [20, Theorem 1.1] implies that $b \in CBMO(\mathbb{R}^n)$. For simplicity, we assume that $\|b\|_{CBMO(\mathbb{R}^n)} = 1$. According to Lemma 2.1, we only need to prove that (2.2a)-(2.2b) holds for b. This consists of two steps. We follow the notation used in [10, Theorem 4.1].

Step 1-proving that b satisfies (2.2a). If not, then there exists a $\tau > 0$ and a sequence of balls $\{B_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} r_i = 0$, such that for any i, $\Phi(b, B_i) > \tau$. Upon writing

$$f_i(y) = \frac{1}{|B_i|^{\frac{1}{p}}} \left[sgn(b(y) - b_{B_i}) - a_0\right] \chi_{B_i}(y), \quad i = 1, 2, \cdots,$$

$$\text{with } a_0 = \frac{1}{|B_i|} \int_{B_i} sgn(b(y) - b_{B_i}) dy,$$

we find

$$\begin{cases} \operatorname{supp} f_i \subset B_i, \quad f_i(y)(b(y) - b_{B_i}) > 0, \\ |f_i(y)| \le 2|B_i|^{-\frac{1}{p}} \quad \text{with } y \in B_i, \\ \|f_i\|_{L^p(\mathbb{R}^n)} \le C, \quad \int_{\mathbb{R}^n} f_i(y) dy = 0. \end{cases} \tag{3.1}$$

The argument is completed by showing that $\{[b, H_{\Omega,\alpha}] f_i\}_{i=1}^{\infty}$ is not a compact set from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. From now on, C_k, $(k \in \mathbb{Z})$ stands for a positive constant depending only on Ω, p, α, τ with C_i, $(1 \le i < k)$. We continue to choose

$$\begin{cases} D = \left\{ x' \in \mathbb{S}^{n-1} : \Omega(x') \ge \dfrac{2A}{(\log(2/C_1))^{\gamma}} \right\} \quad \text{with } A,\ \gamma \text{ be the same as that of in (1.2)}, \\ E = \{x \in \mathbb{R}^n : |x| > C_2 r,\ x' \in D\} \quad \text{with } C_2 = 3C_1^{-1} + 1 > 4. \end{cases}$$

Using (1.1b) and (1.2), we obtain that there exists a $0 < C_1 < 1$ such that

$$\sigma(D) > 0, \quad |x| > C_2|y| \quad \text{for } y \in B_i, \quad x \in E.$$

In view of the fact that

$$\Omega(x') \ge \frac{2A}{(\log(2/C_1))^{\gamma}}$$

and (1.2), we are interested in finding that for $x' \in D$ and $y' \in \mathbb{S}^{n-1}$ with $|x' - y'| \le C_1$,

$$\Omega(y') = \Omega(x') - (\Omega(x') - \Omega(y')) \ge |\Omega(x')| - |\Omega(x') - \Omega(y')| \ge \frac{A}{(\log(2/C_1))^{\gamma}}.$$

This in turn implies that

$$\Omega((x-y)') \ge \frac{A}{(\log(2/C_1))^{\gamma}}.$$

And hence, we get from (3.1) that for $x \in E$,

$$H_{\Omega,\alpha}((b-b_{B_i})f_i)(x) \geq \frac{C}{|B_i|^{\frac{1}{p}}|x|^{n-\alpha}} \int_{B_i} (|b(y)-b_{B_i}| - a_0(b(y)-b_{B_i}))\,dy.$$

Consequently,

$$H_{\Omega,\alpha}\left((b-b_{B_i})f_i\right)(x) \geq \frac{C|B_i|^{1/p'}}{|x|^{n-\alpha}}\Phi(b,B_i) \geq \frac{C\tau|B_i|^{1/p'}}{|x|^{n-\alpha}}.$$

On the other hand, (3.1) and Hölder's inequality allow us to obtain

$$\begin{aligned}
&\left|H_{\Omega,\alpha}\left((b-b_{B_i})f_i\right)(x)\right| \\
\leq&\frac{1}{|x|^{n-\alpha}}\int_{B_i}\left|\Omega((x-y)')(b(y)-b_{B_i})f_i(y)\right|dy \\
\leq&\frac{C|B_i|^{1/p'}}{|x|^{n-\alpha}}\left(\frac{1}{|B_i|}\int_{B_i}|b(y)-b_{B_i}|^{p'}dy\right)^{1/p'}\left(\int_{B_i}|f_i(y)|^p dy\right)^{1/p},
\end{aligned}$$

namely,

$$\left|H_{\Omega,\alpha}\left((b-b_{B_i})f_i\right)(x)\right| \leq \frac{C|B_i|^{1/p'}}{|x|^{n-\alpha}}. \tag{3.2}$$

At the same time, Lemma 2.2(a) and (3.1) shows

$$\begin{aligned}
&\left|(b(x)-b_{B_i})\,H_{\Omega,\alpha}(f_i)(x)\right| \\
\leq&\frac{C|b(x)-b_{B_i}|}{|x|^{n-\alpha}}\int_{B_i}\frac{|f_i(y)|}{(\log(|x|/r_i))^{\gamma}}dy \leq \frac{C|b(x)-B_i||B_i|^{1/p'}}{|x|^{n-\alpha}\,(\log(|x|/r_i))^{\gamma}}.
\end{aligned}$$

This in turn implies that for $a > C_2$,

$$\left(\int_{\{|x|>ar_i\}}|(b(x)-b_{B_i})H_{\Omega,\alpha}(f_i)(x)|^q\,dx\right)^{1/q} \leq C\left(\log a\right)^{1-\gamma}a^{-\frac{n}{p'}},$$

where we used the fact that for $O=\{x: 2^m r_i<|x|<2^{m+1}r_i\}$,

$$\int_O|b(x)-b_{B_i}|^q dx \leq \int_O|b(x)-b_{2^m B_i}|^q dx + \int_O|b_{2^m B_i}-b_{B_i}|^q dx \leq Cm^q|2^m B_i|.$$

Upon setting $W=\{x: ar_i<|x|<br_i\}$, we find according to the above analysis that for $b>a>C_2$

$$\begin{aligned}
&\left(\int_W|[b,H_{\Omega,\alpha}]f_i(x)|^q dx\right)^{1/q} \\
\geq&C\tau|B_i|^{\frac{1}{p'}}\left(\int_{W\cap\{x:x'\in D\}}\frac{dx}{|x|^{(n-\alpha)q}}\right)^{1/q} - C\left(\log a\right)^{1-\gamma}a^{-\frac{n}{p'}} \\
\geq&C\tau\left(a^{-\frac{nq}{p'}}-b^{-\frac{nq}{p'}}\right)^{1/q} - C\left(\log a\right)^{1-\gamma}a^{-\frac{n}{p'}}.
\end{aligned}$$

At the same time, (3.2) shows that

$$
\begin{aligned}
&\left(\int_{\{|x|>br_i\}}|[b,H_{\Omega,\alpha}]f_i(x)|^q dx\right)^{1/q}\\
\leq&\left(\int_{\{|x|>br_i\}}\frac{|B_i|^{\frac{q}{p'}}}{|x|^{(n-\alpha)q}}dx\right)^{1/q}+C\left(\log b\right)^{1-\gamma}b^{-\frac{n}{p'}}\\
\leq& Cb^{-\frac{n}{p'}}+C\left(\log b\right)^{1-\gamma}b^{-\frac{n}{p'}}.
\end{aligned}
$$

Accordingly, there are constants $C_3>C_2,C_5$ and $C:=C(\Omega,p,n,\alpha,\tau)>1$ with $C_4=CC_3$ such that

$$\left(\int_{\{C_3r_i<|x|<C_4r_i\}}|[b,H_{\Omega,\alpha}]f_i(x)|^q dx\right)^{1/q}\geq C_5, \tag{3.3a}$$

$$\left(\int_{\{|x|>C_4r_i\}}|[b,H_{\Omega,\alpha}]f_i(x)|^q dx\right)^{1/q}\leq \frac{C_5}{4}. \tag{3.3b}$$

Set $S\subset\{x: C_3r_i<|x|<C_4r_i\}$ be an arbitrary measurable set. An application of the Minkowski inequality shows that

$$\left(\int_S|[b,H_{\Omega,\alpha}]f_i(x)|^q dx\right)^{1/q}\leq C\left(\frac{|S|}{|B_i|}\right)^{1/q}+C\left(\frac{1}{|B_i|}\int_S|b(x)-b_{B_i}|^q dx\right)^{1/q}. \tag{3.4}$$

Setting

$$g_i(x)=b(x)-b_{B_i}\quad\text{and}\quad \lambda_{g_i}(t)=\left|\{C_5r_i<|x|<C_6r_i:|g_i(x)|>t\}\right|,\quad 0<t<\infty,$$

we obtain from Lemma 2.1 that there are constants C_6, C_7 and C_8 such that

$$\lambda_{g_i}(t+C_6)\leq C_7|B_i|e^{-C_8t}\ \Rightarrow\ \lambda_{g_i}(t)\leq C_7|B_i|e^{-C_8(t-C_6)}.$$

Upon choosing $g_i^*(\mu)=\inf\{t:\lambda_{g_i}(t)\leq\mu\}$, it is easy to check that for $0<\mu<C_7|B_i|$,

$$g_i^*(\mu)\leq\frac{1}{C_8}\ln\frac{C_7|B_i|}{\mu}+C_6.$$

Using Lemma 2.3, we get that for $|S|\ll C_7|B_i|$,

$$
\begin{aligned}
\frac{1}{|B_i|}\int_S|b(x)-b_{B_i}|^q dx\leq&\frac{1}{|B_i|}\int_0^{|S|}|g_i^*(\mu)|^q d\mu\\
\leq&\frac{C|S|}{|B_i|}\left|1+\ln(C_7|B_i|/|S|)\right|^{[q]+1}.
\end{aligned}\tag{3.5}
$$

Eqs. (3.4) and (3.5) imply that there is a $C_9 < \min\{C_7^{\frac{1}{n}}, C_4\}$ such that for $|S|/|B_i| < C_9^n$,

$$\begin{aligned}
&\left(\int_S |[b,H_{\Omega,\alpha}]f_i(x)|^q dx\right)^{1/q}\\
\le& C\left(\frac{|S|}{|B_i|}\right)^{1/q} + C\left(\frac{|S|}{|B_i|}\left(1+\ln\frac{C_7|B_i|}{|S|}\right)^{[q]+1}\right)^{1/q} \le \frac{C_5}{4}.
\end{aligned}$$

Picking a subsequence $\{B_{i(m)}\}_m$ from $\{B_i\}$ with $r_{i(m+1)}/r_{i(m)} < C_9/C_4$, we concluded that for $k>0$,

$$\begin{aligned}
&\|[b,H_{\Omega,\alpha}]f_{i(m)} - [b,H_{\Omega,\alpha}]f_{i(m+k)}\|_{L^q(\mathbb{R}^n)}\\
\ge& \left(\int_{\mathbb{G}_1} |[b,H_{\Omega,\alpha}]f_{i(m)}(x)|^q dx\right)^{1/q} - \left(\int_{\mathbb{G}_2} |[b,H_{\Omega,\alpha}]f_{i(m+k)}(x)|^q dx\right)^{1/q},
\end{aligned}$$

where

$$\begin{cases}
\mathbb{G}_1 = \{x : C_5 r_{i(m)} < |x| < C_6 r_{i(m)}\} \setminus \{x : |x| \le C_6 r_{i(m+k)}\} = \mathbb{G} - (\mathbb{G}_2^c \cap \mathbb{G}),\\
\mathbb{G}_2 = \{x : |x| > C_6 r_{i(m+k)}\},\\
\mathbb{G} = \{x : C_5 r_{i(m)} < |x| < C_6 r_{i(m)}\}.
\end{cases}$$

From (3.3) and what already been proved, we conclude that

$$\|[b,H_{\Omega,\alpha}]f_{i(m)} - [b,H_{\Omega,\alpha}]f_{i(m+k)}\|_{L^q(\mathbb{R}^n)} \ge \left(C_5^p - \left(\frac{C_5}{4}\right)^q\right)^{1/q} - \frac{C_5}{4} \ge \frac{C_5}{4},$$

which clearly shows that $\{[b,H_{\Omega,\alpha}]f_{i(m)}\}_{m=1}^{\infty}$ does not have any convergence subsequence in $L^q(\mathbb{R}^n)$. This in turn implies that $[b,H_{\Omega,\alpha}]$ is not a compact operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Therefore, b satisfies (2.2a) by the contradiction.

Step 2-showing that b satisfies (2.2b). This step can be handled in much the same way as the argument for (2.2a), the only difference being in choosing a sequence $\{B_i\}_i$ such that

$$\Phi(b,B_i) > \tau \quad \text{with} \quad \lim_{i\to\infty} r_i = +\infty.$$

We proceed to show the sufficiency of Theorem 1.1, which can be deduced by the following more general form.

Theorem 3.1. *Suppose that*

$$\begin{cases}
0<\alpha<n, \quad \dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n},\\
\Omega \ \textit{satisfies} \ \text{(1.1a)} \ \textit{and} \ \text{(1.3)},\\
b \in CVMO(\mathbb{R}^n),
\end{cases}$$

then both $[b,H_{\Omega,\alpha}]$ and $[b,H^*_{\Omega,\alpha}]$ are compact operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

To prove Theorem 3.1, the following two lemmas are needed. We first recall the well known Frechet-Kolmogorov theorem as

Lemma 3.1. *Let a set $S \subset L^p(\mathbb{R}^n)$ and $G_\alpha = \{x \in \mathbb{R}^n : |x| > \beta\}$. Then S is strongly pre-compact, if and only if,*

$$\sup_{f\in S} \|f\|_{L^p(\mathbb{R}^n)} < \infty, \tag{3.6a}$$

$$\lim_{|y|\to 0} \|f(\cdot+y) - f(\cdot)\|_{L^p(\mathbb{R}^n)} = 0 \quad \text{uniformly in } f \in S, \tag{3.6b}$$

$$\lim_{\beta\to\infty} \|f\chi_{G_\beta}\|_{L^p(\mathbb{R}^n)} = 0 \quad \text{uniformly in } f \in S. \tag{3.6c}$$

Next, we give the second lemma which can simplify the proof of Theorem 3.1 by considering $b \in C_c^\infty(\mathbb{R}^n)$ ([10, Lemma 4.4]).

Lemma 3.2. *Assume that $[b, T]$ is a compact operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $b \in C_c^\infty(\mathbb{R}^n)$, then $[b, T]$ is also a compact operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $b \in CVMO(\mathbb{R}^n)$.*

We are now in a position to complete the proof of Theorem 3.1. For $b \in C_c^\infty(\mathbb{R}^n)$, we are about to show (3.6a)-(3.6c) for

$$S_1 = \{[b, H_{\Omega,\alpha}]f : f \in Q\} \quad \text{and} \quad S_2 = \left\{[b, H^*_{\Omega,\alpha}]f : f \in Q\right\},$$
$$\text{with } Q = \{f : f \in L^p(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{L^p(\mathbb{R}^n)} \le C\}.$$

The fact $b \in C_c^\infty(\mathbb{R}^n)$ allows us to have

$$\sup_{f\in Q} \|[b, H_{\Omega,\alpha}]f\|_{L^q(\mathbb{R}^n)} \le C\|b\|_{CBMO(\mathbb{R}^n)} \sup_{f\in Q} \|f\|_{L^p(\mathbb{R}^n)} < \infty$$

and to obtain (3.6a).

Next, to show (3.6b), we only need to prove that for any $\varepsilon > 0$ and $|z|$ small enough,

$$\|[b, H_{\Omega,\alpha}]f(\cdot+z) - [b, H_{\Omega,\alpha}]f(\cdot)\|_{L^q(\mathbb{R}^n)} \le C\varepsilon, \quad \forall f \in Q. \tag{3.7}$$

For $0 < \varepsilon < 1/2$, setting

$$\begin{cases} E_1 = \left\{y : |y| < |x+z|,\ |x-y| > e^{\frac{1}{\varepsilon}}|z|\right\}, & E_2 = \left\{y : |y| < |x+z|,\ |x-y| \le e^{\frac{1}{\varepsilon}}|z|\right\}, \\ E_3 = \left\{y : |y| < |x|,\ |x-y| > e^{\frac{1}{\varepsilon}}|z|\right\}, & E_4 = \left\{y : |y| < |x|,\ |x-y| \le e^{\frac{1}{\varepsilon}}|z|\right\}, \end{cases}$$

we achieve that for $z \in \mathbb{R}^n$,

$$|[b, H_{\Omega,\alpha}]f(x+z) - [b, H_{\Omega,\alpha}]f(x)| = K_1^b f + K_2^b f + K_3^b f - K_4^b f,$$

where

$$\begin{cases} K_1^b f = \dfrac{1}{|x|^{n-\alpha}} \displaystyle\int_{E_3} [\Omega(x-y)(b(x+z)-b(x))] f(y)dy, \\ K_2^b f = \dfrac{1}{|x|^{n-\alpha}} \displaystyle\int_{E_3} [\Omega(x-y)(b(y)-b(x+z))] f(y)dy \\ \qquad\quad - \dfrac{1}{|x+z|^{n-\alpha}} \displaystyle\int_{E_1} [\Omega(x+z-y)(b(y)-b(x+z))] f(y)dy, \\ K_3^b f = \dfrac{1}{|x|^{n-\alpha}} \displaystyle\int_{E_4} [\Omega(x-y)(b(y)-b(x))] f(y)dy, \\ K_4^b f = \dfrac{1}{|x+z|^{n-\alpha}} \displaystyle\int_{E_2} [\Omega(x+z-y)(b(y)-b(x+z))] f(y)dy. \end{cases}$$

Combining $b \in C_c^\infty(\mathbb{R}^n)$, $|b(x+z)-b(x)| \leq C|z|$ and Lemma 2.4, we obtain that for $f \in Q$,

$$\|K_1^b f\|_{L^q(\mathbb{R}^n)} \leq C|z| \left\| H_{\Omega,\alpha}^{e^{\frac{1}{\varepsilon}}|z|} f \right\|_{L^q(\mathbb{R}^n)} \leq C|z| \|f\|_{L^p(\mathbb{R}^n)} \leq C|z|.$$

By Lemma 2.2 and Minkoski's inequality, one has

$$\begin{aligned} \|K_2^b f\|_{L^q(\mathbb{R}^n)} \leq & \left(\int_{x:|x-y|>e^{\frac{1}{\varepsilon}}|z|} \left| \frac{1}{|x|^{n-\alpha}} \int_{\widetilde{U_1}} [\Omega(y)-\Omega(y+z)] f(x-y)dy \right|^q dx \right)^{1/q} \\ \leq & C \sum_{k=0}^{\infty} \frac{1}{1+k+\frac{1}{\varepsilon}} \int_{\frac{1}{2^{k+1}e^{\frac{1}{\varepsilon}}}}^{\frac{1}{2^k e^{\frac{1}{\varepsilon}}}} \frac{w_q(\delta)}{\delta} (1+|\log\delta|) d\delta \\ \leq & C\varepsilon, \end{aligned}$$

where

$$\widetilde{E_1} := \left\{ y : 2^{k+1} e^{\frac{1}{\varepsilon}}|z| < |y| < 2^k e^{\frac{1}{\varepsilon}}|z| \right\}.$$

After the observation $|b(x)-b(y)| \leq C|x-y|$ for $|x-y|<1$, we can estimate K_3^b as

$$|K_3^b f| \leq \frac{C}{|x|^{n-\alpha}} \int_{E_4} |\Omega(x-y) f(y)|x-y||dy \leq \frac{C}{|x|^{n-\alpha-1}} \int_{E_4} |\Omega(x-y) f(y)|dy.$$

Hence, a further application of the Minkoski inequality and the Hölder inequality show that

$$\begin{aligned} \|K_3^b f\|_{L^q(\mathbb{R}^n)} \leq & C \left(\int_{x:|x-y|\leq e^{\frac{1}{\varepsilon}}|z|} \left| \frac{1}{|x|^{n-\alpha-1}} \int_{E_4} |\Omega(x-y) f(y)|dy \right|^q dx \right)^{1/q} \\ \leq & C(e^{\frac{1}{\varepsilon}}|z|)^q. \end{aligned}$$

Since

$$|x-y+z| \leq (e^{\frac{1}{\varepsilon}}+1)|z| < 1, \quad |b(x+z)-b(y)| \leq C|x+z-y|,$$

we can estimate $K_4^b f$ as follows,

$$
\begin{aligned}
\|K_4^b f\|_{L^q(\mathbb{R}^n)} \leq & C\left(\int_{\mathbb{R}^n}\left|\int_{E_2}\frac{|\Omega(x+z-y)f(y)|}{|x+z-y|^{n-\alpha-1}}dy\right|^q dx\right)^{1/q}\\
\leq & C\|f\|_{L^p(\mathbb{R}^n)}\left(\int_{E_2}dy\right)^{\frac{1}{p'}}\left(\int_{\left\{x:|x-y|<e^{\frac{1}{\varepsilon}}|z|\right\}}\frac{|\Omega(x+z-y)|^q}{|x+z-y|^{(n-\alpha-1)q}}dx\right)^{\frac{1}{q}}\\
\leq & C\left((e^{\frac{1}{\varepsilon}}+1)|z|\right)^q.
\end{aligned}
$$

So, (3.7) is obtained thanking to

$$
\lim_{|z|\to 0}\|[b,H_{\Omega,\alpha}]f(x)-[b,H_{\Omega,\alpha}]f(x+z)\|_{L^q(\mathbb{R}^n)}=0 \quad \text{uniformly in } f\in Q.
$$

Next, we finish the consideration of S_1 by showing (3.6c). To do so, we first choose β large enough such that

$$
\left(\int_\beta^\infty \frac{1}{t^{(n-\alpha)q-n+1}}dt\right)^{\frac{1}{s}}<\varepsilon, \quad \forall\varepsilon>0, \quad s>1,
$$

and denote by $U:=\operatorname{supp}(b)\subset\{x:|x|<r\}$ for some $r>0$. Then for $|x|>\max\{\beta,4r\}$ and $f\in Q$, apply the Hölder inequality to $\frac{1}{s}+\frac{1}{p}+\frac{1}{q}=1$, one has

$$
\begin{aligned}
|[b,H_{\Omega,\alpha}]f(x)|\leq & \frac{C}{|x|^{n-\alpha}}\int_U |b(y)\Omega(x-y)f(y)|dy\\
\leq & \frac{C\|f\|_{L^p(\mathbb{R}^n)}}{|x|^{n-\alpha}}\left(\int_U|\Omega(x-y)|^q dy\right)^{\frac{1}{q}}.
\end{aligned}
$$

Thereby reaching (3.6c) by the Minkoskin inequality and the fact $|x-y|>3|x|/4$ as

$$
\begin{aligned}
\left(\int_{|x|>\beta}|[b,H_{\Omega,\alpha}]f(x)|^q dx\right)^{1/q}\leq & C\left(\int_{|x|>\beta}\left|\frac{1}{|x|^{n-\alpha}}\int_U|\Omega(x-y)|^q dy\right|dx\right)^{1/q}\\
\leq & C\left(\int_\beta^\infty\frac{dt}{t^{(n-\alpha)q-n+1}}\int_{\mathbb{S}^{n-1}}|\Omega(y')|^q d\sigma(y')\right)^{\frac{1}{q}}\\
\leq & C\varepsilon.
\end{aligned}
$$

Similar arguments apply to S_2, we have

$$
\sup_{f\in Q}\|[b,H^*_{\Omega,\alpha}]f\|_{L^q(\mathbb{R}^n)}\leq C\sup_{f\in Q}\|f\|_{L^p(\mathbb{R}^n)}\leq C<\infty,
$$

whence finding (3.6a). Since $\{y:|y|\geq|x|\}\cap\{y:|y|<R\}=\phi$ if $U:=\operatorname{supp}(b)\subset\{y:|y|<r\}$ for some $r>0$ and x satisfying $|x|>\max\{\beta,4r\}$ in this case, (3.6c) is obviously.

It is sufficient to prove (3.6b) for S_2. We are about to show that for any $\varepsilon > 0$, $f \in Q$ and $|z|$ small enough,

$$\left\|[b, H^*_{\Omega,\alpha}]f(\cdot + z) - [b, H^*_{\Omega,\alpha}]f(\cdot)\right\|_{L^q(\mathbb{R}^n)} \leq C\varepsilon.$$

The rest of the proof runs as that of S_1 with a slight modification. We omit here for the similarity. We completes the proof of Theorem 3.1. □

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