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Borderline Weighted Estimates for Commutators of Fractional Integrals

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. Let $I_{\alpha,\vec{b}}$ be the multilinear commutators of the fractional integrals I_α with the symbol $\vec{b}=(b_1,\cdots,b_k)$. We show that the constant of borderline weighted estimates for I_α is $\frac{1}{\varepsilon}$, and for $I_{\alpha,\vec{b}}$ is $\frac{1}{\varepsilon^{k+1}}$ with each b_i belongs to the Orlicz space $Osc_{\exp L^{s_i}}$.

Key Words: Commutators, fractional integrals, borderline weighted estimates, Fefferman-Stein inequality.

AMS Subject Classifications: 42B25, 47G10

1 Introduction

Let M be the Hardy-Littlewood maximal function, which is defined by

$$Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)|dy,$$

where the supremum is taken over all cubes Q containing x in $\mathbb{R}^n$ with the sides parallel to the coordinate axes. Since the 1930s, there have been many outstanding works in the study of the Hardy-Littlewood maximal function. Among such achievements are the celebrated works of Hardy, Littlewood and Wiener, Fefferman and Stein [9], and Muckenhoupt [13]. Recall that the Hardy-Littlewood-Wiener theorem states that M is bounded

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from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ $(1 < p \le \infty)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, and the Fefferman-Stein inequality [9] can be expressed in the way that

$$\|Mf\|_{L^{1,\infty}(\omega)} \le C\int_{\mathbb{R}^n}|f|M\omega dx. \tag{1.1}$$

The question whether one can extend inequality (1.1) to other type of operators, such as the Hilbert transforms and the Calderón-Zygmund singular intergrals, is known as Muckenhoupt and Wheeden conjecture. In 2012, Reguera and Thiele [22] surprisingly showed that the Muckenhoupt and Wheeden conjecture was not true for the Hilbert transform, which fully indicates that the Hilbert transform does not enjoy the similar weak type inequality as in (1.1). In 1994, Pérez [17] obtained the following less fine inequaltiy for the Calderón-Zygmund singular intergrals.

$$\|Tf\|_{L^{1,\infty}(\omega)} \le C_{\epsilon,T}\int_{\mathbb{R}^n}|f|M_{L(\log L)^\epsilon}\omega dx, \quad \omega\ge 0, \quad \epsilon>0. \tag{1.2}$$

Since then, efforts have been made to clarify and separate the constant $C_{\epsilon,T}$. It was Hytönen and Pérez [10] who first showed that the constant can be gained is ε^{-1} for T and its corresponding maximal singular integral operators T^*. Recently, Domingo-Salazar, Lacey, Rey [8] generalized the results in [10] and further proved that T^* is bounded as a map from $L^1(M_{L\log\log L(\log\log\log L)^\alpha}w)$ into weak-$L^1(w)$ for $1<\alpha<2$ and the constant can be obtained is $(\alpha-1)^{-1}$.

Now we turn to the background of the commutators of T, which can be traced back to the celebrated works of Coifman, Rochberg and Weiss [3]. For a suitable smooth function f, the commutator of T is defined as $[b,T]f=T(bf)-bT(f)$. In [3], the authors proved that if b belongs to BMO($\mathbb{R}^n$), then [b,T] is bounded from $L^p(\mathbb{R}^n)$ onto itself $(1<p<\infty)$. Conversely, if all commutators of Riesz transform $[R_j,b]$, $1\le j\le n$, are L^p bounded, then $b\in$ BMO($\mathbb{R}^n$).

In 1995, Pérez [18] pointed out that the commutators of CZOs are not weak type $(1,1)$ operators. As a replacement, he gave the following $L\log L$ endpoint estimate:

$$\omega\big(\{x\in\mathbb{R}^n:|[b,T]f(x)|>\lambda\}\big) \le C_{\|b\|_{BMO}}[w]_{A_1}\int_{\mathbb{R}^n}\Phi\big(|f|/\lambda\big)\omega dx, \tag{1.3}$$

where $\Phi(t)=t(1+\log^+t)$, $\omega\in A_1$.

Quite naturally, one may ask whether the commutators $[b,T]$ still enjoy the similar inequality as in (1.2) or not. In 2001, Pérez and Pradolin [19] established the following inequality for arbitrary non negative weights w.

$$\omega\big(\{x\in\mathbb{R}^n:|[b,T]f(x)|>\lambda\}\big) \le C_{T,\varepsilon}\int_{\mathbb{R}^n}\Phi\big(|f|\|b\|_{BMO}/\lambda\big)M_{L(\log L)^{1+\varepsilon}}\omega dx. \tag{1.4}$$

In 2017, Pérez et al [20] further figured out that the constant in (1.4) is $\frac{C_T}{\varepsilon^2}$, that is

$$\omega\big(\{x\in\mathbb{R}^n:|[b,T]f(x)|>\lambda\}\big) \le \frac{C_T}{\varepsilon^2}\int_{\mathbb{R}^n}\Phi\big(|f|\|b\|_{BMO}/\lambda\big)M_{L(\log L)^{1+\varepsilon}}\omega dx. \tag{1.5}$$

In this paper, our object of investigation is the fractional integral and its commutators. It is well known that the study of weighted estimates of fractional integrals orginated from the works of Muckenhoupt and Wheeden [14] in 1974. They proved that the fractional integral operator was of strong type $(L^p(w^p), L^q(w^q))$ if $p > 1$ and of weak type $(L^1(w), L^{\frac{n}{n-\alpha},\infty}(w^{\frac{n}{n-\alpha}}))$ if $p = 1$. In 2001, endpoint $L\log L$ type estimates for the commutators of the fractional integral were studied by Ding, Lu and Zhang [7]. Later on, Carro [1] considered borderline weighted estimates for fractional integrals and he proved that

$$\omega\big(\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}\big) \le \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)| M_{\Psi,\alpha} u(x) dx, \tag{1.6}$$

where $\Psi(t) = t\log(e+t)^{1+\epsilon}$. Some other weighted results can be found in the works of Cruz-Uribe and Fiorenza [5], Chen and Xue [2].

This paper is concerned with the borderline weighted estimates of the fractional integral and its commutators, for such particular inequalities as (1.2) and (1.5). We show that the constants in the norm inequalities of I_α and $I_{\alpha,b}$ are still $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^2}$ (just take $k = 1$ in Theorem 1.2), respectively.

The main results of this paper are:

Theorem 1.1. *Let $0 < \epsilon < 1$ and ω be a weight, then for any $\alpha \in (0, \frac{n\epsilon}{2(2+\epsilon)})$, $c = \frac{2n}{2n+\alpha(\epsilon+2)}$, there exists a constant C_{I_α} such that*

$$\|I_\alpha f\|_{L^{1,\infty}(\omega)} \le \frac{C_{I_\alpha}}{\epsilon}\Big[\Big(\int_{\mathbb{R}^n} |f(x)|\big(M^c_{L(\log L)^\epsilon}\omega(x)dx\Big)^{\frac{1}{c}} + \int_{\mathbb{R}^n} |f(x)|M\omega(x))dx\Big].$$

Theorem 1.2. *Let w be a weight, $\vec{b} = (b_1, \cdots, b_k)$, $0 < \epsilon < 1$, $\phi_\rho(t) = t(1+\log^+ t)^\rho$, $\rho > 0$. For $\alpha \in (0, \frac{5n\epsilon}{24(1+\frac{1}{s})+10\epsilon})$,*

$$c = \frac{12n(1+\frac{1}{s})}{12(n+\alpha)\Big(1+\frac{1}{s}\Big)+5\alpha\epsilon}, \quad u = M_{L(\log L)^{\frac{1}{s}+\epsilon},\alpha}\omega(x), \quad v = M_{L(\log L)^{\frac{1}{s}+\epsilon}}\omega(x),$$

there exits a constant C_{I_α} such that

$$\begin{aligned}&\omega\Big(\Big\{x \in \mathbb{R}^n : |I_{\alpha,\vec{b}} f(x)| > \lambda\Big\}\Big)\\ \le&\frac{C_{I_\alpha}}{\epsilon^{k+1}}\Big\{\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\big(\|\vec{b}\||f(x)|/\lambda\big)(u+v)dx + \Big(\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}(\|\vec{b}\||f(x)|/\lambda^c)v^c dx\Big)^{\frac{1}{c}}\\ &+\Big((1+\lambda^{1-c})\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}(\|\vec{b}\||f(x)|/\lambda)v^c dx\Big)^{\frac{1}{c}}\Big\}.\end{aligned}$$

Remark 1.1. When $p = q$, the above results are consistent with the result in [20].

The article is organized as follows. In Section 2, some definitions and basic lemmas will be given. Section 3 will be devoted to demonstrate Theorem 1.1 and Theorem 1.2.

2 Definitions and main lemmas

We begin by introducing some definitions and notations.

Definition 2.1 (Orlicz spaces, [20]). *The Orlicz spaces are defined in the way that*

$$Osc_{\exp L^s} = \Big\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{Osc_{\exp L^s}} < \infty \Big\},$$

where

$$\|f\|_{Osc_{\exp L^s}} = \sup \|f - f_Q\|_{\Psi_s, Q}, \quad \Psi_s(t) = e^{t^s} - 1$$

with $t \geq 0$, $s > 0$.

From the John-Nirenberg's theorem, it is known that $BMO = Osc_{\exp L}$ and for every $s > 1$, it holds that $Osc_{\exp L^s} \subsetneqq BMO$.

Definition 2.2 (Young functions, Orlicz maximal functions, [6]). *A function Φ is called a Young function, if it is a continuous, nonnegative, strictly increasing and convex function defined on $[0,\infty)$ such that*

$$\Phi(0) = 0 \quad and \quad \lim_{t\to\infty} \Phi(t) = \infty.$$

The local Luxembourg norm of a function f with respect to Φ is defined in the way that

$$\|f\|_{\Phi,Q} = \|f\|_{\Phi(L),Q} = \inf \Big\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi\Big(\frac{|f(x)|}{\lambda}\Big) dx \leq 1 \Big\},$$

which is equivalent to

$$\|f\|'_{\Phi,Q} = \inf_{\mu>0} \Big\{ \mu + \frac{\mu}{|Q|} \int_Q \Phi\Big(\frac{|f(x)|}{\mu}\Big) dx \Big\}.$$

The Orlicz maximal function associated to Φ is defined by

$$M_{\Phi,\alpha} f(x) = \sup_{x\in Q} |Q|^{\frac{\alpha}{n}} \|f\|_{\Phi,Q} \quad for \quad \alpha \geq 0.$$

Let

$$\Phi_\rho(t) = t(1 + \log^+(t))^\rho \quad with \quad \log^+(t) = \chi_{(1,\infty)}(t) \log(t)$$

and $\rho > 0$, $t \geq 0$. Then we use the notation

$$\|f\|_{\Phi,Q} = \|f\|_{L(\log L)^\rho, Q}.$$

Definition 2.3 (Commutators of I_α). *Let $b_i \in Osc_{\exp L^{s_i}}, s_i \geq 1$, $i = 1, \cdots, k$, $k \in \mathbb{N}^+$. The symbol-multilinear commutators of the fraction integral I_α with respect to the symbol $\vec{b} = (b_1, \cdots, b_k)$ is defined as follows :*

$$I_{\alpha,\vec{b}} f(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} \prod_{i=1}^k (b_i(x) - b_i(y)) f(y) \, dy.$$

Definition 2.4 (A_p weights, [13]). *A weight ω belongs to the class A_p, $1<p<\infty$, if*

$$[\omega]_{A_p}=\sup_Q\Big(\frac{1}{|Q|}\int_Q\omega\Big)\Big(\frac{1}{|Q|}\int_Q\omega^{-\frac{1}{p-1}}\Big)^{p-1}<\infty.$$

A weight w belongs to the class A_1 if there is a finite constant C such that

$$\frac{1}{|Q|}\int_Q\omega(y)dy\le C\inf_Q\omega,$$

and the infimum of these constants C is called the A_1 constant of w denoted by $[\omega]_{A_1}$.

We will need to use the following Hölder inequality in our proof later.

Lemma 2.1 (Hölder inequality, [21]). *Let $\Phi_0,\Phi_1,,\cdots,\Phi_k$ be Young functions. If*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t)\cdots\Phi_k^{-1}(t)\le\kappa\Phi_0^{-1}(t),$$

then the following inequality holds

$$\|f_1f_2\cdots f_k\|_{\Phi_0,Q}\le\kappa\|f_1\|_{\Phi_1,Q}\|f_2\|_{\Phi_2,Q}\cdots\|f_k\|_{\Phi_k,Q}$$

for all functions $f_1,\cdots,f_m$ and all cubes Q.

In Particular, if $\sum_{i=1}^k\frac{1}{s_i}=\frac{1}{s}$ with each $s_i\ge1$, then it holds that

$$\frac{1}{Q}\int_Q|f_1f_2\cdots f_kg|\le C_s\|f_1\|_{\exp L^{s_1},Q}\|f_2\|_{\exp L^{s_2},Q}\cdots\|f_k\|_{\exp L^{s_k},Q}\|g\|_{L(\log L)^{\frac{1}{s}},Q}.$$

It was known that $M_\delta^\#(I_\alpha f)(x)\le C_\delta M_\alpha f(x)$ pointwisely (see for example [2]). For the commutators of I_α, one may get

Lemma 2.2 (Sharp Estimate). *Let $0<\delta<\epsilon<1$, $M^\#$ be the Fefferman-Stein sharp maximal function and $M_\delta^\#(f)=M^\#(|f|^\delta)^{\frac{1}{\delta}}$. Then there exists a constant $C>0$, depending only on δ and ϵ such that*

$$M_\delta^\#(I_{\alpha,\vec{b}})(x)\le C\|\vec{b}\|\Big(M_{L(\log L)^{\frac{1}{s}},\alpha}f(x)+\sum_{i=1}^m\sum_{\sigma\in C_j^m}\|\vec{\sigma}\|M_\epsilon(I_{\alpha,\vec{b_{\sigma'}}}f)(x)\Big).\tag{2.1}$$

Here $b=\sigma\cup\sigma'$, where σ and σ' are pairwise disjoint sets be a splitting of b.

This lemma can be obtained with small and straightforward modifications in the proof of Lemma 3.1 in [21].

Lemma 2.3 ([15,16]). *Let $0<p<\infty$, $0<\delta<1$ and $w\in A_\infty$, M^d denotes the dyadic maximal function. Then for any function f and $t>0$ satisfying $|\{x:|f(x)|>t\}|<\infty$, it holds that*

$$\|f\|_{L^p(w)}\le C_p[w]_{A_\infty}\|M_\delta^{\#,d}f\|_{L^p(w)}.$$

Lemma 2.4 ([20]). *Let $0 < p < \infty$, $0 < \epsilon \le 1$ and $w \in A_\infty$. Suppose that $|\{x : |f(x)| > t\}| < \infty$ for all $t > 0$. Then there is a constant $C = C_{n,\epsilon}$ such that*

$$\|M_\epsilon^d f\|_{L^p(w)} \le C_p[w]_{A_\infty}\|M_\epsilon^{\#,d} f\|_{L^p(w)}.$$

Lemma 2.5 ([11]). *Let $0 < \alpha < mn$, $1 < p_i < \infty$, $i = 1, \cdots, m$, $m \in \mathbb{N}^+$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{m} < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose that X_i is Banach space and $M_{X_i'}$ is bounded on $L^{p_i}(\mathbb{R}^n)$, u and $v_1, \cdots, v_m$ are weights satisfying*

$$K = \sup_Q \Big(\frac{u(Q)}{|Q|}\Big)^{\frac{1}{q}} \prod_{i=1}^m \|v_i^{-1}\|_{X_i,Q} < \infty,$$

then

$$\|M_\alpha(\vec{f})\|_{L^q(u)} \le CK \prod_{i=1}^m \|M_{X_i'}\|_{L^{p_i}(\mathbb{R}^n)}\|f_i\|_{L^{p_i}(v_i^{p_i})}.$$

Next we will give a key lemma which will be useful in the proof of Lemma 2.8.

Lemma 2.6. *Let $\omega \ge 0$ be a weight,*

$$v = M^{p/q}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}(\omega), \quad s \ge 1 \quad \text{and} \quad 0 < \delta < 1.$$

Then there exists $C > 0$ such that for every $p \in (1, \infty)$

$$\left\| \frac{M_{L(\log L)^{\frac{1}{s}},\alpha} f}{v} \right\|_{L^{p'}(v)} \le Cq^{1+\frac{1}{s}} \left(\frac{q-1}{\delta} q'\right)^{\frac{1}{p'}} \left\| \frac{f}{\omega} \right\|_{L^{q'}(\omega)}. \tag{2.2}$$

Proof. To prove inequality (2.2), it is equivalent to show that

$$\begin{aligned} &\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}},\alpha}(f\omega^{\frac{1}{q}})^{p'} \Big(M^{\frac{p}{q}}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}(\omega)\Big)^{1-p'} \\ \le &C_n^{p'} \Big(q^{1+\frac{1}{s}}\Big)^{p'} \Big(\frac{q-1}{\delta} q'\Big) \Big(\int_{\mathbb{R}^n} |f|^{q'}\Big)^{\frac{p'}{q'}}. \end{aligned} \tag{2.3}$$

We now introduce the notations

$$A_{\frac{1}{s}}(t) = t(1+\log^+(t))^{\frac{1}{s}}, \quad X_{\frac{1}{s}}(t) = \frac{t}{(1+\log^+ t)^{\frac{1}{s}}}.$$

Then by [20], it holds that

$$A^{-1}_{\frac{1}{s}}(t) \ge X_{\frac{1}{s}}(t).$$

Moreover, one can see that

$$\begin{aligned}X_{\frac{1}{s}}(t) =&\Big(\frac{t}{(1+\log^+(t))^{(1+\frac{1}{s})q-1+\delta}}\Big)^{\frac{1}{q}}\big(t(1+\log^+(t))^{1+\delta(q'-1)}\big)^{\frac{1}{q'}}\\ =&:F_1(t)^{\frac{1}{q}}F_2(t)^{\frac{1}{q'}},\end{aligned}$$

where

$$F_1(t)=X_{(1+\frac{1}{s})q-1+\delta}(t),\quad F_2(t)=A_{1+\delta(q'-1)}(t).$$

Again, by [20], one may get

$$M_{L(\log L)^{\frac{1}{s}},\alpha}(fw^{\frac{1}{q}})\le Cq^{1+\frac{1}{s}}M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}),\alpha}(f)\Big(M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}(w)\Big)^{\frac{1}{q}}.$$

Now we give the proof of inequality (2.3).

$$\begin{aligned}&\int_{\mathbb{R}^n}M_{L(\log L)^{\frac{1}{s}},\alpha}(f\omega^{\frac{1}{q}})^{p'}\Big(M^{\frac{p}{q}}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}\omega\Big)^{1-p'}dx\\ \le&\int_{\mathbb{R}^n}\Big(Cq^{1+\frac{1}{s}}M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}),\alpha}(f)\Big(M_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}w\Big)^{\frac{1}{q}}\Big)^{p'}\\ &\times\Big(M^{\frac{p}{q}}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}w\Big)^{1-p'}\Big)dx\\ =&(Cq^{1+\frac{1}{s}})\int_{\mathbb{R}^n}M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}),\alpha}(f)^{p'}dx.\end{aligned}$$

Let $B=\tilde{X}_{1+\delta(p-1)}(L^p)$, by [4] we know

$$\tilde{X}_{1+\delta(p-1)}(L^p)\preccurlyeq t^p.$$

Then

$$|Q|^{\frac{\alpha}{n}}\|f\|_{B,Q}=|Q|^{\frac{\alpha}{n}}\|f\|^{\frac{\alpha p}{n}}_{B,Q}\|f\|^{1-\frac{\alpha p}{n}}_{B,Q}\le C\|f\|_p^{1-\frac{p}{q}}\|f\|^{\frac{p}{q}}_{B,Q},$$

which yields that

$$M_{\alpha,B}f(x)\le C\|f\|_p^{1-\frac{p}{q}}M_{0,B}f(x)^{\frac{p}{q}}.$$

Hence, Lemma 2.1 in [10] gives that

$$\int_{\mathbb{R}^n}(M_{\alpha,B}f(x))^qdx\le C\|f\|_p^{q-p}p\int_1^\infty\frac{B(t)}{t^p}\frac{dt}{t}\|f\|^p_{L^p}\le Cp\frac{cp-1}{\delta}\|f\|^q_p.$$

Therefore,

$$\int_{\mathbb{R}^n}M_{\tilde{X}_{1+\delta(q'-1)}(L^{q'}),\alpha}(f)^{p'}dx\le Cq'\frac{q-1}{\delta}\|f\|^{p'}_{q'}.$$

Consequently, we have shown that

$$\|M_{L(\log)^{\frac{1}{s}}}(fw^{\frac{1}{q}})\|_{L^{p'}(v^{1-p'})}\le Cq^{1+\frac{1}{s}}\Big(\frac{q-1}{\delta}q'\Big)^{\frac{1}{p'}}\|f\|_{L^{q'}(\mathbb{R}^n)}.$$

Thus, we complete the proof. □

Lemma 2.7. *Let ω be a weight and A be a Young function, $p \in (1,\infty)$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, then there exists a constant C_{I_α} such that*

$$\|I_\alpha(f)\|_{L^q(\omega)} \le C_{I_\alpha} p' \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^p(M_{A_q}^{\frac{p}{q}}(\omega))},$$

where $\bar{A}$ is the complementary function of A, $A_q(t)=A(t^{\frac{1}{q}})$. Moreover, if

$$A(t)=t^q(1+\log^+ t)^{q-1+\delta},$$

it holds that

$$\|I_\alpha(f)\|_{L^q(\omega)} \le C_{I_\alpha} p' q^2 \Big(\frac{1}{\delta}\Big)^{\frac{1}{q'}} \|f\|_{L^p\big(M^{\frac{p}{q}}_{L(\log L)^{q-1+\delta}}(\omega)\big)}, \quad 0<\delta<1.$$

Proof. By Proposition 2.2 and Proposition 2.3 of [12], it suffices to show that

$$\|I_\alpha^S(f)\|_{L^q(\omega)} \le C_{I_\alpha} p' \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^p(M_{A_q}^{\frac{p}{q}}(\omega))}, \tag{2.4}$$

where $S \in D$ is a sparse set and D is a standard dyadic grid,

$$I_\alpha^S f = \sum_{Q\in S} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q f dx \cdot \chi_Q.$$

Let $v = M_{A_q}^{\frac{p}{q}}(\omega)$. By duality, there exists $\|g\|_{L^{q'}(\omega)}=1$ such that

$$\|I_\alpha^S(f)\|_{L^q(\omega)} = \int_{\mathbb{R}^n} \frac{I_\alpha^S g\omega}{v} f v dx \le \Big\|\frac{I_\alpha^S g\omega}{v}\Big\|_{L^{p'}(v)} \|f\|_{L^p(v)}.$$

Observe that the adjoint of I_α^S is itself. Then, to prove inequality (2.4), it suffices to show that

$$\Big\|\frac{I_\alpha^S g\omega}{v}\Big\|_{L^{p'}(v)} \le C_{I_\alpha} p' \|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)}. \tag{2.5}$$

First, we consider to prove

$$\Big\|\frac{I_\alpha^S g}{v}\Big\|_{L^{p'}(v)} \le p' \Big\|\frac{M_\alpha g}{v}\Big\|_{L^{p'}(v)}. \tag{2.6}$$

By duality, there exists a nonnegative function h with $\|h\|_{L^p(v)}=1$ such that

$$I = \Big\|\frac{I_\alpha^S g}{v}\Big\|_{L^{p'}(v)} = \int_{\mathbb{R}^n} I_\alpha^S g(x) h(x) dx.$$

Now we consider the operator:

$$S(h)=\frac{M(hv^{\frac{1}{p}})}{v^{\frac{1}{p}}},$$

and use S to build the Rubio de Francia algorithm

$$R(h)=\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{S^k(h)}{\|s\|^k_{L^p(v)}},$$

where R satisfies

(1) $0\le h\le R(h)$;

(2) $\|R(h)\|_{L^p}(v)\le 2\|h\|_{L^p}(v)$;

(3) $R(h)v^{\frac{1}{p}}\in A_1$, $[R(h)v^{\frac{1}{p}}]_{A_1}\le Cp'$.

Note that $[v^{\frac{1}{2p}}]^2_{A_1}\le C_n$, by Lemma 4.2 in [10]. Taking this into account yields that

$$[Rh]_{A_3}=[R(h)v^{\frac{1}{p}}(v^{-\frac{1}{p(1-q)}})^{1-q}]_{A_3}\le[Rhv^{\frac{1}{p}}]_{A_1}[v^{\frac{1}{2p}}]^2_{A_1}\le C_n p'.$$

Therefore $[Rh]_{A_\infty}\le[Rh]_{A_3}\le C_n p'$ gives that

$$I\le\int_{\mathbb{R}^n}I^S_\alpha g(x)Rh(x)dx\le[Rh]_{A_\infty}\int_{\mathbb{R}^n}M_\alpha g\cdot Rhdx\le Cp'\Big\|\frac{M_\alpha g}{v}\Big\|_{L^{p'}(v)}. \tag{2.7}$$

Eq. (2.7) can be obtained by similar reasoning as in Lemma 4.1 of [10] with slight modifications. So we have shown that inequality (2.6) holds. In order to prove (2.5), it suffices to show that

$$\Big\|\frac{M_\alpha f}{v}\Big\|_{L^{p'}(v)}\le C\|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)}\Big(\int_{\mathbb{R}^n}\Big(\frac{f}{\omega^{\frac{1}{q}}}\Big)^{q'}dx\Big)^{\frac{1}{q'}}.$$

Note that Lemma 2.5 and Theorem 6.4 in [6] yield that

$$\begin{aligned}K=&\sup_Q\Big(\frac{1}{|Q|}\int_Q v^{1-p'}\Big)^{\frac{1}{p'}}\|\omega^{\frac{1}{q}}\|_{A,Q}\\=&\sup_Q\Big(\frac{1}{|Q|}\int_Q M^{\frac{p}{q}}_{A_q}(\omega)^{1-p'}dy\Big)^{\frac{1}{p'}}\|\omega^{\frac{1}{q}}\|_{A,Q}\\\le&\|\omega\|^{-\frac{1}{q}}_{A_q,Q}\|\omega^{\frac{1}{q}}\|_{A,Q}=\|\omega^{\frac{1}{q}}\|^{-1}_{A,Q}\|\omega^{\frac{1}{q}}\|_{A,Q}=1.\end{aligned}$$

Moreover, if $A(t)=t^q(1+\log^+t)^{q-1+\delta}$, from the inequality (25) in [10], one may get

$$\|M_{\bar{A}}\|_{L^{q'}(\mathbb{R}^n)}\le C_n\Big(\int_1^\infty\Big(\frac{t}{A(t)}\Big)^{q'}A'(t)dt\Big)^{\frac{1}{q'}}\le q^2\Big(\frac{1}{\delta}\Big)^{\frac{1}{q'}}.$$

Therefore

$$\|I_\alpha(f)\|_{L^q(\omega)} \le C_{I_\alpha} p' q^2 \Big(\frac{1}{\delta}\Big)^{\frac{1}{q'}} \|f\|_{L^p\big(M^{\frac{p}{q}}_{L(\log L)^{q-1+\delta}}(\omega)\big)}.$$

Thus, we complete the proof. □

Lemma 2.8. *Let $\vec{b}=(b_1,\cdots,b_k)$, $k\in\mathbb{N}$, ω be a weight, for $\delta\in(0,1)$,*

$$p\in(1,\infty),\quad \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n},\quad \nu=M^{\frac{p}{q}}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}\omega,$$

there exists a constant C_{I_α} such that

$$\|I_{\alpha,\vec{b}}f\|_{L^q(\omega)} \le C_{I\alpha} {p'}^{k+1} q^{(1+\frac{1}{s})} \left(\frac{q-1}{\delta}q'\right)^{\frac{1}{p'}} \|\vec{b}\| \|f\|_{L^p(\nu)}.$$

Proof. We will divide the proof in two cases, as follows.
Case 1: $k=1$. We denote

$$v=M^{p/q}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}\omega,\quad m=C_{I_\alpha}(p')^2 q^{1+\frac{1}{s}}\Big(\frac{q-1}{\delta}q'\Big)^{\frac{1}{p'}}.$$

By duality, there exists $\|g\|_{L^{q'}(\omega)}=1$ such that

$$\|I_{\alpha,b}(f)\|_{L^q(\omega)}=\int_{\mathbb{R}^n} I_{\alpha,b}f(x)g(x)\omega dx=\int_{\mathbb{R}^n}\frac{I_{\alpha,b}g\omega}{v}fvdx\le\Big\|\frac{I_{\alpha,b}g\omega}{v}\Big\|_{L^{p'}(v)}\|f\|_{L^p(v)}.$$

We only need to show that

$$\Big\|\frac{I^t_{\alpha,b}(f)}{v}\Big\|_{L^{p'}(v)}\le m\|\frac{f}{w}\|_{L^{q'}(w)},$$

where $I^t_{\alpha,b}$ is the adjoint of $I_{\alpha,b}$. The method of duality allows us to find a non-negative function $h\in L^p(v)$ with $\|h\|_{L^p(v)}=1$ such that

$$\Big\|\frac{I^t_{\alpha,b}(f)}{v}\Big\|_{L^{p'}(v)}=\int_{\mathbb{R}^n}\frac{|I^t_{\alpha,b}(f)|}{v}hvdx=\int_{\mathbb{R}^n}|I^t_{\alpha,b}|hdx=I.$$

Now we consider the same operator S as we used in Theorem 2.7 and build the Rubio de Francia algorithm R using the operator S. From the properties of R, we have

$$I\le\int_{\mathbb{R}^n}|I^t_{\alpha,b}f|Rhdx\le C_n[Rh]_{A_\infty}\int_{\mathbb{R}^n}M^\#_r(I^t_{\alpha,b}f)Rhdx\le C_np'\int_{\mathbb{R}^n}M^\#_r(I^t_{\alpha,b}f)Rhdx.$$

Observing that $[b,T]^t=-[b,T^t]$, so $[b,T]^t$ is also a commutator. By Lemma 2.2, it holds that

$$\begin{aligned} I \leq & C_n p' \|b\|_{Osc_{expL^s}} \Big[\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}},\alpha} f(x) Rh(x)dx + \int_{\mathbb{R}^n} M_\epsilon(I^t_{\alpha,}f)(x)Rh(x)dx \Big] \\ = & C_n p' \|b\|_{Osc_{expL^s}} (I_1+I_2). \end{aligned}$$

Applications of Hölder inequality and the second property of operator R lead to the inequalities that

$$\begin{aligned} I_1 \leq & \Big(\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}},\alpha} f^{p'}(x) v^{1-p'}(x)dx \Big)^{\frac{1}{p'}} \Big(\int_{\mathbb{R}^n} Rh^p(x)v(x)dx \Big)^{\frac{1}{p}} \\ \leq & 2 \Big\| \frac{M_{L(\log L)^{\frac{1}{s}},\alpha} f}{v} \Big\|_{L^{p'}(v)}. \end{aligned}$$

Now, Lemma 2.4 will be used to estimate I_2.

$$\begin{aligned} I_2 \leq & C_n [Rh]_{A_\infty} \int_{\mathbb{R}^n} M^\#_\epsilon (I^t_\alpha f)(x)Rh(x)dx \leq C_n p' \int_{\mathbb{R}^n} M^\#_\epsilon (I^t_\alpha f)(x)Rh(x) \\ \leq & c_{n,\epsilon} p' \int_{\mathbb{R}^n} M_\alpha f(x)Rh(x)dx \leq C_n p' \Big\| \frac{M_\alpha f}{v} \Big\|_{L^{p'}(v)}. \end{aligned}$$

The last inequality can be obtained the same as what we have done to deal I_1. Consequently, when

$$v = M^{\frac{p}{q}}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}} \omega,$$

it holds that

$$\Big\| \frac{I^t_{\alpha,b}(f)}{v} \Big\|_{L^{p'}(v)} \leq C_n (p')^2 \|b\|_{Osc_{\exp L^s}} \Big\| \frac{M_{L(\log L)^{\frac{1}{s}},\alpha} f}{v} \Big\|_{L^{p'}(v)}.$$

Therefore by Lemma 2.6, there exists $C_n>0$ such that

$$\Big\| \frac{I^t_{\alpha,b}(f)}{v} \Big\|_{L^{p'}(v)} \leq C_n (p')^2 q^{1+\frac{1}{s}} \Big(\frac{q-1}{\delta} q' \Big)^{\frac{1}{p'}} \Big\| \frac{f}{\omega} \Big\|_{L^{p'}(\omega)}.$$

Case2: $k>1$. The following argument is essentially taken from [20], and we only prepare to give an outline of the proof. We will divide it in two steps.

Step 1. Due to the homogeneity of the operator we may assume that

$$\|b_1\|_{Osc_{\exp L^{s_1}}} = \|b_2\|_{Osc_{\exp L^{s_2}}} = \cdots \|b_k\|_{Osc_{\exp L^{s_k}}} = 1.$$

Repeated reasoning as the case $k=1$ yields that

$$\begin{aligned} I=&\int_{\mathbb{R}^n}|I^t_{\alpha,\vec{b}}f|hdx\\ \leq& C_n p'\Big[\int_{\mathbb{R}^n}M_{L(\log L)^{\frac{1}{s}},\alpha}fRhdx+\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{\mathbb{R}^n}M_\epsilon(I^t_{\alpha,\vec{\sigma}'}f)Rhdx\Big]\\ =&C_n p'(I_1+I_2). \end{aligned}$$

Similarly as the case for $k=1$, one can get

$$I_1\leq 2\Big\|\frac{M_{L(\log L)^{1/s},\alpha}f}{v}\Big\|_{L^{p'}}(v).$$

Step 2. Now we consider the contribution of I_2. Let

$$\Gamma(j)=\begin{cases} 1, & j=0,\\ 2, & j=1,\\ 2+\sum\limits_{i=0}^{j-1}\binom{j}{i}\Gamma(i), & j>1. \end{cases} \tag{2.8}$$

For every $\epsilon\in(0,1)$, we claim that:

$$\int_{\mathbb{R}^n}M_\epsilon(I^t_{\alpha,\vec{\sigma}'}f)(x)Rh(x)dx\leq C\Gamma(\#\sigma')(p')^{\#\sigma'+1}\int_{\mathbb{R}^n}M_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}},\alpha}f(x)Rh(x)dx.$$

Now we turn to the proof of this claim. It will be proved by induction on the number of symbols of $I^t_{\alpha,\vec{\sigma}'}$. We will use the notation $I_{\alpha,\vec{\sigma}'}$ instead of $I^t_{\alpha,\vec{\sigma}'}$ and denote $k=\#\sigma'$.

If $\#\sigma'=0$, since

$$\sum_{i\in\emptyset}\frac{1}{s_i}=0,$$

then

$$\int_{\mathbb{R}^n}M_\epsilon(I_\alpha f)(x)Rh(x)dx\leq Cp'M_\alpha f(x)Rh(x)dx.$$

If $\#\sigma'=1$, then

$$\int_{\mathbb{R}^n}M_\epsilon(I_{\alpha,b}f)Rhdx\leq 2C(p')^2\int_{\mathbb{R}^n}M_{L(\log L)^{\frac{1}{s_1}},\alpha}f(x)Rh(x)dx.$$

Suppose that the claim holds for $0\leq\#\tau<k$, then for every $\epsilon\in(0,1)$:

$$\int_{\mathbb{R}^n}M_\epsilon(I_{\alpha,\vec{\tau}}f)(x)Rh(x)dx\leq C\Gamma(\#\tau)(p')^{\#\tau+1}\int_{\mathbb{R}^n}M_{L(\log L)^{\sum_{i\in\tau}\frac{1}{s_i}},\alpha}f(x)Rh(x)dx.$$

We only need to show the case for $\#\tau' = k$. With Lemma 2.4 and the inductive hypothesis in hand, one may get

$$\begin{aligned}&\int_{\mathbb{R}^n} M_\epsilon(I_{\alpha,\vec{\sigma}'}f)(x)Rh(x)dx\\ \leq&\Big(1+\sum_{j=1}^{k}\sum_{\tau\in C_j^k}\Gamma(\#\tau')\Big)Cp'^{k+1}\int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}},\alpha}f(x)Rh(x)dx.\end{aligned}$$

Denoting

$$\Big(1+\sum_{j=1}^{k}\sum_{\tau\in C_j^k}\Gamma(\#\tau')\Big)=\Gamma(k),$$

then

$$I_2\leq C_{n,\delta,\epsilon}(p')^{m+1}\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}},\alpha}fRhdx.$$

Therefore, the same reasoning as what we have done in dealing with I_1 yields that

$$I_2\leq C_{n,\delta,\epsilon}(p')^{m+1}\Big\|\frac{M_{L(\log L)^{\frac{1}{s}},\alpha}f}{v}\Big\|_{L^{p'}(v)}.$$

Consequently, by Lemma 2.6, as already noted, it holds that

$$\Big\|\frac{I^t_{\alpha,\vec{b}}f}{v}\Big\|_{L^{p'}(v)}\leq C_n(p')^{k+1}q^{(1+\frac{1}{s})}\Big(\frac{q-1}{\delta}q'\Big)^{\frac{1}{p'}}\Big\|\frac{M_{L(\log L)^{\frac{1}{s}},\alpha}f}{\omega}\Big\|_{L^{p'}(\omega)}.$$

Thus, we complete the proof. □

3 The proof of borderline estimates

Now we turn to prove Theorem 1.1. The following argument is essentially taken from [10].

Proof of Theorem 1.1. By Proposition 2.2 and 2.3 in [12], it suffices to show that

$$\|I^S_\alpha f\|_{L^{1,\infty}(\omega)}\leq\frac{C_{I_\alpha}}{\epsilon}\Big[\int_{\mathbb{R}^n}|f(x)|(M^c_{L(\log L)^\epsilon,\alpha}\omega(x)dx)^{\frac{1}{c}}+\int_{\mathbb{R}^n}|f(x)|M\omega(x)dx\Big],$$

where $S\in D$ is the sparse set, D is the dyadic grid,

$$I^S_\alpha f=\sum_{Q\in S}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q fdx\cdot\chi_Q.$$

Thanks to the homogeneity of the operator, it is enough to show

$$\begin{aligned}&\omega(\{x\in\mathbb{R}^n:|I_\alpha^S f|>2\})\\ \le&\frac{C_{I_\alpha}}{\epsilon}\Big[\int_{\mathbb{R}^n}|f(x)|(M^c_{L(\log L)^\epsilon,\alpha}\omega(x)dx)^{\frac{1}{c}}+\int_{\mathbb{R}^n}|f(x)|M\omega(x)dx\Big].\end{aligned}$$

Decompose $f=g+b$ at height $\lambda=1$ with

$$b=\sum_j(f-f_{Q_j})\chi_{Q_j}.$$

Introduce the notations

$$\Omega=\bigcup_j Q_j,\quad \widetilde{\Omega}=\bigcup_j 3Q_j,$$

then $\|g\|_{L^\infty}\le 2^n$ and

$$\begin{aligned}&\omega(\{x\in\mathbb{R}^n:|I_\alpha^S f(x)|>2\})\\ \le&\omega(\widetilde{\Omega})+\omega\Big(\{(\widetilde{\Omega})^c:I_\alpha^S g(x)>1\}\Big)+\omega\Big(\{(\widetilde{\Omega})^c:I_\alpha^S b(x)>1\}\Big)\\ =&:I+II+III.\end{aligned}$$

Consider first the contribution made by I. The Chebyshev inequality gives that

$$I\le\sum_j|3Q_j|\frac{1}{|3Q_j|}\int_{3Q_j}\omega(x)dx\le\sum_j|3Q_j|\inf_{z\in Q_j}M\omega(z)\le\int_{\mathbb{R}^n}f(y)M\omega(y)dy.$$

By [10], it is easy to see III=0. Moreover, by the fact that $\|g\|_{L^\infty}\le 2^n$, Then

$$\begin{aligned}II\le&\|I_\alpha^S g\|^q_{L^q(\omega\chi_{(\widetilde{\Omega})^c})}\\ \le&C\Big(q^2p'\Big(\frac{1}{\delta}\Big)^{\frac{1}{q'}}\Big)^q\Big(\int_{(\widetilde{\Omega})^c}|g|^{\frac{p}{q}}M^{\frac{p}{q}}_{L(\log L)^{q-1+\delta}}\Big)^{\frac{q}{p}}\\ \le&Cq^{2q}\Big(\frac{1}{\delta}\Big)^{\frac{q}{q'}}(p')^q\Big(\int_{(\mathbb{R}^n)^c}|f|M^{\frac{p}{q}}_{L(\log L)^{q-1+\delta}}\Big)^{\frac{q}{p}}.\end{aligned}$$

Let $q-1=\frac{\varepsilon}{2}=\delta<1$, then

$$q=\frac{\varepsilon}{2}+1,\quad q^{2q}\Big(\frac{1}{\delta}\Big)^{\frac{q}{q'}}(p')^q\le\frac{1}{\varepsilon},\quad c=\frac{p}{q}=\frac{2n}{2n+\alpha(\epsilon+2)}.$$

Therefore in all,

$$\|I_\alpha^S f\|_{L^{1,\infty}(\omega)}\le\frac{C_{I_\alpha}}{\epsilon}\Big[\int_{\mathbb{R}^n}|f(x)|(M^c_{L(\log L)^\epsilon}\omega(x)dx)^{\frac{1}{c}}+\int_{\mathbb{R}^n}|f(x)|M\omega(x)dx\Big].$$

Thus, we complete the proof. □

We now begin the proof of Theorem 1.2.

Proof of Theorem 1.2. We will divide it in two cases.
Case 1: $k=1$. Due to the homogeneity of the operator, we may assume that $\|b_1\|_{Osc_{\exp L^s}}=1$. Consider the Calderón-Zygmund decomposition of f at height λ. It bears emphasis that there exists a family of dyadic cubes Q_j which are pairwise disjoint such that

$$\lambda \le \frac{1}{|Q_j|}\int_{Q_j}|f| \le 2^n\lambda.$$

Denote $\Omega=\bigcup_j Q_j$, and write $f=g+h$, where g is denoted to be the "good" part of f, defined by

$$g(x)=\begin{cases} f(x), & x\in\Omega^c,\\ f_{Q_j}, & x\in Q_j,\end{cases}$$

with the property that $|g(x)|\le 2^n\lambda$ a.e., and $h=\sum h_j$ with

$$h_j=(f-f_{Q_j})\chi_{Q_j} \quad \text{and} \quad f_{Q_j}=\frac{1}{|Q_j|}\int_{Q_j}|f|dx.$$

Now we introduce the notations

$$w^*(x)=w(x)\chi_{\mathbb{R}^n\backslash\tilde{\Omega}}(x) \quad \text{and} \quad w_j(x)=w(x)\chi_{\mathbb{R}^n\backslash\tilde{Q}_j}(x),$$

where $\tilde{Q}_j=5\sqrt{n}Q_j$ and $\tilde{\Omega}=\bigcup_j\tilde{Q}_j$. The Calderón-Zygmund decomposition allows us to write

$$\begin{aligned}&w(\{x\in\mathbb{R}^n:|I_{\alpha,b}f(x)|>\lambda\})\\ \le& w\left(\left\{x\in\mathbb{R}^n\backslash\tilde{\Omega}:|I_{\alpha,b}g(x)|>\frac{\lambda}{2}\right\}\right)+w(\tilde{\Omega})+w\left(\left\{x\in\mathbb{R}^n\backslash\tilde{\Omega}:|I_{\alpha,b}h(x)|>\frac{\lambda}{2}\right\}\right)\\ =:& I+II+III.\end{aligned}$$

Consider first the contribution of I, Chebyschev's inequality gives that

$$I\le\frac{2^q}{\lambda^q}\int_{\mathbb{R}^n}|I_{\alpha,b}g(x)|^q w^*(x)dx.$$

Now we choose

$$q=1+\frac{5\epsilon}{12\left(1+\frac{1}{s}\right)} \quad \text{and} \quad \delta=\epsilon-\left(1+\frac{1}{s}\right)(q-1).$$

Then it follows that

$$(p')^{2q}p^{(1+\frac{1}{s})q}\Big(\frac{q-1}{\delta}\Big)^{\frac{q}{p'}} \le C_s\frac{1}{\epsilon^2}, \quad \Big(1+\frac{1}{s}\Big)q-1+\delta=\frac{1}{s}+\epsilon,$$
$$c=\frac{p}{q}=\frac{12n(1+\frac{1}{s})}{12(n+\alpha)(1+\frac{1}{s})+5\alpha\epsilon}.$$

By Lemma 2.8, one may obtain

$$\begin{aligned}&\frac{2^q}{\lambda^q}\int_{\mathbb{R}^n}|I_{\alpha,b}g(x)|^q\omega^*(x)dx\\ \le&\frac{2^q}{\lambda^q}C\Big((p')^2q^{(1+\frac{1}{s})}\Big(\frac{q-1}{\delta}q'\Big)^{\frac{1}{p'}}\Big)^q\Big(\int_{\mathbb{R}^n}|g(x)|^pM^{\frac{p}{q}}_{L(\log L)^{(1+\frac{1}{s})q-1+\delta}}\omega^*(x)dx\Big)^{\frac{q}{p}}\\ \le&C\frac{1}{\epsilon^2}\frac{1}{\lambda^{\frac{q}{p}}}\Big(\int_{\mathbb{R}^n\backslash\Omega}|f(x)|M^{\frac{p}{q}}_{L(\log L)^{\frac{1}{s}+\epsilon}}\omega^*(x)dx\Big)^{\frac{q}{p}}.\end{aligned}$$

This means, of course, that there exists $C>0$ such that

$$I\le C\frac{1}{\epsilon^2}\Big(\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}M^c_{L(\log L)^{\frac{1}{s}+\epsilon}}w(y)dy\Big)^{\frac{1}{c}}.$$

We have the following standard estimate for II

$$II\le\sum_j\int_{5\sqrt{n}Q_j}w(x)dx\le\sum_j(5\sqrt{n})^n|Q_j|\inf_{z\in Q_j}Mw(x)\le(5\sqrt{n})^n\int\limits_{\mathbb{R}^n}\frac{f(y)}{\lambda}Mw(y)dy.$$

Now we turn to the discussion of III. Since

$$I_{\alpha,b}h=\sum_j I_{\alpha,b}h_j=\sum_j(b(x)-b_{Q_j})I_\alpha h_j(x)-I_\alpha(b-b_{Q_j})h_j,$$

we may split III into two parts

$$\begin{aligned}III\le&w\bigg(\Big\{x\in\mathbb{R}^n\backslash\tilde{\Omega}:|\sum_j(b(x)-b_{Q_j})I_\alpha h_j(x)|>\frac{\lambda}{4}\Big\}\bigg)\\&+w\bigg(\Big\{x\in\mathbb{R}^n\backslash\tilde{\Omega}:|\sum_j I_\alpha[b-b_{Q_j}]h_j(x)|>\frac{\lambda}{4}\Big\}\bigg)\\ :=&A+B.\end{aligned}$$

The Chebyschev's inequality and the cancellation of h_j yield that

$$\begin{aligned} A \leq& \frac{C}{\lambda}\sum_j \int_{\mathbb{R}^n\setminus\tilde{Q}_j} |b(x)-b_{Q_j}|w(x)\int_{Q_j}|h_j(y)|\Big|\Big(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x-y'|^{n-\alpha}}\Big)\Big|dydx \\ \leq& \frac{C}{\lambda}\sum_j \int_{Q_j}|h_j(y)|\int_{\mathbb{R}^n\setminus\tilde{Q}_j}\frac{|y-y'|^{n-\alpha}}{|x-y|^{n-\alpha}|x-y'|^{n-\alpha}}|b(x)-b_{Q_j}|w_j(x)dxdy \\ \leq& \frac{C}{\lambda}\sum_j \int_{Q_j}|h_j|\sum_{k=1}^{\infty}\int_{2^k l(Q_j)\leq|x-y'|\leq 2^{k+1}l(Q_j)}\frac{(2l(Q_j))^{n-\alpha}|b-b_{Q_j}|w_j}{(2^{k-1}l(Q_j))^{n-\alpha}(2^k l(Q_j))^{n-\alpha}}dxdy \\ \leq& \frac{C}{\lambda}\sum_j \int_{Q_j}|h_j(y)|dy\sum_{k=1}^{\infty}2^{-k(n-\alpha)}\frac{1}{|2^{k+1}Q_j|^{n-\alpha}}\int_{2^{k+1}Q_j}|b(x)-b_{Q_j}|w_j(x)dx. \end{aligned}$$

By Lemma 2.1 and

$$|b(x)-b_{Q_j}|\leq|b(x)-b_{2^{k+1}Q_j}|+|b_{2k+1Q_j}-b_{Q_j}|,$$

it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty}\frac{2^{-k(n-\alpha)}}{|2^{k+1}Q_j|^{n-\alpha}}\int_{2^{k+1}Q_j}|b(x)-b_{Q_j}|w_j(x)dx \\ \leq& \sum_{k=1}^{\infty}2^{-k(n-\alpha)}\inf_{z\in Q_j}M_{L\log L^{\frac{1}{s}},\alpha}w_j(z)+\sum_{k=1}^{\infty}2^{-k(n-\alpha)}(k+1)\inf_{z\in Q_j}M_{L\log L^{\frac{1}{s}},\alpha}w_j(z) \\ \leq& \inf_{z\in Q_j}M_{L\log L^{\frac{1}{s}},\alpha}w_j(z). \end{aligned}$$

Therefore,

$$A\leq\frac{C}{\lambda}\sum_j\int_{Q_j}M_{L\log^{\frac{1}{s}},\alpha}w_j(y)|h_j(y)|dy\leq\frac{C}{\lambda}\int_{\mathbb{R}^n}|f(y)|M_{L\log L^{\frac{1}{s}},\alpha}w(y)dy.$$

On the other hand, B may be split into four parts

$$\begin{aligned} B \leq& \frac{C}{\lambda\epsilon}\sum_j\inf_{z\in Q_j}(M_{L\log^{\epsilon}}w_j(z))\Big(\int_{Q_j}|b(x)-b_{Q_j}||f(x)|dx+\int_{Q_j}|b(x)-b_{Q_j}||f_{Q_j}|dx\Big)^{\frac{1}{c}} \\ &+\frac{C}{\lambda\epsilon}\sum_j\inf_{z\in Q_j}Mw_j(z)\Big(\int_{Q_j}|b(x)-b_{Q_j}||f(x)|dx+\int_{Q_j}|b(x)-b_{Q_j}||f_{Q_j}|dx\Big) \\ \leq& \frac{1}{\epsilon}(B_1+B_2+B_3+B_4). \end{aligned}$$

By the generalized Hölder inequality and definition of Orlicz maximal function, one may obtain

$$\begin{aligned} B_1 \leq& C\sum_j\inf_{z\in Q_j}(M_{L\log L^{\epsilon}}w_j(z))\frac{1}{\lambda}|Q_j|^{\frac{1}{c}}\|f\|^{\frac{1}{c}}_{L(\log L)^{\frac{1}{s}},Q_j} \\ \leq& \Big(\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\Big(\frac{|f(x)|}{\lambda^c}\Big)M^c_{L\log L^{\epsilon}}w(x)dx+\lambda^{1-c}\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\Big(\frac{|f(x)|}{\lambda}\Big)M^c_{L\log L^{\epsilon}}w(x)dx\Big)^{\frac{1}{c}}. \end{aligned}$$

Similarly, it holds that

$$B_2 \le \Big(\int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\Big(\frac{|f(y)|}{\lambda^c}\Big) M^c_{L\log L^{\epsilon}} w_j(y)dy \Big)^{\frac{1}{c}},$$

$$B_3 + B_4 \le C \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\Big(\frac{|f(y)|}{\lambda}\Big) Mw(x)dx.$$

This completes the proof of Case 1.

Case 2 : $k \ge 2$. We will use inductive discussion here, i.e., suppose that the inequality holds for $l \le k-1$ symbols, we need to show it holds for $l = k$. Due to the homogeneity of the operator, we may assume that

$$\|b_1\|_{Osc_{\exp L^{s_1}}} = \cdots = \|b_1\|_{Osc_{\exp L^{s_k}}} = 1.$$

We decompose f like in the same way as the case $k = 1$. Then

$$\begin{aligned} & w\left(\left\{x \in \mathbb{R}^n : |I_{\alpha,\vec{b}} f(x)| > \lambda\right\}\right) \\ \le & w\left(\left\{x \in \mathbb{R}^n \backslash \tilde{\Omega} : |I_{\alpha,\vec{b}} g(x)| > \frac{\lambda}{2}\right\}\right) + w(\tilde{\Omega}) + w\left(\left\{x \in \mathbb{R}^n \backslash \tilde{\Omega} : |I_{\alpha,\vec{b}} h(x)| > \frac{\lambda}{2}\right\}\right) \\ =: & I + II + III. \end{aligned}$$

The Chebyschev's inequality gives that

$$I \le \frac{2^q}{\lambda^q} \int_{\mathbb{R}^n} |I_{\alpha,\vec{b}} g(x)|^q \omega^*(x)dx. \tag{3.1}$$

Choosing

$$q = 1 + \frac{5\epsilon}{12(1+\frac{1}{s})}, \quad \delta = \epsilon - \Big(1 + \frac{1}{s}\Big)(q-1),$$

and therefore

$$c := \frac{p}{q} = \frac{12n(1+\frac{1}{s})}{12(n+\alpha)(1+\frac{1}{s}) + 5\alpha\epsilon}, \quad (p')^{(k+1)q} p^{(1+\frac{1}{s})q} \Big(\frac{q-1}{\delta}\Big)^{\frac{q}{p'}} \le C_s \frac{1}{\epsilon^{k+1}},$$

$$\Big(1 + \frac{1}{s}\Big)q - 1 + \delta = \frac{1}{s} + \epsilon.$$

Then (3.1) is controlled by

$$C_n (p')^{(k+1)q} p^{(1+\frac{1}{s})q} \Big(\frac{q-1}{\delta}\Big)^{\frac{q}{p'}} \Big(\int_{\mathbb{R}^n} |g(x)|^q M^c_{L(\log L)^{(1+\frac{1}{s})q-1+\delta},\alpha} \omega^*(x)dx \Big)^{1/c}.$$

Proceeding as $k = 1$ yields that

$$I \le C_n \frac{1}{\epsilon^{k+1}} \Big(\int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M^c_{L(\log L)^{\frac{1}{s}+\epsilon},\alpha} \omega(y)dy \Big)^{\frac{1}{c}},$$

$$II \le (5\sqrt{n})^n \int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_\omega(y)dy.$$

We continue to estimate III. It can been written in the following way

$$\begin{aligned}III \leq&\omega\Bigg(\Bigg\{y\in\mathbb{R}^n\backslash\tilde{\Omega}:\Big|\sum_j(b_1(x)-(b_1)_{Q_j})\cdots(b_k(x)-(b_k)_{Q_j})I_\alpha h_j(x)\Big|>\frac{\lambda}{6}\Bigg\}\Bigg)\\&+\omega\Bigg(\Bigg\{y\in\mathbb{R}^n\backslash\tilde{\Omega}:\Big|C_kI_\alpha((b_1-(b_1)_{Q_j})\cdots(b_k-(b_k)_{Q_j})h_j)(x)\Big|>\frac{\lambda}{6}\Bigg\}\Bigg)\\&+\omega\Bigg(\Bigg\{y\in\mathbb{R}^n\backslash\tilde{\Omega}:\Big|\sum_j\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}C_\sigma I_\alpha^{\vec{\sigma}}((b-\vec{b_{Q_j}})_{\sigma'}h_j)(x)\Big|>\frac{\lambda}{6}\Bigg\}\Bigg)\\=:&L_1+L_2+L_3.\end{aligned}$$

Now we deal with L_1. Denote

$$\omega_j=\omega\cdot\chi_{\mathbb{R}^n\backslash5\sqrt{n}Q_j}\quad\text{and}\quad B(x)=\prod_{i=1}^k|b_i(x)-(b_i)_{Q_j}|.$$

Same reasoning as what we have done in dealing with $k=1$ gives that

$$\begin{aligned}L_1\leq&\sum_j\frac{C}{\lambda}\int_{Q_j}|h_j(y)|\sum_m\frac{8^{n-\alpha}}{2^{m(n-\alpha)}}\frac{1}{(2^{m+1}l(Q_j))^{n-\alpha}}\int_{|x-y'|\leq2^{m+1}l(Q_j)}B(x)\omega_j(x)dxdy\\\leq&\frac{C_k}{\lambda}\sum_j\int_{Q_j}|h_j(y)|M_{L(\log L)^{\frac{1}{s}},\alpha}\omega_j(y)dy\\\leq&\frac{C_k}{\lambda}\int_{\mathbb{R}^n}|f(y)|M_{L(\log L)^{\frac{1}{s}},\alpha}\omega(y)dy.\end{aligned}$$

Consider now the contribution of L_2. By Theorem 1.1, L_2 can be written as

$$\begin{aligned}L_2\leq&\frac{C}{\lambda}\frac{1}{\epsilon}\Bigg\{\sum_j\Bigg[\int_{Q_j}B(x)|f-f_{Q_j}|M^c_{L(\log L)^\epsilon}\omega_j(x)dx\Bigg]^{\frac{1}{c}}+\int_{Q_j}B(x)|f-f_{Q_j}|M\omega_j(x)dx\Bigg\}\\\leq&\frac{C}{\lambda}\frac{1}{\epsilon}\sum_j\inf_{z\in Q_j}M_{L(\log L)^\epsilon}\omega_j(z)\Big(\int_{Q_j}B(x)|f(x)|dx+\int_{Q_j}B(x)|f_{Q_j}|dx\Big)^{\frac{1}{c}}\\&+\frac{C}{\lambda}\frac{1}{\epsilon}\sum_j\inf_{z\in Q_j}M\omega_j(z)\Bigg(\int_{Q_j}B(x)|f(x)|dx+\int_{Q_j}B(x)|f_{Q_j}|dx\Bigg)\\\leq&\frac{1}{\epsilon}(B_1+B_2+B_3+B_4).\end{aligned}$$

Lemma 2.1 tells us that

$$\frac{1}{|Q_j|}\int_{Q_j}B(x)dx\leq C\prod_{i=1}^m\|b_i-(b_i)_{Q_j}\|_{expL^{s_i},Q_j}\leq C\|\vec{b}\|=C,$$

therefore

$$B_2 \le \frac{C}{\lambda}\Big(\int_{\mathbb{R}^n}\phi\Big(\frac{|f(x)|}{\lambda^c}\Big)(M^c_{L(\log L)^\epsilon}\omega_j(x))dx\Big)^{\frac{1}{c}}.$$

Proceeding similarly as before, one may obtain

$$\begin{aligned}B_1 \le &\frac{C}{\lambda}\sum_j \inf_{z\in Q_j}\Big(M_{L(\log L)^\epsilon}\omega_j\Big)\,|Q_j|^{\frac{1}{c}}\,\|f(x)\|^{\frac{1}{c}}_{L(\log L)^{\frac{1}{s}},Q_j}\\ \le &C\Big(\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\Big(\frac{|f|}{\lambda^c}\Big)\Big(M^c_{L(\log L)^\epsilon}\omega\Big)dx+\lambda^{1-c}\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\Big(\frac{|f|}{\lambda}\Big)\Big(M^c_{L(\log L)^\epsilon}\omega\Big)dx\Big)^{\frac{1}{c}},\\ B_3+B_4\le &C\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\Big(\frac{|f(y)|}{\lambda}\Big)M\omega(x)dx,\end{aligned}$$

which means that we have shown that the desired estimate holds for L_2.

Finally, we consider the last term L_3.

$$\begin{aligned}L_3 \le &\omega\Bigg(\Big\{y\in\mathbb{R}^n\backslash\tilde{\Omega}:\Big|\sum_{i=1}^{k-1}\sum_{\sigma\in C_i(b)}C_\sigma I^{\vec{\sigma}}_\alpha\Big(\sum_j(b-\vec{\lambda})_{\sigma'}f\chi_{Q_j}\Big)(x)\Big|>\frac{\lambda}{12}\Big\}\Bigg)\\ &+\omega\Bigg(\Big\{y\in\mathbb{R}^n\backslash\tilde{\Omega}:\Big|\sum_{i=1}^{k-1}\sum_{\sigma\in C_i(b)}C_\sigma I^{\vec{\sigma}}_\alpha\Big(\sum_j(b-\vec{\lambda})_{\sigma'}f_{Q_j}\chi_{Q_j}\Big)(x)\Big|>\frac{\lambda}{12}\Big\}\Bigg)\\ =:&L_{31}+L_{32}.\end{aligned}$$

By the inductive hypothesis and the Chebyshev's inequality, L_{31} can be estimated as

$$\begin{aligned}L_{31}\le &C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i(b)}\sum_j\frac{1}{\epsilon^{\#\sigma+1}}\Big\{\inf_{z\in Q_j}\Big(M_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}+\epsilon},\alpha}\omega_j(z)+M_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}+\epsilon}}\omega_j(z)\Big)\\ &\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\Big(\frac{|f(x)|}{\lambda}(b(x)-b_{Q_j})_{\sigma'}\Big)dx\\ &+\Big(\inf_{z\in Q_j}M^c_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}+\epsilon}}\omega_j(z)\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\Big(\frac{|f(x)|}{\lambda^c}(b(x)-b_{Q_j})_{\sigma'}\Big)\Big)^{\frac{1}{c}}\\ &+\Big(\inf_{z\in Q_j}M^c_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}+\epsilon}}\omega_j(z)(1+\lambda^{1-c})\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\Big(\frac{|f(x)|}{\lambda}(b(x)-b_{Q_j})_{\sigma'}\Big)dx\Big)^{\frac{1}{c}}\Big\}.\\ =&L_{311}+L_{312}+L_{313}+L_{314}.\end{aligned}$$

By Lemma 2 of [17], we have

$$\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\Big(\frac{|f(x)|}{\lambda}(b(x)-b_{Q_j})_{\sigma'}\Big)dx\le Ck\int_{Q_j}\Phi_{\frac{1}{s}}\Big(\frac{|f(x)|}{\lambda}\Big)dx.$$

We may continue with the estimate

$$\begin{aligned} L_{311}0 \leq & C_k \sum_{i=1}^{k-1} \sum_{\sigma\in c_i^k} \sum_j \frac{1}{\epsilon^{\#\sigma+1}} \inf_{z\in Q_j} M_{L(\log L)^{\sum_{i\in\sigma'} \frac{1}{s_i}+\epsilon},\alpha} \omega_j(z) \int_{Q_j} \Phi_{\frac{1}{s}}\Big(\frac{|f(x)|}{\lambda}\Big)dx \\ \leq & C_k \frac{1}{\epsilon^k} \sum_j \inf_{z\in Q_j} M_{L(\log L)^{\sum_{i\in\sigma'} \frac{1}{s_i}+\epsilon},\alpha} \omega_j(z) \int_{Q_j} \Phi_{\frac{1}{s}}\Big(\frac{|f(x)|}{\lambda}\Big)dx \\ \leq & C_k \frac{1}{\epsilon^k} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\Big(\frac{|f(x)|}{\lambda}\Big) M_{L(\log L)^{\frac{1}{s}+\epsilon},\alpha}\omega(x)dx. \end{aligned}$$

The remainders of L_{31} can be treated in the similar way. Repeated reasoning as $L_{3.1}$ may lead to our desired estimate for $L_{3.2}$ This completes the proof for Case 2 and therefore also finishes the proof of Theorem 1.2. □

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