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Intrinsic Square Function Characterizations of Several Hardy–Type Spaces–a Survey

Xianjie Yan, Dachun Yang and Wen Yuan*

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

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Dedicated to Prof. Shanzhen Lu with admiration on the occasion of his 80th birthday

Abstract. In this article, the authors give a survey about the recent developments of intrinsic square function characterizations and their applications on several Hardy-type spaces, including (weak) Musielak–Orlicz Hardy spaces, variable (weak) Hardy spaces, and Hardy spaces associated with ball quasi-Banach function spaces. The authors also present some open problems.

Key Words: Intrinsic square function, (weak) Musielak–Orlicz Hardy space, variable (weak) Hardy space, ball quasi-Banach function space, Campanato space.

AMS Subject Classifications: 42B25, 42B30, 42B35, 46E30

1 Introduction

In order to settle a conjecture proposed by Fefferman and Stein [8] on the boundedness of the Lusin area function $S(f)$ from the weighted Lebesgue space $L^2_{\mathcal{M}(v)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_v(\mathbb{R}^n)$ with $0 \le v \in L^1_{\rm loc}(\mathbb{R}^n)$, where $\mathcal{M}(v)$ denotes the Hardy–Littlewood maximal function of v, Wilson originally introduced the intrinsic square functions in [43] and obtained their boundedness on the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ in [44], where $p \in (1,\infty)$ and w belongs to the Muckenhoupt weight $A_p(\mathbb{R}^n)$. Later, Huang and Liu [15] established the intrinsic square function characterizations of the weighted Hardy space $H^1_w(\mathbb{R}^n)$ with $\alpha \in (0,1]$ and $w \in A_{1+\alpha/n}(\mathbb{R}^n)$, under the additional assumption that $f \in L^1_w(\mathbb{R}^n)$. This was further generalized to the weighted Hardy space $H^p_w(\mathbb{R}^n)$ with $\alpha \in (0,1]$, $p \in (n/(n+\alpha),1)$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$ by Wang and Liu [41], under the additional assumption that $f \in (\mathrm{Lip}(\alpha,1,0))^*$, where $(\mathrm{Lip}(\alpha,1,0))^*$ denotes the dual space of the Lipschitz space $\mathrm{Lip}(\alpha,1,0)$. In addition, Wang and Liu

*Corresponding author. *Email addresses:* xianjieyan@mail.bnu.edu.cn (X. Yan), dcyang@bnu.edu.cn (D. Yang), wenyuan@bnu.edu.cn (W. Yuan)

in [40] proved some weak type estimates of intrinsic square functions on the weighted Hardy space $H_w^p(\mathbb{R}^n)$ with $\alpha \in (0,1)$, $p = n/(n+\alpha)$, and $w \in A_1(\mathbb{R}^n)$; Wang [38] obtained the boundedness of the intrinsic square functions including the g_λ^*-function on the weighted weak Hardy space $WH_w^p(\mathbb{R}^n)$ with $\lambda \in (3+2\alpha/n,\infty)$, where $\alpha \in (0,1]$, $p \in (n/(n+\alpha),1]$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$. Indeed, these intrinsic square functions can be thought of as "grand maximal" square functions in the style of the "grand maximal function" of Fefferman and Stein from [8]; they dominate all the square functions of the form $S(f)$ (and the classical ones as well), but are not essentially bigger than any one of them. Especially, the intrinsic Lusin area function has the distinct advantage of being pointwisely comparable at different cone openings, which is a property long known not to be true for the classical Lusin area function; see Wilson [43–46] and also Lerner [20,21].

With the development of the real-variable theories of several Hardy-type spaces on Euclidean spaces, the study of the intrinsic square functions on these spaces has attracted a lot of attentions in recent years. Liang and Yang in [25] first introduced the s-order intrinsic square functions and characterized the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ in terms of the related intrinsic Lusin area function, the intrinsic g-function, and the intrinsic g_λ^*-function with the best-known range $\lambda \in (2+2(\alpha+s)/n,\infty)$, which essentially improve the known results in [15] and [41]. Motivated by this, Zhuo et al. [57] generalized the corresponding results in [25] to the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with $\lambda \in (3+2(\alpha+s)/n,\infty)$; Yan [47, 48] obtained similar characterizations on the weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ and the variable weak Hardy space $WH^{p(\cdot)}(\mathbb{R}^n)$. Very recently, Yan et al. [49] continued the above line of research and established the intrinsic square function characterizations of the Hardy type space $H_X(\mathbb{R}^n)$ related to a ball quasi-Banach function space X satisfying some mild additional assumptions. For more applications of such intrinsic square functions, we refer the reader to [10,11,23,37,39,52].

In this article, we first give a survey on the recent developments of intrinsic square function characterizations and their applications on several Hardy-type spaces, including (weak) Musielak–Orlicz Hardy spaces, variable (weak) Hardy spaces, and Hardy spaces associated with ball quasi-Banach function spaces. To be precise, the main results that we review include: the (finite) atomic characterizations of the Musielak–Orlicz Hardy space, the atomic characterization of the variable Hardy space, the (finite) atomic characterizations of the Hardy space associated with ball quasi-Banach function space, Campanato type spaces, and duality theories related to the above three kinds of function spaces as well as their intrinsic square function characterizations. We also correct some errors and seal some gaps existing in [49, Theorems 1.10, 1.12, 1.15, and 1.16]. Finally, we present some open problems.

To be precise, the remainder of this survey is organized as follows.

In Section 2, we recall the definitions of intrinsic square functions and some notation which are used throughout this article.

The aim of Section 3 is to review the intrinsic square function characterizations of $H^\varphi(\mathbb{R}^n)$ and $WH^\varphi(\mathbb{R}^n)$, where $\varphi:\ \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfies that, for any given $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt $\mathbb{A}_\infty(\mathbb{R}^n)$ weight uniformly in $t \in (0,\infty)$. To this end, via using the non-tangential grand maximal function, we first recall the definitions of $H^{\varphi}(\mathbb{R}^n)$ and $WH^{\varphi}(\mathbb{R}^n)$, which were introduced, respectively, by Ky [19] and Liang et al. [26]. Then we present (finite) atomic characterizations of $H^{\varphi}(\mathbb{R}^n)$ established in [19] and the duality theory of $H^{\varphi}(\mathbb{R}^n)$ established in [24]. Moreover, the intrinsic square function characterizations including the intrinsic g_λ^*-functions of $H^{\varphi}(\mathbb{R}^n)$ and $WH^{\varphi}(\mathbb{R}^n)$ with $\lambda \in (2+2(\alpha+s)/n,\infty)$, which were obtained, respectively, in [25] and [47], are also presented.

In Section 4, we aim to recall the intrinsic square function characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ and $WH^{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot):\ \mathbb{R}^n \to (0,\infty)$ being a variable exponent function and satisfying the globally log-Hölder continuous condition. For this purpose, we need to recall the definitions of $H^{p(\cdot)}(\mathbb{R}^n)$ and $WH^{p(\cdot)}(\mathbb{R}^n)$, which were introduced, respectively, by Nakai and Sawano [29] and Yan et al. [50]. Furthermore, the atomic characterization and the duality theory of $H^{p(\cdot)}(\mathbb{R}^n)$, established in [29], are presented. As an application, the intrinsic square function characterizations including the intrinsic g_λ^*-functions of $H^{p(\cdot)}(\mathbb{R}^n)$ and $WH^{p(\cdot)}(\mathbb{R}^n)$ with $\lambda \in (3+2(\alpha+s)/n,\infty)$, obtained, respectively, in [57] and [48], are also reviewed.

Section 5 is devoted to the intrinsic square function characterizations of Hardy spaces $H_X(\mathbb{R}^n)$ related to ball quasi-Banach function spaces X. The space $H_X(\mathbb{R}^n)$ was originally introduced and characterized in terms of atoms by Sawano et al. [33]. We then recall their finite atomic characterization, duality theory, and intrinsic square function characterizations, obtained originally in [49]. Since $H_X(\mathbb{R}^n)$ includes various known Hardy type spaces, the results obtained in [49] have a wide range of generality and essentially improve the known results for $H^{p(\cdot)}(\mathbb{R}^n)$ in [57] by widening the range of the parameter λ into the best known one. Some errors and gaps existing in [49, Theorems 1.10, 1.12, 1.15, and 1.16] are also corrected in this section.

To state the main results of this article, we first make some conventions on notation. Let $\mathbb{N} := \{1,2,\cdots\}$ and $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. We denote by C a positive constant which is independent of the main parameters, but may vary from line to line. We use $C_{(\alpha,\cdots)}$ to denote a positive constant depending on the indicated parameters $\alpha,\cdots$. The symbol $A \lesssim B$ means $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. If E is a subset of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its characteristic function and by $E^{\complement}$ the set $\mathbb{R}^n\backslash E$. For any $r\in(0,\infty)$ and $x \in \mathbb{R}^n$, we denote by $B(x,r)$ the ball centered at x with the radius r, namely, $B(x,r) := \{y \in \mathbb{R}^n:\ |x-y|<r\}$. For any ball B, we use x_B to denote its center and r_B its radius, and denote by λB for any $\lambda \in (0,\infty)$ the ball concentric with B having the radius λr_B. For any index $q \in [1,\infty]$, we denote by q' its conjugate index, namely, $1/q+1/q'=1$.

2 Intrinsic square functions

In this section, we give some notation and recall some notions on intrinsic square functions which are used throughout this article. For any measurable set $E \subset \mathbb{R}^n$ and

$r \in (0,\infty)$, let $L^r(E)$ be the set of all measurable functions f on E such that

$$\|f\|_{L^r(E)} := \left[\int_E |f(x)|^r dx\right]^{1/r} < \infty.$$

For any $r \in (0,\infty)$, we denote by $L^r_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all r-locally integrable functions on $\mathbb{R}^n$.

For any given $s \in \mathbb{Z}_+$, $C^s(\mathbb{R}^n)$ denotes the set of all functions having continuous classical derivatives up to order s. For any given $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, let $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ be the set of all functions $\phi \in C^s(\mathbb{R}^n)$ satisfying that

$$\mathrm{supp}\phi \subset \{x \in \mathbb{R}^n:\ |x| \le 1\}, \quad \int_{\mathbb{R}^n} \phi(x) x^\gamma dx = 0,$$

for any $\gamma \in \mathbb{Z}^n_+ := (\mathbb{Z}_+)^n$ with $|\gamma| \le s$, and, for any $x_1, x_2 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}^n_+$ with $|\nu| = s$,

$$|D^\nu \phi(x_1) - D^\nu \phi(x_2)| \le |x_1 - x_2|^\alpha.$$

Here and thereafter, for any $\gamma := (\gamma_1, \cdots, \gamma_n) \in \mathbb{Z}^n_+$ and $x := (x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$|\gamma| := \gamma_1 + \cdots + \gamma_n, \quad x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}, \quad \text{and} \quad D^\gamma := \left(\frac{\partial}{\partial x_1}\right)^{\gamma_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\gamma_n}.$$

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$, let

$$A_{\alpha,s}(f)(y,t) := \sup_{\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)} |f * \phi_t(y)|.$$

Here and thereafter, for any $t \in (0,\infty)$, $\phi_t(\cdot) := t^{-n}\phi(\cdot/t)$. Then the intrinsic Littlewood–Paley g-function $g_{\alpha,s}(f)$, the intrinsic Littlewood–Paley g^*_λ-function $g^*_{\lambda,\alpha,s}(f)$ with $\lambda \in (0,\infty)$, and the intrinsic Lusin-area function $S_{\alpha,s}(f)$ of f are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) := \left\{\int_0^\infty [A_{\alpha,s}(f)(x,t)]^2 \frac{dt}{t}\right\}^{1/2},$$

$$g^*_{\lambda,\alpha,s}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_{\alpha,s}(f)(y,t)]^2 \frac{dydt}{t^{n+1}}\right\}^{1/2},$$

$$S_{\alpha,s}(f)(x) := \left\{\int_{\Gamma(x)} [A_{\alpha,s}(f)(y,t)]^2 \frac{dydt}{t^{n+1}}\right\}^{1/2},$$

here and thereafter, $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+:\ |y-x| < t\}$.

We also recall another kind of similar-looking square functions, defined via convolutions with kernels which might have unbounded supports. For any given $\alpha \in (0,1]$,

$\epsilon \in (0,\infty)$, and $s \in \mathbb{Z}_+$, let $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ be the set of all functions $\phi \in C^s(\mathbb{R}^n)$ satisfying that, for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le s$, and for any $x \in \mathbb{R}^n$,

$$|D^\gamma \phi(x)| \le (1+|x|)^{-n-\epsilon}, \quad \int_{\mathbb{R}^n} \phi(x) x^\gamma dx = 0$$

and, for any $\nu \in \mathbb{Z}_+^n$ with $|\nu| = s$, and for any $x_1, x_2 \in \mathbb{R}^n$,

$$|D^\nu \phi(x_1) - D^\nu \phi(x_2)| \le |x_1 - x_2|^\alpha \left[(1+|x_1|)^{-n-\epsilon} + (1+|x_2|)^{-n-\epsilon}\right].$$

Note that, in what follows, the parameter ϵ usually has to be chosen large enough. For any f satisfying

$$|f(\cdot)|(1+|\cdot|)^{-n-\epsilon} \in L^1(\mathbb{R}^n), \tag{2.1}$$

and for any $(y,t) \in \mathbb{R}_+^{n+1}$, let

$$\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) := \sup_{\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)} |f * \phi_t(y)|.$$

Then, for any $\lambda \in (0,\infty)$ and $x \in \mathbb{R}^n$, we let

$$\begin{aligned}
\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) &:= \left\{\int_0^\infty \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(x,t)\right]^2 \frac{dt}{t}\right\}^{1/2},\\
\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x) &:= \left\{\int_0^\infty\int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 \frac{dydt}{t^{n+1}}\right\}^{1/2},\\
\widetilde{S}_{(\alpha,\epsilon),s}(f)(x) &:= \left\{\int_{\Gamma(x)} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)\right]^2 \frac{dydt}{t^{n+1}}\right\}^{1/2}.
\end{aligned}$$

When $s = 0$, these intrinsic square functions were originally introduced by Wilson [43] and he observed that they were pointwisely equivalent to each other by their generic natures; Liang and Yang generalized these observations to $s \in \mathbb{Z}_+$ in [25, Proposition 2.4 and Theorem 2.6], which read as follows.

Lemma 2.1. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (0,\infty)$. Then, for any f satisfying* (2.1)*, and for any $x \in \mathbb{R}^n$, it holds true that*

$$\begin{aligned}
&g_{\alpha,s}(f)(x) \sim S_{\alpha,s}(f)(x),\\
&\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \sim \widetilde{S}_{(\alpha,\epsilon),s}(f)(x),
\end{aligned}$$

with the positive equivalence constants independent of f and x.

Lemma 2.2. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (\max\{\alpha,s\},\infty)$. Then there exists a positive constant C such that, for any f satisfying* (2.1)*, and for any $x \in \mathbb{R}^n$,*

$$\frac{1}{C} g_{\alpha,s}(f)(x) \le \widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \le C g_{\alpha,s}(f)(x).$$

3 Intrinsic square function characterizations of (weak) Musielak–Orlicz Hardy spaces

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it is non-decreasing, $\Phi(0)=0$, $\Phi(t)>0$ for any $t\in(0,\infty)$, and $\lim_{t\to\infty}\Phi(t)=\infty$. The function Φ is said to be of upper (resp. lower) type p for some $p\in[0,\infty)$ if there exists a positive constant C such that, for any $s\in[1,\infty)$ (resp. $s\in[0,1]$) and $t\in[0,\infty)$, $\Phi(st)\le Cs^p\Phi(t)$.

For a given function $\varphi:\ \mathbb{R}^n\times[0,\infty)\to[0,\infty)$ such that, for any $x\in\mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of uniformly upper (resp. lower) type p for some $p\in[0,\infty)$ if there exists a positive constant C such that, for any $x\in\mathbb{R}^n$, $s\in[1,\infty)$ (resp. $s\in[0,1]$), and $t\in[0,\infty)$, $\varphi(x,st)\le Cs^p\varphi(x,t)$. The critical uniformly lower type index and the critical uniformly upper type index of φ are defined, respectively, by setting

$$i(\varphi):=\sup\{p\in(0,\infty):\ \varphi \text{ is of uniformly lower type } p\} \tag{3.1}$$

and

$$I(\varphi):=\inf\{p\in(0,\infty):\ \varphi \text{ is of uniformly upper type } p\}.$$

Observe that $i(\varphi)$ and $I(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ or of uniformly upper type $I(\varphi)$ (see [22]).

Let $\varphi:\ \mathbb{R}^n\times[0,\infty)\to[0,\infty)$ satisfy that $x\mapsto\varphi(x,t)$ is measurable for any $t\in[0,\infty)$. The function $\varphi(\cdot,t)$ is said to satisfy the uniformly Muckenhoupt condition for some $q\in[1,\infty)$, denoted by $\varphi\in\mathbb{A}_q(\mathbb{R}^n)$, if, when $q\in(1,\infty)$,

$$[\varphi]_{\mathbb{A}_q(\mathbb{R}^n)}:=\sup_{t\in(0,\infty)}\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|^q}\int_B\varphi(x,t)dx\left\{\int_B[\varphi(y,t)]^{-q'/q}\,dy\right\}^{q/q'}<\infty,$$

or, when $q=1$,

$$[\varphi]_{\mathbb{A}_1(\mathbb{R}^n)}:=\sup_{t\in(0,\infty)}\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B\varphi(x,t)dx\left(\operatorname*{esssup}_{y\in B}[\varphi(y,t)]^{-1}\right)<\infty,$$

where the first suprema are taken over all $t\in(0,\infty)$ and the second ones over all balls $B\subset\mathbb{R}^n$. Let

$$\mathbb{A}_\infty(\mathbb{R}^n):=\bigcup_{q\in[1,\infty)}\mathbb{A}_q(\mathbb{R}^n).$$

The critical weight index $q(\varphi)$ of $\varphi\in\mathbb{A}_\infty(\mathbb{R}^n)$ is defined by setting

$$q(\varphi):=\inf\left\{q\in[1,\infty):\ \varphi\in\mathbb{A}_q(\mathbb{R}^n)\right\}. \tag{3.2}$$

Now, we recall the notion of growth functions (see [19]).

Definition 3.1. *A function $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a growth function if the following conditions are satisfied:*

(i) φ is a Musielak–Orlicz function, namely,

(i)$_1$ the function $\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$ is an Orlicz function for almost every $x \in \mathbb{R}^n$;

(ii)$_1$ the function $\varphi(\cdot,t)$ is a measurable function on $\mathbb{R}^n$ for any $t \in [0,\infty)$.

(ii) $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$.

(iii) φ is of positive uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

Throughout the article, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0,\infty)$, define

$$\varphi(E,t) := \int_E \varphi(x,t)dx.$$

Let us now introduce the Musielak–Orlicz space and the weak Musielak–Orlicz space.

Definition 3.2. *Let φ be a growth function as in Definition* 3.1.

(i) The Musielak–Orlicz space $L^\varphi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi(x,|f(x)|)dx < \infty$ equipped with the quasi-norm

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

(ii) The weak Musielak–Orlicz space $WL^\varphi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\sup_{\alpha\in(0,\infty)} \varphi(\{x \in \mathbb{R}^n : |f(x)| > \alpha\},\alpha) < \infty$$

equipped with the quasi-norm

$$\|f\|_{WL^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \sup_{\alpha\in(0,\infty)} \varphi\left(\{x \in \mathbb{R}^n : |f(x)| > \alpha\}, \frac{\alpha}{\lambda}\right) \le 1\right\}.$$

In what follows, denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions equipped with the well-known topology determined by a countable family of norms, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual space equipped with the weak-$*$ topology. For any $m \in \mathbb{N}$, let

$$\mathcal{S}_m(\mathbb{R}^n) := \left\{\psi \in \mathcal{S}(\mathbb{R}^n) : \sup_{\beta\in\mathbb{Z}_+^n,|\beta|\le m+1}\ \sup_{x\in\mathbb{R}^n} \left[(1+|x|)^{(m+2)(n+1)} \left|D^\beta\psi(x)\right|\right] \le 1\right\}.$$

Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, the non-tangential grand maximal function f_m^* of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f_m^*(x) := \sup_{\psi \in \mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t, t\in(0,\infty)} |f * \psi_t(y)|. \tag{3.3}$$

When

$$m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1] \rfloor, \tag{3.4}$$

where $i(\varphi)$ and $q(\varphi)$ are, respectively, as in (3.1) and (3.2), we denote f_m^* as in (3.3) with $m := m(\varphi)$ simply by f^*. Here and thereafter, for any $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor$ denotes the largest integer not greater than α.

Now, we recall the definitions of the Musielak–Orlicz Hardy space (see [19,22,24,25,51]) and the weak Musielak–Orlicz Hardy space (see [26,51]).

Definition 3.3. *Let φ be a growth function as in Definition* 3.1.

(i) The Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$, equipped with the quasi-norm

$$\|f\|_{H^\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}.$$

(ii) The weak Musielak–Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in WL^\varphi(\mathbb{R}^n)$, equipped with the quasi-norm

$$\|f\|_{WH^\varphi(\mathbb{R}^n)} := \|f^*\|_{WL^\varphi(\mathbb{R}^n)}.$$

In what follows, for any $s \in \mathbb{Z}_+$, $\mathcal{P}_s(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$ of degree not greater than s; for any ball $B \subset \mathbb{R}^n$ and any locally integrable function g on $\mathbb{R}^n$, we use $P_B^s g$ to denote the minimizing polynomial of g on the ball B with degree not greater than s, which means that $P_B^s g$ is the unique polynomial $f \in \mathcal{P}_s(\mathbb{R}^n)$ such that, for any $h \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\int_B [g(x) - f(x)]\, h(x)dx = 0.$$

The following Musielak–Orlicz Campanato space was originally introduced in [24, Definition 1.1].

Definition 3.4. *Let φ be a growth function as in Definition* 3.1, *$q \in [1,\infty)$, and $s \in \mathbb{Z}_+$. Then the Musielak–Orlicz Campanato space $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \left\{ \int_B \left[\frac{|f(x) - P_B^s f(x)|}{\varphi(x, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})} \right]^q \varphi\left(x, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right) dx \right\}^{1/q} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and $P_B^s f$ denotes the minimizing polynomial of f on B with degree not greater than s.

Let $q \in [1,\infty)$ and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a nonnegative function. Then $w \in A_q(\mathbb{R}^n)$ means if, when $q \in (1,\infty)$,

$$[w]_{A_q(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B w(x)dx \left\{ \int_B [w(y)]^{-q'/q}\, dy \right\}^{q/q'} < \infty,$$

or, when $q = 1$,

$$[w]_{A_1(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B w(x)dx \left(\operatorname*{esssup}_{y\in B} [w(y)]^{-1} \right) < \infty,$$

where the suprema are taken over all balls $B \subset \mathbb{R}^n$ (see [13,28]). Let

$$A_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} A_q(\mathbb{R}^n).$$

Observe that, in the definition of $\mathbb{A}_q(\mathbb{R}^n)$, for any given $q \in [1,\infty]$, if φ is independent of t, then $\mathbb{A}_q(\mathbb{R}^n)$ just becomes the classical Muckenhoupt weight $A_q(\mathbb{R}^n)$.

Remark 3.1. (i) Recall that, for any given $\beta \in [0,\infty)$, $q \in [1,\infty)$, and $s \in \mathbb{Z}_+$, the Campanato space $L_{\beta,q,s}(\mathbb{R}^n)$, introduced by Campanato [2], is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L_{\beta,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} |B|^{-\beta} \left\{ \frac{1}{|B|} \int_B |f(x) - P^s_B f(x)|^q \, dx \right\}^{1/q} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and $P^s_B f$ denotes the minimizing polynomial of f on B with degree not greater than s. When

$$\varphi(x,t) := t^p \quad \text{for any } x \in \mathbb{R}^n \quad \text{and} \quad t \in (0,\infty), \tag{3.5}$$

where $p \in (0,1]$, via some trivial computations, we know that

$$\|f\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)} = \|f\|_{L_{\frac{1}{p}-1,q,s}(\mathbb{R}^n)}.$$

In this case, $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ just becomes the classical Campanato space $L_{\frac{1}{p}-1,q,s}(\mathbb{R}^n)$ (see also [24, Remark 1.2(i)]).

(ii) Let $p \in (0,\infty)$ and $w \in A_\infty(\mathbb{R}^n)$. Recall that the weighted Campanato space $L_{w,\beta,q,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L_{w,\beta,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{[w(B)]^\beta} \left\{ \frac{1}{w(B)} \int_B |f(x) - P^s_B f(x)|^q \, [w(x)]^{1-q} dx \right\}^{1/q} < \infty,$$

where

$$w(B) := \int_B w(x)dx,$$

the supremum is taken over all balls B of $\mathbb{R}^n$, and $P_B^s f$ denotes the minimizing polynomial of f on B with degree not greater than s. When

$$\varphi(x,t) := w(x)t^p \quad \text{for any } x \in \mathbb{R}^n \quad \text{and} \quad t \in (0,\infty), \tag{3.6}$$

where $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$, it is easy to see that

$$\|f\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)} = \|f\|_{L_{w,\frac{1}{p}-1,q,s}(\mathbb{R}^n)}.$$

In this case, $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ is just the weighted Campanato space $L_{w,\frac{1}{p}-1,q,s}(\mathbb{R}^n)$ (see also [24, Remark 1.2(ii)]).

The following John–Nirenberg inequality for functions in $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ was obtained in [24, Theorem 2.5].

Theorem 3.1. *Let φ be a growth function as in Definition 3.1, $s \in \mathbb{Z}_+$, and $f \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$. Then there exist positive constants C_1, C_2, and C_3, independent of f, such that, for any ball $B \subset \mathbb{R}^n$ and $\alpha \in (0,\infty)$, when $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$,*

$$\begin{aligned}&\varphi\left(\left\{x \in B:\ \frac{|f(x)-P_B^s f(x)|}{\varphi(x,\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})} > \alpha\right\}, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\\ \le& C_1 \exp\left\{-\frac{C_2\alpha}{\|f\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}}\right\},\end{aligned}$$

and, when $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$ for some $q \in (1,\infty)$,

$$\begin{aligned}&\varphi\left(\left\{x \in B:\ \frac{|f(x)-P_B^s f(x)|}{\varphi(x,\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})} > \alpha\right\}, \|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\right)\\ \le& C_3 \left[1+\frac{\alpha}{\|f\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}}\right]^{-q'},\end{aligned}$$

where $1/q+1/q'=1$.

Via using Theorem 3.1, Liang and Yang in [24, Theorem 2.7] proved the following equivalent characterization for $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$; the details are omitted here.

Theorem 3.2. *Let φ be a growth function as in Definition 3.1, $s \in \mathbb{Z}_+$, and $q \in (1,[q(\varphi)]')$ with $q(\varphi)$ as in (3.2). Then the following statements are mutually equivalent:*

(i) $f \in \mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$.

(ii) $f \in L^1_{\rm loc}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x) - P^s_B f(x)|\, dx < \infty,$$

where the supremum is taken over all balls B *of* $\mathbb{R}^n$ *and* $P^s_B f$ *denotes the minimizing polynomial of* f *on* B *with degree not greater than* s*.*

Moreover, $\|\cdot\|_{\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)}$ *and* $\|\cdot\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)}$ *are equivalent quasi-norms.*

Let φ be a growth function as in Definition 3.1. Recall that, for any measurable subset E of $\mathbb{R}^n$, the space $L^q_\varphi(E)$ for any given $q \in [1,\infty]$ is defined to be the set of all measurable functions f on E such that

$$\|f\|_{L^q_\varphi(E)} := \begin{cases} \sup\limits_{t\in(0,\infty)} \left[\dfrac{1}{\varphi(E,t)} \displaystyle\int_E |f(x)|^q \varphi(x,t)dx\right]^{1/q} < \infty, & \text{when } q \in [1,\infty), \\ \|f\|_{L^\infty(E)} < \infty & \text{when } q = \infty. \end{cases}$$

Let $q \in (q(\varphi),\infty]$ and $s \in [m(\varphi),\infty) \cap \mathbb{Z}_+$, where $q(\varphi)$ and $m(\varphi)$ are, respectively, as in (3.2) and (3.4). A measurable function a on $\mathbb{R}^n$ is called a (φ,q,s)-atom if there exists a ball B such that

(i) $\operatorname{supp} a := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset B$;

(ii) $\|a\|_{L^q_\varphi(B)} \le \|\mathbf{1}_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\delta dx = 0$ for any $\delta \in \mathbb{Z}^n_+$ with $|\delta| \le s$.

Recall that the atomic Musielak–Orlicz Hardy space $H^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f = \sum_{j\in\mathbb{N}} b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{b_j\}_{j\in\mathbb{N}}$ is a sequence of multiples of (φ,q,s)-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$ satisfying that

$$\sum_{j\in\mathbb{N}} \varphi\left(B_j, \|b_j\|_{L^q_\varphi(B_j)}\right) < \infty.$$

For any given sequence $\{b_j\}_{j\in\mathbb{N}}$ of multiples of (φ,q,s)-atoms, let

$$\Lambda_q(\{b_j\}_{j\in\mathbb{N}}) := \inf\left\{\lambda \in (0,\infty) : \sum_{j\in\mathbb{N}} \varphi\left(B_j, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \le 1\right\},$$

$$\|f\|_{H^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)} := \inf\left\{\Lambda_q(\{b_j\}_{j\in\mathbb{N}}) : f = \sum_{j\in\mathbb{N}} b_j \text{ in } \mathcal{S}'(\mathbb{R}^n)\right\},$$

where the last infimum is taken over all representations of f as above. We use $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ to denote the set of all finite combinations of (φ,q,s)-atoms. The norm of f in $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ is defined by setting

$$\|f\|_{H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)} := \inf\left\{\Lambda_q(\{b_j\}_{j=1}^k):\ f=\sum_{j=1}^k b_j \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n)\right\},$$

where the infimum is taken over all finite decompositions of f as above.

The following atomic characterization of $H^\varphi(\mathbb{R}^n)$ is just [19, Theorem 3.1].

Theorem 3.3. *Let φ be a growth function as in Definition 3.1, $q\in(q(\varphi),\infty]$, and $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$, where $q(\varphi)$ and $m(\varphi)$ are, respectively, as in (3.2) and (3.4). Then $H^\varphi(\mathbb{R}^n)=H^{\varphi,q,s}_{\rm atom}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Definition 3.5. *A growth function φ is said to satisfy the uniformly locally dominated convergence condition if the following holds true:*

For any compact set $K\subset\mathbb{R}^n$ and any sequence $\{f_m\}_{m\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$, if $f_m\to f$ almost everywhere as $m\to\infty$, and, for any $m\in\mathbb{N}$, $|f_m|\le g$ almost everywhere for some nonnegative measurable function g satisfying that

$$\sup_{t\in(0,\infty)}\int_K g(x)\frac{\varphi(x,t)}{\varphi(K,t)}dx<\infty,$$

then

$$\sup_{t\in(0,\infty)}\int_K |f_m(x)-f(x)|\,\frac{\varphi(x,t)}{\varphi(K,t)}dx\to 0$$

as $m\to\infty$.

Remark 3.2. We point that, as was mentioned in [19, p. 125], when $w\in A_\infty(\mathbb{R}^n)$ and Φ is an Orlicz function, the growth function $\varphi(x,t):=w(x)\Phi(t)$ for any $(x,t)\in\mathbb{R}^n\times[0,\infty)$ satisfies the uniformly locally dominated convergence condition.

In what follows, the symbol $\mathcal{C}(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$. The following finite atomic characterization of $H^\varphi(\mathbb{R}^n)$ is just [19, Theorem 3.4].

Theorem 3.4. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition, $q\in(q(\varphi),\infty]$, and $s\in[m(\varphi),\infty)\cap\mathbb{Z}_+$, where $q(\varphi)$ and $m(\varphi)$ are, respectively, as in (3.2) and (3.4).*

- *(i) If $q\in(q(\varphi),\infty)$, then $\|\cdot\|_{H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^\varphi(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$;*
- *(ii) If $q=\infty$, then $\|\cdot\|_{H^{\varphi,\infty,s}_{\rm fin}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^\varphi(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^{\varphi,\infty,s}_{\rm fin}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$.*

Combining Theorems 3.2, 3.3, and 3.4, Liang and Yang obtained the following conclusion in [24, Theorem 3.5]; the details of its proof are omitted here.

Theorem 3.5. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition, and $s \in [m(\varphi),\infty) \cap \mathbb{Z}_+$ with $m(\varphi)$ as in* (3.4). *Then the dual space of $H^\varphi(\mathbb{R}^n)$, denoted by $(H^\varphi(\mathbb{R}^n))^*$, is $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ in the following sense:*

(i) Suppose that $f \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$. Then the linear functional

$$T_f:\ g \to T_f(g) := \int_{\mathbb{R}^n} f(x)g(x)dx, \tag{3.7}$$

initially defined for all $g \in H^{\varphi,q,s}_{\rm fin}(\mathbb{R}^n)$ with some $q \in (q(\varphi),\infty)$ and $q(\varphi)$ as in (3.2), *has a bounded extension to $H^\varphi(\mathbb{R}^n)$.*

(ii) Conversely, every continuous linear functional on $H^\varphi(\mathbb{R}^n)$ arises as in (3.7) *with a unique $f \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$.*

Moreover,

$$\|f\|_{\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)} \sim \|T_f\|_{(H^\varphi(\mathbb{R}^n))^*},$$

where the positive equivalence constants are independent of f.

As an immediate corollary of Theorems 3.2 and 3.5, the following conclusion was obtained in [24, Corollary 3.7].

Corollary 3.1. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition. Then, for any given $q \in [1,[q(\varphi)]')$ and $s \in [m(\varphi),\infty) \cap \mathbb{Z}_+$, $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$ and $\mathcal{L}_{\varphi,1,m(\varphi)}(\mathbb{R}^n)$ coincide with equivalent quasi-norms, where $q(\varphi)$ and $m(\varphi)$ are, respectively, as in* (3.2) *and* (3.4), *and $1/q(\varphi)+1/[q(\varphi)]'=1$.*

By Remarks 3.1 and 3.2, and Theorem 3.5, we have the following observations.

Remark 3.3. (i) For any given $p \in (0,1]$, we denote the classical Lebesgue space and the Hardy space, respectively, by $L^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$. Let $p \in (0,1]$ and $s \in [\lfloor n(1/p-1)\rfloor,\infty) \cap \mathbb{Z}$. When φ is defined as in (3.5), then $H^\varphi(\mathbb{R}^n)$ and $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ are, respectively, $H^p(\mathbb{R}^n)$ and $L_{\frac{1}{p}-1,1,s}(\mathbb{R}^n)$. In this case, the conclusion of Theorem 3.5 was obtained by Taibleson and Weiss in [34] (see also [24, Remark 3.6(i)]).

(ii) Let $p \in (0,\infty)$ and $w \in A_\infty(\mathbb{R}^n)$. Recall that the weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right\}^{1/p} < \infty.$$

Let $p \in (0,1]$, $q_0 \in [1,\infty)$, $s \in [\lfloor n(q_0/p-1)\rfloor,\infty)\cap\mathbb{Z}$, and $w \in A_\infty(\mathbb{R}^n)$ with the critical weight index q_0. When φ is defined as in (3.6), then $H^\varphi(\mathbb{R}^n)$ and $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ become, respectively, the weighted Hardy space $H^p_w(\mathbb{R}^n)$ and its dual space $L_{w,\frac{1}{p}-1,1,s}(\mathbb{R}^n)$. In this case, the conclusion of Theorem 3.5 was obtained by García-Cuerva in [12] (see also [24, Remark 3.6(ii)]).

(iii) Let Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. We denote the Orlicz space and the Orlicz–Hardy space, respectively, by $L^\Phi(\mathbb{R}^n)$ and $H^\Phi(\mathbb{R}^n)$. Recall that the Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda>0:\ \int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\right\}<\infty.$$

For any given $q\in[1,\infty)$ and $s\in\mathbb{Z}_+$, the Campanato space $\mathcal{L}_{\Phi,q,s}(\mathbb{R}^n)$ is defined to be the set of all $f\in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}_{\Phi,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n}\frac{|B|}{\|\mathbf{1}_B\|_{L^\Phi(\mathbb{R}^n)}}\left\{\frac{1}{|B|}\int_B|f(x)-P^s_Bf(x)|^q\,dx\right\}^{1/q}<\infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and P^s_Bf denotes the minimizing polynomial of f on B with degree not greater than s. Let $p_\Phi^+\in(0,1]$ and $s\ge\lfloor n(1/p_\Phi^- -1)\rfloor$. If

$$\varphi(x,t):=\Phi(t)\quad\text{for any }\ x\in\mathbb{R}^n\quad\text{and}\quad t\in(0,\infty),$$

where Φ is an Orlicz function, then $H^\varphi(\mathbb{R}^n)$ and $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ are, respectively, $H^\Phi(\mathbb{R}^n)$ and $\mathcal{L}_{\Phi,1,s}(\mathbb{R}^n)$. A corresponding conclusion of Theorem 3.5 for Orlicz–Hardy spaces was obtained by Nakai and Sawano [30].

The following lemma is from [34, p. 83].

Lemma 3.1. *Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$, $s\in\mathbb{Z}_+$, and B be a ball in $\mathbb{R}^n$. Then there exists a positive constant C, independent of f and B, such that*

$$\sup_{x\in B}|P^s_Bf(x)|\le\frac{C}{|B|}\int_B|f(x)|dx,$$

where P^s_Bf denotes the minimizing polynomial of f on B with degree not greater than s.

When $f\in\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, Liang and Yang in [25, Proposition 2.3] obtained the following conclusion for $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, which implies that the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$. We generalize the corresponding result to $f\in\mathcal{S}(\mathbb{R}^n)$ by an argument similar to that used in the proof of [25, Proposition 2.3]. In what follows, we use $\vec{0}_n$ to denote the origin of $\mathbb{R}^n$.

Lemma 3.2. *Let φ be a growth function as in Definition* 3.1, $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p \in (n/(n+\alpha+s),1]$, *and* $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$. *If* $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ *or* $f \in \mathcal{S}(\mathbb{R}^n)$, *then* $f \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$.

Proof. We only consider $\mathcal{S}(\mathbb{R}^n)$ here. For any $f \in \mathcal{S}(\mathbb{R}^n)$, any ball $B := B(x_0,r) \subset \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ and $r \in (0,\infty)$, and any $x \in \mathbb{R}^n$, let

$$p_B(x) := \sum_{\gamma\in\mathbb{Z}_+^n,\ |\gamma|\le s} \frac{D^\gamma f(x_0)}{\gamma!}(x-x_0)^\gamma \in \mathcal{P}_s(\mathbb{R}^n).$$

Then, from Lemma 3.1 and the Taylor remainder theorem, we deduce that, for any $x \in B$, there exists an $\eta(x) \in B$ such that

$$\begin{aligned}\int_B |f(x)-P_B^s f(x)|\,dx &\le \int_B [|f(x)-p_B(x)| + |P_B^s(p_B-f)(x)|]\,dx \\ &\lesssim \int_B |f(x)-p_B(x)|\,dx \\ &\lesssim \int_B \left| \sum_{\gamma\in\mathbb{Z}_+^n,\ |\gamma|=s+1} \frac{D^\gamma f(\eta(x))}{\gamma!}(x-x_0)^\gamma \right| dx. \end{aligned} \tag{3.8}$$

If $|x_0|+r\le 1$, namely, $B\subset B(\vec{0}_n,1)$, then, by (3.8) and [25, (2.2)], we obtain

$$\frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}}\int_B |f(x)-P_B^s f(x)|\,dx \lesssim \frac{|B|^{(n+s+1)/n}}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \lesssim \frac{|B(\vec{0}_n,1)|^{(n+s+1)/n}}{\|\mathbf{1}_{B(\vec{0}_n,1)}\|_{L^\varphi(\mathbb{R}^n)}} \lesssim 1. \tag{3.9}$$

If $|x_0|+r>1$ and $|x_0|\le 2r$, then $r>1/3$ and $|B|\sim |B(\vec{0}_n,|x_0|+r)|$. Since f is bounded, from Lemma 3.1 and [25, (2.2)], we deduce that

$$\begin{aligned}&\frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}}\int_B |f(x)-P_B^s f(x)|\,dx \\ \lesssim&\frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}}\int_B |f(x)|dx \\ \lesssim&\frac{|B|}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \lesssim \frac{|B(\vec{0}_n,|x_0|+r)|}{\|\mathbf{1}_{B(\vec{0}_n,|x_0|+r)}\|_{L^\varphi(\mathbb{R}^n)}} \lesssim 1. \end{aligned} \tag{3.10}$$

If $|x_0|+r>1$ and $|x_0|>2r$, then, for any $x\in B$, it holds true that $|x|\sim|x_0|\gtrsim 1$ and $1+|x_0|\sim|x_0|+r$, which, together with the fact that

$$|D^\gamma f(x)| \lesssim (1+|x|)^{-n-\epsilon}$$

for any $x \in \mathbb{R}^n$ and $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| = s+1$, (3.8), and [25, (2.2)], further implies that

$$\begin{aligned}
&\frac{1}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x) - P_B^s f(x)|\, dx \\
\lesssim &(1+|x_0|)^{-n-\epsilon} \frac{|B|^{(n+s+1)/n}}{\|\mathbf{1}_B\|_{L^\varphi(\mathbb{R}^n)}} \\
\lesssim &|B(\vec{0}_n, |x_0|+r)|^{-(n+\epsilon)/n} \frac{|B(\vec{0}_n, |x_0|+r)|^{(n+s+1)/n}}{\|\mathbf{1}_{B(\vec{0}_n,|x_0|+r)}\|_{L^\varphi(\mathbb{R}^n)}} \\
\lesssim &\frac{|B(\vec{0}_n, |x_0|+r)|^{(s+1-\epsilon)/n}}{\|\mathbf{1}_{B(\vec{0}_n,|x_0|+r)}\|_{L^\varphi(\mathbb{R}^n)}} \lesssim 1.
\end{aligned} \tag{3.11}$$

Combining (3.9), (3.10), and (3.11), we obtain $f \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$, which completes the proof of Lemma 3.2. □

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to vanish weakly at infinity if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$ (see, for instance, [9, p. 50]). We point out that the definition of the vanishing weakly at infinity in this article is the same as those in [33, p. 29] or [36, p. 13]. The following conclusion was obtained in [14, Lemma 4.14].

Lemma 3.3. *Let φ be a growth function as in Definition* 3.1. *If $f \in H^\varphi(\mathbb{R}^n)$, then f vanishes weakly at infinity.*

Next, we state the main results of this section. The following intrinsic square function characterizations of the Musielak–Orlicz Hardy space were obtained in [25, Theorems 1.6 and 1.8, and Corollary 1.7].

Theorem 3.6. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g_{\alpha,s}(f) \in L^\varphi(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H^\varphi(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H^\varphi(\mathbb{R}^n)} \le \|g_{\alpha,s}(f)\|_{L^\varphi(\mathbb{R}^n)} \le C\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

The same is true if $g_{\alpha,s}(f)$ is replaced, respectively, by $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$, and $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s,\infty)$.

Theorem 3.7. *Let φ be a growth function satisfying the uniformly locally dominated convergence condition. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$, and $\lambda \in (2+2(\alpha+s)/n,\infty)$. Then $f \in H^\varphi(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g^*_{\lambda,\alpha,s}(f) \in L^\varphi(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H^\varphi(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H^\varphi(\mathbb{R}^n)} \le \left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^\varphi(\mathbb{R}^n)} \le C\|f\|_{H^\varphi(\mathbb{R}^n)}.$$

The same is true if $g^*_{\lambda,\alpha,s}(f)$ is replaced by $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s,\infty)$.

Remark 3.4. (i) We point out that there exists a positive constant $\widetilde{C}$ such that, for any $\psi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)$, $\widetilde{C}\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ and hence $\psi \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ (see Lemma 3.2). Thus, all the above intrinsic square functions appearing in Theorems 3.6 and 3.7 are well defined for functionals in $(\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$. Moreover, if $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $\psi \in \mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ (see also Lemma 3.2). Therefore, if $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$, then $f \in \mathcal{S}'(\mathbb{R}^n)$, which implies that f vanishing weakly at infinity makes sense in Theorems 3.6 and 3.7. On the other hand, for any $f \in H^\varphi(\mathbb{R}^n)$, by Theorem 3.5 and Lemma 3.3, we know that it naturally holds true that $f \in (\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n))^*$ and f vanishes weakly at infinity.

(ii) Let $\alpha \in (0,1]$, $p \in (n/(n+\alpha),1]$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$. Huang and Liu [15] and Wang and Liu [41] established the intrinsic Littlewood–Paley g_λ^*-characterization of $H_w^p(\mathbb{R}^n)$ with $\lambda > 3+2\alpha/n$ under some additional assumptions. This corresponds to the case when $s=0$ and φ is defined as in (3.6) of Theorem 3.7, in which we remove those additional assumptions and also improve the range of λ to $\lambda > 2+2\alpha/n$ which coincides with the best known range of λ. Moreover, via using Lemma 2.1, Liang and Yang in [25] gave simpler proofs of [15, Theorem 3] and [41, Theorem 3].

(iii) As applications of Theorem 3.6, Liang and Yang [25] and Fu and Yang [11] obtained, respectively, the φ-Carleson measure characterization of $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ and the wavelet characterization of $H^\varphi(\mathbb{R}^n)$.

The following intrinsic square function characterizations of the weak Musielak–Orlicz Hardy space were obtained in [47, Theorems 1.6 and 1.7].

Theorem 3.8. *Let φ be a growth function as in Definition* 3.1*. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, and $\varphi \in \mathbb{A}_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$.*

(i) If $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g_{\alpha,s}(f) \in WL^\varphi(\mathbb{R}^n)$, then $f \in WH^\varphi(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of f, such that

$$\|f\|_{WH^\varphi(\mathbb{R}^n)} \le C\,\|g_{\alpha,s}(f)\|_{WL^\varphi(\mathbb{R}^n)};$$

(ii) There exists a positive constant C such that, for any $f \in WH^\varphi(\mathbb{R}^n) \cap (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$,

$$\|g_{\alpha,s}(f)\|_{WL^\varphi(\mathbb{R}^n)} \le C\|f\|_{WH^\varphi(\mathbb{R}^n)}.$$

The same is true if $g_{\alpha,s}(f)$ is replaced, respectively, by $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$, and $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\max\{\alpha,s\},\infty)$.

Theorem 3.9. *Let φ be a growth function as in Definition* 3.1*. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/(n+\alpha+s),1]$, $\varphi \in \mathbb{A}_{p[1+(\alpha+s)/n]}(\mathbb{R}^n)$, and $\lambda \in (2+2(\alpha+s)/n,\infty)$.*

(i) If $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g^*_{\lambda,\alpha,s}(f) \in WL^\varphi(\mathbb{R}^n)$, then $f \in WH^\varphi(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of f, such that

$$\|f\|_{WH^\varphi(\mathbb{R}^n)} \le C\,\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{WL^\varphi(\mathbb{R}^n)};$$

(ii) There exists a positive constant C such that, for any $f \in WH^{\varphi}(\mathbb{R}^n) \cap (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$,

$$\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{WL^{\varphi}(\mathbb{R}^n)} \leq C\|f\|_{WH^{\varphi}(\mathbb{R}^n)}.$$

The same is true if $g^*_{\lambda,\alpha,s}(f)$ *is replaced by* $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ *with* $\epsilon \in (\max\{\alpha, s\}, \infty)$.

Remark 3.5. (i) From Remark 3.4(i), we deduce that all the intrinsic square functions appearing in Theorems 3.8(ii) and 3.9(ii) are well defined for functionals in $(\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$.

(ii) Let $\alpha \in (0,1]$, $p \in (n/(n+\alpha), 1]$, and $w \in A_{p(1+\alpha/n)}(\mathbb{R}^n)$. Recall that Wang [38] established the boundedness of the intrinsic Littlewood–Paley g^*_λ-function on the weighted weak Hardy space $WH^p_w(\mathbb{R}^n)$ with $\lambda \in (3+2\alpha/n, \infty)$. This corresponds to the case when $s = 0$ and φ is defined as in (3.6) of Theorem 3.9(ii), in which we improve the range of λ to the best-known range $\lambda \in (2+2\alpha/n, \infty)$.

(iii) Differently from Theorems 3.6 and 3.7, we, in Theorems 3.8 and 3.9, need some additional assumptions, this is essentially because the dual space of $WH^{\varphi}(\mathbb{R}^n)$ is not known so far. It is still a challenging problem to remove these additional assumptions.

4 Intrinsic square function characterizations of variable (weak) Hardy spaces

We first recall some notation about variable Lebesgue spaces. For an exposition of these concepts, we refer the reader to the monographs [4, 6]. A measurable function $p(\cdot): \mathbb{R}^n \to (0,\infty)$ is called a variable exponent. Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all variable exponents $p(\cdot)$ satisfying

$$0 < p_- := \operatorname*{essinf}_{x\in\mathbb{R}^n} p(x) \leq \operatorname*{esssup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty. \tag{4.1}$$

Here and thereafter, define

$$\underline{p} := \min\{p_-, 1\}, \tag{4.2}$$

where p_- is as in (4.1).

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The modular functional (or, simply, the modular) $\varrho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting, for any measurable function f on $\mathbb{R}^n$,

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$$

and the Luxembourg (also known as the Luxembourg–Nakano) quasi-norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ of any measurable function f on $\mathbb{R}^n$ is defined by setting

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \ \varrho_{p(\cdot)}(f/\lambda) \leq 1\right\}.$$

Definition 4.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

(i) The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$.

(ii) The variable weak Lebesgue space $WL^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{WL^{p(\cdot)}(\mathbb{R}^n)} := \sup_{\alpha \in (0,\infty)} \left[\alpha \left\|\mathbf{1}_{\{x \in \mathbb{R}^n:\ |f(x)| > \alpha\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right] < \infty.$$

A function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is said to satisfy the globally log-Hölder continuous condition, denoted by $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, if there exist positive constants $C_{\log}(p)$ and C_∞, and $p_\infty \in \mathbb{R}$ such that, for any $x,\ y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/|x-y|)},$$
$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

For any $N \in \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \sum_{\beta \in \mathbb{Z}_+^n,\ |\beta| \le N} \sup_{x \in \mathbb{R}^n} \left[(1+|x|)^N \left|D^\beta \psi(x)\right|\right] \le 1 \right\}.$$

Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, the radial grand maximal function $f^*_{N,+}$ of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f^*_{N,+}(x) := \sup \left\{ |f * \psi_t(x)| :\ t \in (0,\infty) \text{ and } \psi \in \mathcal{F}_N(\mathbb{R}^n) \right\}. \tag{4.3}$$

Now, we recall the definitions of the variable Hardy space (see [29,57]) and the variable weak Hardy space (see [48,50,58]).

Definition 4.2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $N \in (\frac{n}{\underline{p}} + n + 1, \infty)$ be a positive integer, where $\underline{p}$ is as in* (4.2).

(i) The variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^*_{N,+} \in L^{p(\cdot)}(\mathbb{R}^n)$, equipped with the quasi-norm

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \left\|f^*_{N,+}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Here and thereafter, $f^*_{N,+}$ is as in (4.3).

(ii) *The variable weak Hardy space $WH^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^*_{N,+} \in WL^{p(\cdot)}(\mathbb{R}^n)$, equipped with the quasi-norm*

$$\|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)} := \left\|f^*_{N,+}\right\|_{WL^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 4.1. (i) The spaces $H^{p(\cdot)}(\mathbb{R}^n)$ and $WH^{p(\cdot)}(\mathbb{R}^n)$ are independent of the choice of $N \in (\frac{n}{\underline{p}} + n + 1, \infty) \cap \mathbb{Z}_+$; see [29, Theorem 3.3] and [50, Remark 2.14(i)].

(ii) If $p(\cdot) \equiv p \in (0, \infty)$, the spaces $H^{p(\cdot)}(\mathbb{R}^n)$ and $WH^{p(\cdot)}(\mathbb{R}^n)$ are, respectively, the classical Hardy space $H^p(\mathbb{R}^n)$ and the classical weak Hardy space $WH^p(\mathbb{R}^n)$.

The following Campanato type space was originally introduced in [29, Definition 6.1].

Definition 4.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in [1, \infty)$, and $s \in \mathbb{Z}_+$. Then the variable exponent Campanato space $\mathcal{L}_{p(\cdot),q,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{L}_{p(\cdot),q,s}(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{|B|}{\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[\frac{1}{|B|}\int_B |f(x) - P^s_B f(x)|^q \, dx\right]^{1/q} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and $P^s_B f$ denotes the minimizing polynomial of f on B with degree not greater than s.

The following John–Nirenberg inequality for functions in $\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)$ was obtained in [57, Lemma 2.21].

Theorem 4.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $s \in (\frac{n}{p_-} - n - 1, \infty) \cap \mathbb{Z}_+$, and $f \in \mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)$, where p_- and p_+ are as in* (4.1)*. Then there exist positive constants c_1 and c_2, independent of f, such that, for any ball $B \subset \mathbb{R}^n$ and $\lambda \in (0, \infty)$,*

$$|\{x \in B: \ |f(x) - P^s_B f(x)| > \lambda\}| \le c_1 \exp\left\{-\frac{c_2|B|\lambda}{\|f\|_{\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)}\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\}|B|.$$

Via using Theorem 4.1, Zhuo et al. in [57, Corollary 2.22] obtained the following equivalent characterization for $\mathcal{L}_{p(\cdot),q,s}(\mathbb{R}^n)$.

Theorem 4.2. *Let $p(\cdot)$, s be as in Theorem* 4.1*, and $q \in (1, \infty)$. Then $f \in \mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)$ if and only if $f \in \mathcal{L}_{p(\cdot),q,s}(\mathbb{R}^n)$.*

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in [1, \infty]$, and $s \in \mathbb{Z}_+$. A measurable function a on $\mathbb{R}^n$ is called a $(p(\cdot), q, s)$-atom if there exists a ball B such that

(i) $\operatorname{supp} a := \{x \in \mathbb{R}^n: \ a(x) \neq 0\} \subset B$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le \frac{|B|^{1/q}}{\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\delta dx=0$ for any $\delta\in\mathbb{Z}_+^n$ with $|\delta|\le s$.

Recall that the variable atomic Hardy space $H_{\rm atom}^{p(\cdot),q,s}(\mathbb{R}^n)$ is defined to be the set of all $f\in\mathcal{S}'(\mathbb{R}^n)$ which can be represented as

$$f=\sum_{j\in\mathbb{N}}\lambda_j a_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

where $\{\lambda_j\}_{j\in\mathbb{N}}$ is a sequence of non-negative numbers, satisfying

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j\mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}<\infty,$$

and $\{a_j\}_{j\in\mathbb{N}}$ a sequence of $(p(\cdot),q,s)$-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$. Moreover, for any $f\in H_{\rm atom}^{p(\cdot),q,s}(\mathbb{R}^n)$, let

$$\|f\|_{H_{\rm atom}^{p(\cdot),q,s}(\mathbb{R}^n)}=\inf\left\{\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j\mathbf{1}_{B_j}}{\|\mathbf{1}_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\},$$

where $\underline{p}$ is as in (4.2) and the infimum is taken over all admissible decompositions of f as above.

The following conclusion is just [32, Theorem 1.1].

Theorem 4.3. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$, $q\in[1,\infty]\cap(p_+,\infty]$, and $s\in(\frac{n}{p_-}-n-1,\infty)\cap\mathbb{Z}_+$, where p_+ and p_- are as in* (4.1). *Then $H^{p(\cdot)}(\mathbb{R}^n)=H_{\rm atom}^{p(\cdot),q,s}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Let $q\in[1,\infty]$ and $s\in\mathbb{Z}_+$. Denote by $L_{\rm comp}^q(\mathbb{R}^n)$ the set of all functions $f\in L^q(\mathbb{R}^n)$ with compact support and

$$L_{\rm comp}^{q,s}(\mathbb{R}^n):=\left\{f\in L_{\rm comp}^q(\mathbb{R}^n):\ \int_{\mathbb{R}^n}f(x)x^\delta dx=0 \text{ for any } \delta\in\mathbb{Z}_+^n \text{ with } |\delta|\le s\right\}.$$

The following conclusion was obtained in [29, Theorem 7.5].

Theorem 4.4. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ and $p_+\in(0,1]$. Assume that $q\in[1,\infty]\cap(p_+,\infty]$ and $s\in(n/p_--n-1,\infty)\cap\mathbb{Z}_+$. Then the dual space of $H_{\rm atom}^{p(\cdot),q,s}(\mathbb{R}^n)$, denoted by $(H_{\rm atom}^{p(\cdot),q,s}(\mathbb{R}^n))^*$, is $\mathcal{L}_{p(\cdot),q',s}(\mathbb{R}^n)$ in the following sense: for any $f\in\mathcal{L}_{p(\cdot),q',s}(\mathbb{R}^n)$, the linear functional*

$$T_f:\ g\to T_f(g):=\int_{\mathbb{R}^n}f(x)g(x)dx, \tag{4.4}$$

initially defined for any $g \in L^{q,s}_{\rm comp}(\mathbb{R}^n)$, has a bounded extension to $H^{p(\cdot),q,s}_{\rm atom}(\mathbb{R}^n)$; conversely, if T is a bounded linear functional on $H^{p(\cdot),q,s}_{\rm atom}(\mathbb{R}^n)$, then T has the form as in (4.4) *with a unique $f \in \mathcal{L}_{p(\cdot),q',s}(\mathbb{R}^n)$. Moreover,*

$$\|f\|_{\mathcal{L}_{p(\cdot),q',s}(\mathbb{R}^n)} \sim \|T_f\|_{(H^{p(\cdot),q,s}_{\rm atom}(\mathbb{R}^n))^*},$$

where the positive equivalence constants are independent of f.

By Theorems 4.3 and 4.4, we obtain the following equivalence on the spaces $\mathcal{L}_{p(\cdot),q,s}(\mathbb{R}^n)$, whose proof is omitted.

Corollary 4.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $q \in (1,\infty)$, $s \in (n/p_- - n - 1,\infty) \cap \mathbb{Z}_+$, and $s_0 := \lfloor n/p_- - n \rfloor$. Then $\mathcal{L}_{p(\cdot),q,s}(\mathbb{R}^n)$ and $\mathcal{L}_{p(\cdot),1,s_0}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

The following conclusion, which was obtained in [57, Lemma 2.8], implies that the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n))^*$.

Lemma 4.1. *Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (\alpha+s,\infty)$. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_- \in (n/(n+\alpha+s),1]$. If $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ or $f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in \mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)$.*

The following conclusion was obtained in [57, Corollary 2.2].

Lemma 4.2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then f vanishes weakly at infinity.*

As the main results of this section, the following intrinsic square function characterizations of variable Hardy spaces were obtained in [57, Theorems 1.8, 1.10, and Corollary 1.9].

Theorem 4.5. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $p_- \in (n/(n+\alpha+s),1]$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g_{\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le \|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

The same is true if $g_{\alpha,s}(f)$ is replaced, respectively, by $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$, and $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s,\infty)$.

Theorem 4.6. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p_- \in (n/(n+\alpha+s),1]$, and $\lambda \in (3+2(\alpha+s)/n,\infty)$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g^*_{\lambda,\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le \left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

The same is true if $g^*_{\lambda,\alpha,s}(f)$ is replaced by $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s,\infty)$.

Remark 4.2. (i) By Lemma 4.1 and an argument similar to that used in Remark 3.4(i), we conclude that all the above intrinsic square functions appearing in Theorems 4.5 and 4.6 are well defined for functionals in $(\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n))^*$ and, for any $f \in (\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n))^*$, f vanishing weakly at infinity makes sense. On the other hand, by Theorems 4.3 and 4.4, Corollary 4.1, and Lemma 4.2, we know that, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$, it naturally holds true that $f \in (\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n))^*$ and f vanishes weakly at infinity.

(ii) Recall that Liang and Yang [25] established the intrinsic g_λ^*-function characterization of $H^\varphi(\mathbb{R}^n)$ with the best-known range $\lambda \in (2+2(\alpha+s)/n, \infty)$ (see also Theorem 3.7). However, the proof of Theorem 3.7 requires an aperture estimate of Musielak–Orlicz spaces, which strongly depends on the properties of uniformly Muckenhoupt weights, and hence we can not prove Theorem 4.6 in a similar way. Thus, in [57, Remark 1.11(ii)], Zhuo et al. pointed that it was unclear whether or not the intrinsic g_λ^*-function, with $\lambda \in (2+2(\alpha+s)/n, 3+2(\alpha+s)/n]$, can characterize the space $H^{p(\cdot)}(\mathbb{R}^n)$. To this question, Yan et al. gave a positive answer in [49, Remark 1.19(ii)] [see also Remark 5.8(ii)].

(iii) As applications of Theorem 4.5, Zhuo et al. [57] and Fu [10] obtained, respectively, the $p(\cdot)$-Carleson measure characterization of $\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)$ and the wavelet characterization of $H^{p(\cdot)}(\mathbb{R}^n)$.

The following intrinsic square function characterizations of variable weak Hardy spaces were obtained in [48, Theorems 1.6 and 1.7].

Theorem 4.7. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $p_- \in (n/(n+\alpha+s), 1]$.*

(i) If $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g_{\alpha,s}(f) \in WL^{p(\cdot)}(\mathbb{R}^n)$, then f belongs to $WH^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of f, such that

$$\|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)} \le C\, \|g_{\alpha,s}(f)\|_{WL^{p(\cdot)}(\mathbb{R}^n)};$$

(ii) There exists a positive constant C such that, for any $f \in WH^{p(\cdot)}(\mathbb{R}^n) \cap (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$,

$$\|g_{\alpha,s}(f)\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)}.$$

The same is true if $g_{\alpha,s}(f)$ is replaced, respectively, by $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$, and $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\max\{\alpha, s\}, \infty)$.

Theorem 4.8. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p_- \in (n/(n+\alpha+s), 1]$, and $\lambda \in (3+2(\alpha+s)/n, \infty)$.*

(i) If $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g^*_{\lambda,\alpha,s}(f) \in WL^{p(\cdot)}(\mathbb{R}^n)$, then f belongs to $WH^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C, independent of f, such that

$$\|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)} \le C \left\|g^*_{\lambda,\alpha,s}(f)\right\|_{WL^{p(\cdot)}(\mathbb{R}^n)};$$

(ii) There exists a positive constant C such that, for any $f \in WH^{p(\cdot)}(\mathbb{R}^n) \cap (\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$,

$$\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)}.$$

The same is true if $g^*_{\lambda,\alpha,s}(f)$ is replaced by $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\max\{\alpha, s\}, \infty)$.

Remark 4.3. (i) As was mentioned in Remark 3.4(i), all the above intrinsic square functions appearing in Theorems 4.7(ii) and 4.8(ii) are well defined for functionals in $(\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n))^*$.

(ii) Recall that Yan [48] characterized the weak Musielak–Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$ in terms of the intrinsic Littlewood–Paley g^*_λ-function with $\lambda \in (2+2(\alpha+s)/n, \infty)$ (see also Theorem 3.9). However, as was mentioned in Remark 4.2(ii), we can not prove Theorem 4.8 by an argument similar to that used in the proof of Theorem 3.9 because it strongly depends on the properties of uniformly Muckenhoupt weights. Thus, it is still unknown whether or not the intrinsic g^*_λ-function, with $\lambda \in (2+2(\alpha+s)/n, 3+2(\alpha+s)/n]$, can characterize $WH^{p(\cdot)}(\mathbb{R}^n)$ (see also [48, Remark 1.8(iii)]).

(iii) Differently from Theorems 4.5 and 4.6, we, in Theorems 4.7 and 4.8, need some additional assumptions, this is essentially because the dual space of $WH^{p(\cdot)}(\mathbb{R}^n)$ is not known so far. It is still a challenging problem to remove these additional assumptions.

5 Intrinsic square function characterizations of Hardy spaces associated with ball quasi-Banach function spaces

We first recall the definition of ball quasi-Banach function spaces (see, for instance, [5,16, 33,35,53]).

Definition 5.1. *A quasi-Banach space Y, consisting of measurable functions on $\mathbb{R}^n$, is called a ball quasi-Banach function space if it satisfies*

(i) $\|f\|_Y = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \le |f|$ in the sense of almost everywhere implies that $\|g\|_Y \le \|f\|_Y$;

(iii) $0 \le f_m \uparrow f$ in the sense of almost everywhere implies that $\|f_m\|_Y \uparrow \|f\|_Y$;

(iv) $\mathbf{1}_B \in Y$ *for any ball* $B \subset \mathbb{R}^n$.

Moreover, a ball quasi-Banach function space Y *is called a ball Banach function space if the norm of* Y *satisfies*

(v) for any $f,\ g \in Y$,

$$\|f+g\|_Y \le \|f\|_Y + \|g\|_Y;$$

(vi) for any ball $B \subset \mathbb{R}^n$*, there exists a positive constant* $C_{(B)}$ *such that, for any* $f \in Y$,

$$\int_B |f(x)|dx \le C_{(B)}\|f\|_Y.$$

Remark 5.1. Observe that, in Definition 5.1, if we replace any ball B by any bounded measurable set E, we obtain its another equivalent formulation.

The following Hardy type space was first introduced by Sawano et al. [33].

Definition 5.2. *Let* X *be a ball quasi-Banach function space,* $\Phi \in \mathcal{S}(\mathbb{R}^n)$ *satisfy*

$$\int_{\mathbb{R}^n} \Phi(x)dx \neq 0,$$

and $b \in (0,\infty)$ *be sufficiently large. Then the Hardy space* $H_X(\mathbb{R}^n)$ *associated with* X *is defined by setting*

$$H_X(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{H_X(\mathbb{R}^n)} := \left\|M_b^{**}(f,\Phi)\right\|_X < \infty \right\},$$

where $M_b^{**}(f,\Phi)$ *is defined by setting, for any* $x \in \mathbb{R}^n$,

$$M_b^{**}(f,\Phi)(x) := \sup_{(y,t)\in\mathbb{R}^{n+1}_+} \frac{|\Phi_t * f(x-y)|}{(1+t^{-1}|y|)^b}.$$

Recall that the Hardy–Littlewood maximal operator $\mathcal{M}$ is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|dy,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x. For any given $\theta \in (0,\infty)$, the powered Hardy–Littlewood maximal operator $\mathcal{M}^{(\theta)}$ is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\theta)}(f)(x) := \left\{ \mathcal{M}\left(|f|^\theta\right)(x) \right\}^{\frac{1}{\theta}}. \tag{5.1}$$

Remark 5.2. Assume that $r \in (0,\infty)$ and $\mathcal{M}$ is bounded on $X^{1/r}$. Then, by [33, Theorem 3.1], we know that, in Definition 5.2, it is enough to choose $b \in (\frac{n}{r},\infty)$.

We also need the notions of the p-convexification and the concavity of X (see, for instance, [27, Definition 1.d.3] and [31, Chapter 2] for more details).

Definition 5.3. *Let X be a ball quasi-Banach function space and $p \in (0,\infty)$.*

(i) The p-convexification X^p of X is defined by setting

$$X^p := \{f \text{ is measurable on } \mathbb{R}^n :\ |f|^p \in X\},$$

equipped with the quasi-norm $\|f\|_{X^p} := \||f|^p\|_X^{1/p}$.

(ii) The space X is said to be p-concave if there exists a positive constant C such that, for any $\{f_k\}_{k\in\mathbb{N}} \subset X^{1/p}$,

$$\sum_{k=1}^{\infty} \|f_k\|_{X^{1/p}} \le C \left\| \sum_{k=1}^{\infty} |f_k| \right\|_{X^{1/p}}.$$

In particular, when $C = 1$, X is said to be strictly p-concave.

We recall the notion of absolutely continuous quasi-norms as follows (see, for instance, [1, Definition 3.1] and [36, Definition 3.2]).

Definition 5.4. *Let X be a ball quasi-Banach function space. A function $f \in X$ is said to have an absolutely continuous quasi-norm in X if $\|f\mathbf{1}_{E_j}\|_X \downarrow 0$ whenever $\{E_j\}_{j=1}^{\infty}$ is a sequence of measurable sets that satisfy $E_j \supset E_{j+1}$ for any $j \in \mathbb{N}$ and $\bigcap_{j=1}^{\infty} E_j = \emptyset$. Moreover, X is said to have an absolutely continuous quasi-norm if, for any $f \in X$, f has an absolutely continuous quasi-norm in X.*

In this section, we need some basic additional assumptions on the ball quasi-Banach function space X as follows.

Assumption 5.1. *Let X be a ball quasi-Banach function space. Assume that, for some $\theta, h \in (0,1]$ and $\theta < h$, there exists a positive constant C such that, for any $\{f_k\}_{k=1}^{\infty} \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\left\| \left\{ \sum_{k=1}^{\infty} \left[\mathcal{M}^{(\theta)}(f_k) \right]^h \right\}^{\frac{1}{h}} \right\|_X \le C \left\| \left\{ \sum_{k=1}^{\infty} |f_k|^h \right\}^{\frac{1}{h}} \right\|_X, \tag{5.2}$$

where $\mathcal{M}^{(\theta)}$ is as in (5.1).

For any ball Banach function space X, its associate space (also called the Köthe dual) X' is defined by setting

$$X' := \left\{ f \in M(\mathbb{R}^n) :\ \|f\|_{X'} := \sup\left\{ \|fg\|_{L^1(\mathbb{R}^n)} :\ g \in X,\ \|g\|_X = 1 \right\} < \infty \right\},$$

where $M(\mathbb{R}^n)$ denotes the set of all measurable functions on $\mathbb{R}^n$.

Assumption 5.2. *Assume that X is a ball quasi-Banach function space and there exists an $h \in (0,1]$ such that $X^{1/h}$ is a ball Banach function space satisfying that there exist $q_0 \in (1,\infty)$ and $C \in (0,\infty)$ such that, for any $f \in (X^{1/h})'$,*

$$\left\|\mathcal{M}^{((q_0/h)')}(f)\right\|_{(X^{1/h})'} \le C\|f\|_{(X^{1/h})'}.$$

In what follows, let $\theta \in (0,1]$ be as in (5.2) and

$$d_X := \lfloor n(1/\theta - 1)\rfloor. \tag{5.3}$$

Definition 5.5. *Assume that X is a ball quasi-Banach function space satisfying* (5.2) *for some $0 < \theta < h \le 1$ and Assumption* 5.2 *for some $q_0 \in (1,\infty)$ and the same h as in* (5.2)*. Let $q \in [q_0,\infty]$ and $s \in \mathbb{Z}_+$ satisfy $s \ge d_X$, where d_X is as in* (5.3)*. A measurable function a on $\mathbb{R}^n$ is called an (X,q,s)-atom if there exists a ball B such that*

- *(i)* $\operatorname{supp} a := \{x \in \mathbb{R}^n :\ a(x) \neq 0\} \subset B$;
- *(ii)* $\|a\|_{L^q(\mathbb{R}^n)} \le \frac{|B|^{1/q}}{\|\mathbf{1}_B\|_X}$;
- *(iii)* $\int_{\mathbb{R}^n} a(x)x^\delta dx = 0$ *for any $\delta \in \mathbb{Z}_+^n$ with $|\delta| \le s$.*

The atomic Hardy space $H^{X,q,s}_{\mathrm{atom}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \quad \text{in } \mathcal{S}'(\mathbb{R}^n), \tag{5.4}$$

where $\{\lambda_j\}_{j\in\mathbb{N}}$ is a sequence of non-negative numbers, satisfying

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X}\right]^h \mathbf{1}_{B_j}\right\}^{\frac{1}{h}}\right\|_X < \infty,$$

and $\{a_j\}_{j\in\mathbb{N}}$ a sequence of (X,q,s)-atoms supported, respectively, in balls $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$. Moreover, for any $f \in H^{X,q,s}_{\mathrm{atom}}(\mathbb{R}^n)$, let

$$\|f\|_{H^{X,q,s}_{\mathrm{atom}}(\mathbb{R}^n)} := \inf\left\{\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X}\right)^h \mathbf{1}_{B_j}\right\}^{\frac{1}{h}}\right\|_X\right\},$$

where the infimum is taken over all admissible decompositions of f as above.

The finite atomic Hardy space $H^{X,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$, associated with X, is defined to be the set of all finite linear combinations of (X,q,s)-atoms. The quasi-norm $\|\cdot\|_{H^{X,q,s}_{\mathrm{fin}}(\mathbb{R}^n)}$ in $H^{X,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is defined by *setting, for any* $f \in H_{\rm fin}^{X,q,s}(\mathbb{R}^n)$,

$$\|f\|_{H_{\rm fin}^{X,q,s}(\mathbb{R}^n)} := \inf\left\{\left\|\left\{\sum_{j=1}^N \left(\frac{\lambda_j}{\|\mathbf{1}_{B_j}\|_X}\right)^h \mathbf{1}_{B_j}\right\}^{\frac{1}{h}}\right\|_X :\ N\in\mathbb{N},\right.$$
$$\left. f=\sum_{j=1}^N \lambda_j a_j,\ \{\lambda_j\}_{j=1}^N \subset [0,\infty)\right\},$$

where the infimum is taken over all finite linear combinations of f *in terms of* (X,q,s)*-atoms* $\{a_j\}_{j=1}^N$ *supported, respectively, in balls* $\{B_j\}_{j=1}^N$ *as above.*

The following atomic characterizations of $H_X(\mathbb{R}^n)$ are just [33, Theorems 3.6 and 3.7].

Theorem 5.1. *Let* X, q, *and* s *be as in Definition* 5.5. *Then*

$$H_X(\mathbb{R}^n) = H_{\rm atom}^{X,q,s}(\mathbb{R}^n)$$

with equivalent quasi-norms.

The following conclusion, obtained in [33, Corollary 3.11(ii)], implies that $H_{\rm fin}^{X,q,s}(\mathbb{R}^n)$ is dense in $H_{\rm atom}^{X,q,s}(\mathbb{R}^n)$.

Proposition 5.1. *Let* X *be as in Definition* 5.5. *Assume further that* X *has an absolutely continuous quasi-norm. Then the convergence of* (5.4) *holds true in* $H_X(\mathbb{R}^n)$.

The following finite atomic characterization of $H_X(\mathbb{R}^n)$ is a corrected version of [49, Theorem 1.10] and the proof is almost the same as that of [49, Theorem 1.10]; the details of its proof are omitted here.

Theorem 5.2. *Let* X, q_0, q, *and* s *be as in Definition* 5.5.

(i) If $q \in [q_0,\infty)$, *then* $\|\cdot\|_{H_{\rm fin}^{X,q,s}(\mathbb{R}^n)}$ *and* $\|\cdot\|_{H_X(\mathbb{R}^n)}$ *are equivalent quasi-norms on* $H_{\rm fin}^{X,q,s}(\mathbb{R}^n)$;

(ii) If $q=\infty$, *then* $\|\cdot\|_{H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)}$ *and* $\|\cdot\|_{H_X(\mathbb{R}^n)}$ *are equivalent quasi-norms on* $H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)\cap \mathcal{C}(\mathbb{R}^n)$.

Via using some arguments similar to those used in [19, Lemma 8.1] and [54, Proposition 2.22], we obtain the following conclusion.

Theorem 5.3. *Let* X *and* s *be as in Definition* 5.5. *Assume further that* X *has an absolutely continuous quasi-norm. Then* $H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ *is dense in* $H_X(\mathbb{R}^n)$.

Proof. By Theorem 5.1 and Proposition 5.1, we know that $H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)$ is dense in $H_X(\mathbb{R}^n)$. Thus, it suffices to prove that $H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ is dense in $H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{H_X(\mathbb{R}^n)}$. To this end, let $q\in(q_0,\infty)$ with q_0 as in Definition 5.5, a be an (X,∞,s)-atom supported in a ball $B:=B(x_B,r_B)\subset\mathbb{R}^n$, and $\phi\in\mathcal{S}(\mathbb{R}^n)$ satisfy

$$\mathrm{supp}\phi\subset B(\vec{0}_n,1)\quad\text{and}\quad\int_{\mathbb{R}^n}\phi(x)dx=1.$$

Then, by [7, Theorem 2.1], we know that there exists a sequence $\{t_k\}_{k\in\mathbb{N}}\subset(0,\infty)$ such that $a*\phi_{t_k}\in H_{\rm fin}^{X,\infty,s}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ and

$$\lim_{k\to\infty}\|a*\phi_{t_k}-a\|_{L^q(\mathbb{R}^n)}=0. \tag{5.5}$$

Notice that

$$\frac{(a*\phi_{t_k}-a)|B(x_B,r_B+1)|^{1/q}}{\|a*\phi_{t_k}-a\|_{L^q(\mathbb{R}^n)}\|\mathbf{1}_{B(x_B,r_B+1)}\|_X}$$

is an (X,q,s)-atom. From this, Theorem 5.1, and (5.5), we deduce that

$$\lim_{k\to\infty}\|a*\phi_{t_k}-a\|_{H_X(\mathbb{R}^n)}=\frac{\|\mathbf{1}_{B(x_B,r_B+1)}\|_X}{|B(x_B,r_B+1)|^{1/q}}\lim_{k\to\infty}\|a*\phi_{t_k}-a\|_{L^q(\mathbb{R}^n)}=0.$$

This finishes the proof of Theorem 5.3. $\square$

To describe the dual space of $H_X(\mathbb{R}^n)$, we need the following Campanato space introduced in [49, Definition 1.11].

Definition 5.6. *Let X be a ball quasi-Banach function space, $q\in[1,\infty)$, and $s\in\mathbb{Z}_+$. Then the Campanato space $\mathcal{L}_{X,q,s}(\mathbb{R}^n)$, associated with X, is defined to be the set of all $f\in L^q_{\rm loc}(\mathbb{R}^n)$ such that*

$$\|f\|_{\mathcal{L}_{X,q,s}(\mathbb{R}^n)}:=\sup_{B\subset\mathbb{R}^n}\frac{|B|}{\|\mathbf{1}_B\|_X}\left\{\frac{1}{|B|}\int_B|f(x)-P_B^sf(x)|^q\,dx\right\}^{1/q}<\infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and P_B^sf denotes the minimizing polynomial of f on B with degree not greater than s.

Remark 5.3. As was displayed in Theorems 3.1 and 4.1, the John–Nirenberg inequalities for functions in $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ and $\mathcal{L}_{p(\cdot),1,s}(\mathbb{R}^n)$ are obtained, respectively, in [24] and [57]. However, the John–Nirenberg inequality on $\mathcal{L}_{X,1,s}(\mathbb{R}^n)$ is still unknown.

Using Lemma 3.1, Yan et al. obtained the following equivalent characterization for $\mathcal{L}_{X,q,s}(\mathbb{R}^n)$ in [49, Lemma 2.6].

Lemma 5.1. *Let X be a ball quasi-Banach function space, $q\in[1,\infty)$, and $s\in\mathbb{Z}_+$. Then the following statements are mutually equivalent:*

(i) $f\in\mathcal{L}_{X,q,s}(\mathbb{R}^n)$.

(ii) $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|^*_{\mathcal{L}_{X,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \inf_{Q\in\mathcal{P}_s(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_X} \left\{\frac{1}{|B|}\int_B |f(x)-Q(x)|^q dx\right\}^{1/q} < \infty.$$

Moreover, $\|\cdot\|_{\mathcal{L}_{X,q,s}(\mathbb{R}^n)}$ *and* $\|\cdot\|^*_{\mathcal{L}_{X,q,s}(\mathbb{R}^n)}$ *are equivalent quasi-norms.*

Combining Theorems 5.1, 5.2, and 5.3, and Proposition 5.1, we have the following dual theorem, which is a corrected version of [49, Theorem 1.12] and the proof is almost the same as that of [49, Theorem 1.12]; the details of its proof are omitted here.

Theorem 5.4. *Assume that X is a ball quasi-Banach function space having an absolutely continuous quasi-norm and satisfying* (5.2) *for some $0<\theta<h\le 1$ and Assumption* 5.2 *for some $q_0\in(1,\infty)$ and the same h as in* (5.2). *Let X be m-concave with some $m\in[h,1]$, $q\in[q_0,\infty]$, and $s\ge d_X$ a fixed integer, where d_X is as in* (5.3). *Then the dual space of $H_X(\mathbb{R}^n)$, denoted by $(H_X(\mathbb{R}^n))^*$, is $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q+1/q'=1$ in the following sense:*

(i) Let $f\in\mathcal{L}_{X,q',s}(\mathbb{R}^n)$. Then the linear functional

$$T_f:\ g\to T_f(g) := \int_{\mathbb{R}^n} f(x)g(x)dx, \tag{5.6}$$

originally defined for any $g\in H^{X,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$, has a bounded extension to $H_X(\mathbb{R}^n)$.

(ii) Conversely, every continuous linear functional on $H_X(\mathbb{R}^n)$ arises as in (5.6) *with a unique $f\in\mathcal{L}_{X,q',s}(\mathbb{R}^n)$.*

Moreover,

$$\|f\|_{\mathcal{L}_{X,q',s}(\mathbb{R}^n)} \sim \|T_f\|_{(H_X(\mathbb{R}^n))^*},$$

where the positive equivalence constants are independent of f.

Remark 5.4. (i) We point out that there exists an error in [49, Theorem 1.12]. Indeed, the proof of Theorem 5.4 strongly depends on the fact that $H^{X,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^{X,q,s}_{\mathrm{atom}}(\mathbb{R}^n)$, which is guaranteed by X having absolutely continuous quasi-norm (see Proposition 5.1). However, in [49, Theorem 1.12], this condition is missing.

(ii) Although the Morrey space and the Musielak–Orlicz space are both ball quasi-Banach function spaces, the result of Theorem 5.4 is not applicable to the Hardy–Morrey space and the Musielak–Orlicz Hardy space. Indeed, the Morrey space does not have an absolutely continuous quasi-norm, which makes that the dual space of the Hardy–Morrey space is still unknown, while the growth variable and the space variable of a given Musielak–Orlicz function are in general not separable, which makes Assumption 5.2 fail on the Musielak–Orlicz space.

We point out that there exists a gap in the proof of [49, Theorem 1.12] when $q = \infty$. Indeed, by Theorem 5.2, we know that $\|\cdot\|_{H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_X(\mathbb{R}^n)}$ are equivalent quasi-norms on $H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ instead of $H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)$. Thus, using Theorem 5.3 and repeating the proof of [49, Theorem 1.12], we find that any $f\in\mathcal{L}_{X,1,s}(\mathbb{R}^n)$ induces a linear functional T_f as in (5.6), which is initially defined on $H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ and has a bounded extension to $H_X(\mathbb{R}^n)$. To seal this gap, by an argument similar to that used in [54, (2.31)], we obtain the following conclusion.

Lemma 5.2. *Let X and s be as in Theorem* 5.4. *Let $f\in\mathcal{L}_{X,1,s}(\mathbb{R}^n)$ and T_f be as in* (5.6) *initially defined on $H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$. Then, for any $g\in H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)$,*

$$T_f(g)=\int_{\mathbb{R}^n}f(x)g(x)dx.$$

Proof. Indeed, suppose $g\in H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)$ and ${\rm supp}\, g\subset B(\vec{0}_n,R)$ with $R\in(0,\infty)$. Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ satisfy

$${\rm supp}\phi\subset B(\vec{0}_n,1)\quad\text{and}\quad\int_{\mathbb{R}^n}\phi(x)dx=1.$$

By the proof of Theorem 5.3, we know that there exists a sequence $\{t_k\}_{k\in\mathbb{N}}\subset(0,\infty)$ such that

$$g*\phi_{t_k}\in H^{X,\infty,s}_{\rm fin}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n),\quad \lim_{k\to\infty}g*\phi_{t_k}(x)=g(x)$$

for almost every $x\in\mathbb{R}^n$, and

$$\lim_{k\to\infty}\|g*\phi_{t_k}-g\|_{H_X(\mathbb{R}^n)}=0.$$

From this, the fact that

$$|g*\phi_{t_k}f|\lesssim\|g\|_{L^\infty(\mathbb{R}^n)}\mathbf{1}_{B(\vec{0}_n,R+1)}|f|,$$

and the Lebesgue dominated convergence theorem, we deduce that

$$T_f(g)=\lim_{k\to\infty}T_f(g*\phi_{t_k})=\lim_{k\to\infty}\int_{\mathbb{R}^n}f(x)g*\phi_{t_k}(x)dx=\int_{\mathbb{R}^n}f(x)g(x)dx,$$

which completes the proof of Lemma 5.2. $\square$

Remark 5.5. Using [36, Remarks 2.4, 2.7 and 3.4], [3, p. 25], and Theorem 5.4, we obtain the following conclusions.

(i) Let $p\in(0,1]$, $q\in(1,\infty]$, and $s\in[\lfloor n(1/p-1)\rfloor,\infty)\cap\mathbb{Z}$. When $X:=L^p(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q+1/q'=1$ are, respectively, $H^p(\mathbb{R}^n)$ and $L_{\frac{1}{p}-1,q',s}(\mathbb{R}^n)$. In this case, the conclusion of Theorem 5.4 was obtained by Taibleson and Weiss in [34] [see also Remark 3.3(i)].

(ii) Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, $s \in [\lfloor n(1/p_- - 1)\rfloor, \infty) \cap \mathbb{Z}$, and $q \in (1,\infty]$. When $X := L^{p(\cdot)}(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q + 1/q' = 1$ are, respectively, $H^{p(\cdot)}(\mathbb{R}^n)$ and $\mathcal{L}_{p(\cdot),q',s}(\mathbb{R}^n)$. In this case, the conclusion of Theorem 5.4 was obtained by Nakai and Sawano [29] (see also Theorem 4.4).

(iii) Assume that Φ is an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. Let $p_\Phi^+ \in (0,1]$, $q \in (1,\infty]$, and $s \in [\lfloor n(1/p_\Phi^- - 1)\rfloor, \infty) \cap \mathbb{Z}$. When $X := L^\Phi(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q + 1/q' = 1$ are, respectively, $H^\Phi(\mathbb{R}^n)$ and $\mathcal{L}_{\Phi,q',s}(\mathbb{R}^n)$. A corresponding conclusion of Theorem 5.4 was obtained by Nakai and Sawano [30] [see also Remark 3.3(iii)].

(iv) Let $t,\ r \in (0,\infty)$ and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. We denote the Orlicz-slice space and the Orlicz-slice Hardy space, respectively, by $(E_\Phi^r)_t(\mathbb{R}^n)$ and $(HE_\Phi^r)_t(\mathbb{R}^n)$. Recall that the Orlicz-slice space $(E_\Phi^r)_t(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E_\Phi^r)_t(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{B(x,t)}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{B(x,t)}\|_{L^\Phi(\mathbb{R}^n)}}\right]^r dx\right\}^{1/r} < \infty.$$

Let $t \in (0,\infty)$, $r \in (0,1]$, $p_\Phi^+ \in (0,1]$, $q \in [1,\infty)$, $d \in (0, \min\{p_\Phi^-, r\})$, and $s \in [\lfloor n(1/d - 1)\rfloor, \infty) \cap \mathbb{Z}_+$. The Campanato space $\mathcal{L}_{\Phi,t}^{r,q,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in L_{\rm loc}^q(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}_{\Phi,t}^{r,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \inf_{P\in\mathcal{P}_s(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_{(E_\Phi^r)_t(\mathbb{R}^n)}} \left\{\frac{1}{|B|}\int_B |f(x) - P(x)|^q\, dx\right\}^{1/q} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$. Let $q \in (1,\infty]$. When $X := (E_\Phi^r)_t(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q + 1/q' = 1$ are, respectively, $(HE_\Phi^r)_t(\mathbb{R}^n)$ and $\mathcal{L}_{\Phi,t}^{r,q',s}(\mathbb{R}^n)$ (see Lemma 5.1). In this case, the conclusion of Theorem 5.4 was obtained by Zhang et al. in [55].

(v) Let $\vec{p} := (p_1, \cdots, p_n) \in (0,\infty]^n$. We denote the mixed-norm Lebesgue space and the mixed-norm Hardy space, respectively, by $L^{\vec{p}}(\mathbb{R}^n)$ and $H^{\vec{p}}(\mathbb{R}^n)$. Recall that the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} \left\{\int_{\mathbb{R}} |f(x_1, \cdots, x_n)|^{p_1} dx_1\right\}^{\frac{p_2}{p_1}} dx_2\right]^{\frac{p_3}{p_2}} \cdots dx_n\right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \cdots, n\}$. Let $\vec{p} \in (0,\infty]^n$, $q \in [1,\infty)$, and $s \in \mathbb{Z}_+$. The mixed-norm Campanato space $\mathcal{L}_{\vec{p},q,s}(\mathbb{R}^n)$

is defined to be the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}_{\vec{p},q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \inf_{P\in\mathcal{P}_s(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)}} \left\{\frac{1}{|B|}\int_B |f(x)-P(x)|^q\,dx\right\}^{1/q} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$. Let $\vec{p} \in (0,1]^n$, $q \in (1,\infty]$, and $s \in [\lfloor n(1/p_- - 1)\rfloor, \infty) \cap \mathbb{Z}$ with $p_- := \min\{p_1,\cdots,p_n\}$. When $X := L^{\vec{p}}(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q+1/q'=1$ are, respectively, $H^{\vec{p}}(\mathbb{R}^n)$ and $\mathcal{L}_{\vec{p},q',s}(\mathbb{R}^n)$ (see Lemma 5.1). In this case, the conclusion of Theorem 5.4 was obtained by Huang et al. in [17].

(vi) Let $p \in (0,\infty)$ and $r \in (0,\infty]$. We denote the Lorentz space and the Hardy–Lorentz space, respectively, by $L^{p,r}(\mathbb{R}^n)$ and $H^{p,r}(\mathbb{R}^n)$. Recall that the Lorentz space $L^{p,r}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,r}(\mathbb{R}^n)} := \begin{cases} \left\{\displaystyle\int_0^\infty \left[t^{1/p} f^*(t)\right]^r \frac{dt}{t}\right\}^{1/r} < \infty, & \text{when } r \in (0,\infty), \\ \displaystyle\sup_{t\in(0,\infty)} \left[t^{1/p} f^*(t)\right] < \infty, & \text{when } r = \infty, \end{cases}$$

where f^*, the decreasing rearrangement function of f, is defined by setting, for any $t \in [0,\infty)$,

$$f^*(t) := \inf\{t_1 \in (0,\infty) : \mu_f(t_1) \le t\} \quad \text{with } \mu_f(t_1) := |\{x \in \mathbb{R}^n : |f(x)| > t_1\}|.$$

Let $p \in (0,\infty)$, $q \in [1,\infty)$, and $s \in [\lfloor n(1/p-1)\rfloor, \infty) \cap \mathbb{Z}_+$. The BMO space $\mathrm{BMO}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{BMO}_{p,q}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{|B|}{\|\mathbf{1}_B\|_{L^p(\mathbb{R}^n)}} \left\{\frac{1}{|B|}\int_B |f(x)-P^s_B f(x)|^q\,dx\right\}^{1/q} < \infty,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ and $P^s_B f$ denotes the minimizing polynomial of f on B with degree not greater than s. Let $p \in (0,1]$, $r \in (p,1]$, and $q \in (1,\infty]$. When $X := L^{p,r}(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q + 1/q' = 1$ are, respectively, $H^{p,r}(\mathbb{R}^n)$ and $\mathrm{BMO}_{p,q'}(\mathbb{R}^n)$. In this case, the conclusion of Theorem 5.4 was obtained by Jiao et al. in [18].

(vii) Let $p \in (0,1]$, $q_0 \in [1,\infty)$, $s \in [\lfloor n(q_0/p-1)\rfloor, \infty) \cap \mathbb{Z}$, $q \in (1,\infty]$, and $w \in A_\infty(\mathbb{R}^n)$ with the critical weight index q_0. When $X := L^p_w(\mathbb{R}^n)$, then $H_X(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',s}(\mathbb{R}^n)$ with $1/q+1/q'=1$ become, respectively, the weighted Hardy space $H^p_w(\mathbb{R}^n)$ and its dual space $L_{w,\frac{1}{p}-1,q',s}(\mathbb{R}^n)$. In this case, the conclusion of Theorem 5.4 was obtained by García-Cuerva in [12] [see also Remark 3.3(ii)].

As an immediate corollary of Theorem 5.4, we have the following equivalence on the space $\mathcal{L}_{X,q,s}(\mathbb{R}^n)$ (see also [49, Corollary 1.13]).

Corollary 5.1. *Let X, q_0, d_X, and s be as in Theorem* 5.4 *and* $1/q_0 + 1/q_0' = 1$. *If $q \in (1, q_0']$, then $\mathcal{L}_{X,q,s}(\mathbb{R}^n) = \mathcal{L}_{X,1,d_X}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Remark 5.6. (i) For the classical Hardy space $H^p(\mathbb{R}^n)$, Corollary 5.1 is a special case of Corollary 3.1.

(ii) For the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, Corollary 5.1 is just Corollary 4.1.

(iii) For the Orlicz–Hardy space $H^\Phi(\mathbb{R}^n)$, a corresponding conclusion of Corollary 5.1 was obtained by Nakai and Sawano [30].

(iv) For the Orlicz-slice Hardy space $(HE^r_\Phi)_t(\mathbb{R}^n)$, Corollary 5.1 is an immediate corollary of [55, Theorem 5.7].

(v) For the mixed-norm Hardy space $H^{\vec{p}}(\mathbb{R}^n)$, Corollary 5.1 is a special case of [17, Corollary 3.12].

(vi) For the Lorentz–Hardy space $H^{p,q}(\mathbb{R}^n)$, Corollary 5.1 is a special case of [18, Corollary 7.5].

(vii) For the weighted Hardy space $H^p_w(\mathbb{R}^n)$, Corollary 5.1 is a special case of Corollary 3.1.

The following conclusion implies that the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{X,1,s}(\mathbb{R}^n))^*$, which was obtained in [49, Lemma 2.15].

Lemma 5.3. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (\alpha + s, \infty)$. Assume that X is as in Theorem* 5.4. *If $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ or $f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in \mathcal{L}_{X,1,s}(\mathbb{R}^n)$.*

The following conclusion was obtained in [3, pp. 18-19].

Lemma 5.4. *Let X be as in Theorem* 5.4. *If $f \in H_X(\mathbb{R}^n)$, then f vanishes weakly at infinity.*

Next, we state the main results of this section. The following intrinsic square function characterizations of $H_X(\mathbb{R}^n)$ were obtained in [49, Theorems 1.15 and 1.16].

Theorem 5.5. *Let $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$. Assume that X is a ball quasi-Banach function space having absolutely continuous quasi-norm and satisfying* (5.2) *with $\frac{n}{n+\alpha+s} < \theta < h \le 1$ and Assumption* 5.2 *for some $q_0 \in (1,\infty)$ and the same h as in* (5.2). *If X is m-concave for some $m \in [h,1]$, then $f \in H_X(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{X,1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g_{\alpha,s}(f) \in X$. Moreover, there exists a positive constant C such that, for any $f \in H_X(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H_X(\mathbb{R}^n)} \le \|g_{\alpha,s}(f)\|_X \le C\|f\|_{H_X(\mathbb{R}^n)}.$$

The same is true if $g_{\alpha,s}(f)$ is replaced, respectively, by $S_{\alpha,s}(f)$, $\widetilde{g}_{(\alpha,\epsilon),s}(f)$, and $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha + s, \infty)$.

Theorem 5.6. *Let $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$. Assume that X, θ, h, and q_0 are as in Theorem 5.5. If X is m-concave for some $m \in [h,1]$, and $\lambda \in (\max\{2/\theta, 2/\theta+1-2/q_0\}, \infty)$, then $f \in H_X(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{X,1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $g^*_{\lambda,\alpha,s}(f) \in X$. Moreover, there exists a positive constant C such that, for any $f \in H_X(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H_X(\mathbb{R}^n)} \le \|g^*_{\lambda,\alpha,s}(f)\|_X \le C\|f\|_{H_X(\mathbb{R}^n)}.$$

The same is true if $g^*_{\lambda,\alpha,s}(f)$ is replaced by $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s, \infty)$.

Remark 5.7. (i) By Lemma 5.3 and an argument similar to that used in Remark 3.4(i), we conclude that all the above intrinsic square functions appearing in Theorems 5.5 and 5.6 are well defined for functionals in $(\mathcal{L}_{X,1,s}(\mathbb{R}^n))^*$ and, for any $f \in (\mathcal{L}_{X,1,s}(\mathbb{R}^n))^*$, f vanishing weakly at infinity makes sense. On the other hand, by Theorem 5.4, Corollary 5.1, and Lemma 5.4, we know that, for any $f \in H_X(\mathbb{R}^n)$, it naturally holds true that $f \in (\mathcal{L}_{X,1,s}(\mathbb{R}^n))^*$ and f vanishes weakly at infinity.

(ii) As was mentioned in Remark 5.4(i), the condition that X has an absolutely continuous quasi-norm is still needed in [49, Theorems 1.15 and 1.16]. In the present article, we seal this gap in Theorems 5.5 and 5.6.

(iii) As was mentioned in Remark 5.4(ii), the results of Theorems 5.5 and 5.6 are not applicable to the Hardy–Morrey space and the Musielak–Orlicz Hardy space.

(iv) Recently, Zhang et al. introduced the weak Hardy-type space $WH_X(\mathbb{R}^n)$, associated with ball quasi-Banach function space X, and developed a complete real-variable theory of these spaces; see [42, 56]. However, the intrinsic square function characterizations of $WH_X(\mathbb{R}^n)$ are still unknown.

(v) As was mentioned in Remark 3.4(iii) and Remark 4.2(iii), the intrinsic square function characterizations have been used to establish the Carleson measure characterizations of Campanato type spaces and the wavelet characterizations of Hardy type spaces. Therefore, it is natural to consider the wavelet characterizations of $H_X(\mathbb{R}^n)$ and the Carleson measure characterization of $\mathcal{L}_{X,1,s}(\mathbb{R}^n)$, which will be presented in a forthcoming article.

Remark 5.8. Using [36, Remarks 2.4, 2.7 and 3.4], Remark 5.5, [3, p. 25], and Theorems 5.5 and 5.6, we immediately obtain the following conclusions.

(i) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $p \in (n/[n+\alpha+s], 1]$. If X is defined as in Remark 5.5(i), then Theorem 5.6 is a special case of Theorem 3.7.

(ii) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $p_+ \in (0,1]$ and $p_- \in (n/[n+\alpha+s], 1]$. If X is defined as in Remark 5.5(ii), then Theorem 5.6 is just Theorem 4.6, where the range of λ is replaced by the best-known range $\lambda \in (2+2(\alpha+s)/n, \infty)$.

(iii) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and Φ be an Orlicz function with $p_\Phi^- \in (n/[n+\alpha+s],1]$ and $p_\Phi^+ \in (0,1]$. If X is defined as in Remark 5.5(iii), then Theorem 5.6 is a special case of Theorem 3.7.

(iv) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and Φ be an Orlicz function with $p_\Phi^- \in (n/[n+\alpha+s],1]$ and $p_\Phi^+ \in (0,1]$. Let $t \in (0,\infty)$ and $r \in (p_\Phi^-,1]$. If X is defined as in Remark 5.5(iv), then $f \in (HE_\Phi^r)_t(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{\Phi,t}^{r,1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $\|g_{\alpha,s}(f)\|_{(E_\Phi^r)_t(\mathbb{R}^n)} < \infty$ or $\|g_{\lambda,\alpha,s}^*(f)\|_{(E_\Phi^r)_t(\mathbb{R}^n)} < \infty$ for any $\lambda \in (2+2(\alpha+s)/n,\infty)$. Moreover, for any $f \in (HE_\Phi^r)_t(\mathbb{R}^n)$,

$$\|f\|_{(HE_\Phi^r)_t(\mathbb{R}^n)} \sim \|g_{\alpha,s}(f)\|_{(E_\Phi^r)_t(\mathbb{R}^n)} \sim \left\|g_{\lambda,\alpha,s}^*(f)\right\|_{(E_\Phi^r)_t(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f. In this case, the conclusions of Theorems 5.5 and 5.6 were obtained in [49, Corollary 1.18(iv)].

(v) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\vec{p} \in (0,1]^n$ with $p_- \in (n/[n+\alpha+s],1]$. If X is defined as in Remark 5.5(v), then $f \in H^{\vec{p}}(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{\vec{p},1,s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $\|g_{\alpha,s}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty$ or $\|g_{\lambda,\alpha,s}^*(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty$ for any $\lambda \in (2+2(\alpha+s)/n,\infty)$. Moreover, for any $f \in H^{\vec{p}}(\mathbb{R}^n)$,

$$\|f\|_{H^{\vec{p}}(\mathbb{R}^n)} \sim \|g_{\alpha,s}(f)\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\|g_{\lambda,\alpha,s}^*(f)\right\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f. In this case, the conclusions of Theorems 5.5 and 5.6 were obtained in [49, Corollary 1.18(v)].

(vi) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/[n+\alpha+s],1]$, and $r \in (p,1]$. If X is defined as in Remark 5.5(vi), then $f \in H^{p,r}(\mathbb{R}^n)$ if and only if $f \in (\mathrm{BMO}_{p,1}(\mathbb{R}^n))^*$, f vanishes weakly at infinity, and $\|g_{\alpha,s}(f)\|_{L^{p,r}(\mathbb{R}^n)} < \infty$ or $\|g_{\lambda,\alpha,s}^*(f)\|_{L^{p,r}(\mathbb{R}^n)} < \infty$ for any $\lambda \in (2+2(\alpha+s)/n,\infty)$. Moreover, for any $f \in H^{p,r}(\mathbb{R}^n)$,

$$\|f\|_{H^{p,r}(\mathbb{R}^n)} \sim \|g_{\alpha,s}(f)\|_{L^{p,r}(\mathbb{R}^n)} \sim \left\|g_{\lambda,\alpha,s}^*(f)\right\|_{L^{p,r}(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f. In this case, the conclusions of Theorems 5.5 and 5.6 were obtained in [49, Corollary 1.18(vi)].

(vii) Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p \in (n/[n+\alpha+s],1]$, and $w \in A_{p(1+(\alpha+s)/n)}(\mathbb{R}^n)$. If X is defined as in Remark 5.5(vii), then Theorem 5.6 is a special case of Theorem 3.7.

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