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Nodal Solutions of the Brezis-Nirenberg Problem in Dimension 6

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Abstract. We show that the classical Brezis-Nirenberg problem

$$\begin{aligned} -\Delta u &= u|u| + \lambda u \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

when Ω is a bounded domain in $\mathbb{R}^6$ has a sign-changing solution which blows-up at a point in Ω as λ approaches a suitable value $\lambda_0 > 0$.

Key Words: Sign-changing solutions, blow-up phenomenon, Ljapunov-Schmidt reduction, Transversality theorem.

AMS Subject Classifications: 35B44, 58C15

1 Introduction

Brezis and Nirenberg in their famous paper [6] introduced the problem

$$-\Delta u = |u|^{\frac{4}{n-2}} u + \lambda u \qquad \text{in } \Omega, \tag{1.1a}$$
$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $n \geq 3$. A huge number of results concerning (1.1) has been obtained since then. Let us summarize the most relevant results which are also connected with the topic of the present paper.

First of all, the classical Pohozaev's identity ensures that (1.1) does not have any solutions if $\lambda \leq 0$ and Ω is a star-shaped domain. A simple argument shows that problem

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(1.1) does not have any positive solutions if $\lambda \geq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. The existence of a least energy positive solution u_λ to (1.1), i.e., a solution which achieves the infimum

$$m_\lambda := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_{\Omega_\theta} \left(|\nabla u|^2 - \lambda u^2\right) dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}$$

has been proved by Brezis and Nirenberg in [6] when $\lambda \in (0, \lambda_1(\Omega))$ in dimension $n \geq 4$ and when $\lambda \in (\lambda^*(\Omega), \lambda_1(\Omega))$ in dimension $n = 3$, where $\lambda^*(\Omega) > 0$ depends on the domain Ω. If Ω is the ball then $\lambda^*(\Omega) = \frac{1}{4}\lambda_1(\Omega)$ (see [6]), while the general case has been treated by Druet in [12]. The existence of a sign-changing solution has been proved by Cerami, Solimini and Struwe in [9] when $\lambda \in (0, \lambda_1(\Omega))$ and $n \geq 6$ and by Capozzi, Fortunato and Palmieri in [8] when $\lambda \geq \lambda_1(\Omega)$ and $n \geq 4$.

There is a wide literature about the study of the asymptotic profile of the solutions when the parameter λ approaches either zero or some strictly positive values depending on the dimension n and the domain Ω. In the following, we will focus on the existence of solutions which exhibit a positive or negative blow-up phenomenon as λ approaches some particular values.

When the parameter λ approaches zero, positive and sign-changing solutions which blow-up positively or negatively at one or more points in Ω do exist provided the dimension $n \geq 4$. Rey in [24], Musso and Pistoia in [19] and Esposito, Pistoia and Vétois in [13] built solutions to (1.1) with simple positive or negative blow-up points, i.e., around each point the solution looks like a positive or a negative standard bubble. Here the standard bubbles are the functions

$$U_{\delta,\xi}(x) := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{\left(\delta^2 + |x-\xi|^2\right)^{\frac{n-2}{2}}} \quad \text{with } \delta > 0, \quad \xi \in \mathbb{R}^n, \tag{1.2a}$$

$$\alpha_n := \left(n(n-2)\right)^{\frac{n-2}{4}}, \tag{1.2b}$$

which are the only positive solutions of the equation

$$-\Delta U = U^{\frac{n+2}{n-2}}$$

in $\mathbb{R}^n$ (see [4,7,28]) More precisely, if λ is small enough problem (1.1) has a positive solution which blows-up at one point (see [24] if $n \geq 5$ and [13] if $n = 4$) and a sign-changing solution which blows-up positively and negatively at two different points (see [19] if $n \geq 5$). As far as we know the existence of multiple concentration in the case $n = 4$ is still open. If $n = 3$ positive solutions of (1.1) blowing-up at a single point when the parameter λ approaches a strictly positive number have been found by Del Pino, Dolbeault and Musso in [11]. Moreover, sign-changing solutions having both positive and negative blow-up points can be constructed arguing as Musso and Salazar in [20], where they found solutions which blow-up at more points when λ is close to a suitable strictly positive number. In higher dimension $n \geq 7$ Premoselli [22] found an arbitrary large number of sign-changing solutions to (1.1) with a towering blow-up point in Ω, i.e., around the point the solution likes look like the superposition of bubbles of alternating sign (see also Iacopetti and Vaira [16]). In particular, if Ω is a ball these solutions are nothing but the radially symmetric nodal solutions. Conversely, if Ω is the ball in low dimension $n = 3,4,5,6$, Atkinson, Brezis and Peletier in [3] proved that problem (1.1) does not have any sign-changing radial solutions when $\lambda \in (0, \lambda_*)$ where λ_* depends on the dimension n (see also Iacopetti and Pacella [15] and Dammak [10]). In particular, we expect that in low dimension the blowing-up phenomenon takes places when λ approaches a positive value different from zero. In fact Iacopetti and Vaira in [17] proved that if $n = 4,5$ and λ approaches the first eigenvalue $\lambda_1(\Omega)$ the problem (1.1) has a sign-changing solution which blows-up at the origin and shares the shape of the positive first eigenfunction associated with $\lambda_1(\Omega)$ far away. So a natural question arises: *is it possible to find a sign-changing blowing-up solution of* (1.1) *in dimension* $n = 6$ *when* λ *approaches some strictly positive number?*

In the present paper, we give a positive answer. In order to state our result, we need to introduce some notation and the assumptions.

Let u_0 be a solution to

$$\begin{cases} -\Delta u_0 = |u_0|u_0 + \lambda_0 u_0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

If $\xi_0 \in \Omega$ is such that $\max_\Omega u_0 = u_0(\xi_0) > 0$, we suppose that

$$\boxed{\lambda_0 = 2u_0(\xi_0).} \tag{1.4}$$

We assume that u_0 is non-degenerate, i.e.,

$$\boxed{\begin{cases} -\Delta v = (2|u_0| + \lambda_0)v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad \Rightarrow \quad v \equiv 0.} \tag{1.5}$$

If v_0 solves

$$\begin{cases} -\Delta v_0 - (2|u_0| + \lambda_0)v_0 = u_0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

we require that

$$\boxed{2v_0(\xi_0) \neq 1.} \tag{1.7}$$

We will show that the problem

$$\begin{cases} -\Delta u = u|u| + (\lambda_0 + \epsilon)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^6$ has a sign-changing solution which blows-up at ξ_0 as $|\epsilon|$ approaches zero (note that ϵ is not necessarily positive) and shares the shape of u_0 far away from ξ_0. More precisely our existence result reads as follows.

Theorem 1.1. *Assume* (1.4), (1.5) *and* (1.7). *There exists* $\varepsilon_0 > 0$ *such that*

1. *if* $1 - 2v_0(\xi_0) > 0$ *and* $\epsilon \in (0, \varepsilon_0)$,
2. *if* $1 - 2v_0(\xi_0) < 0$ *and* $\epsilon \in (-\varepsilon_0, 0)$,

then there exists a sign-changing solution u_ϵ *of the problem* (1.8), *which blows-up at the point* ξ_0 *as* $\epsilon \to 0$. *More precisely*

$$u_\epsilon(x) = u_0(x) + \epsilon v_0(x) - PU_{\delta_\epsilon,\xi_\epsilon}(x) + \phi_\epsilon(x)$$

with as $\epsilon \to 0$

$$\delta_\epsilon|\epsilon|^{-1} \to d > 0, \quad \xi_\epsilon \to \xi_0 \quad \textit{and} \quad \|\phi_\epsilon\|_{H_0^1(\Omega)} = \mathcal{O}\left(\epsilon^2|\ln|\epsilon||^{\frac{2}{3}}\right).$$

Here $PU_{\delta,\xi}$ denotes the projection onto $H_0^1(\Omega)$ of the standard bubble $U_{\delta,\xi}$ defined in (1.2), i.e., $-\Delta PU_{\delta,\xi} = U_{\delta,\xi}^2$ in Ω with $PU_{\delta,\xi} = 0$ on $\partial\Omega$.

It is natural to ask for which domains Ω the assumptions (1.4), (1.5) and (1.7) hold true. If Ω is the ball and u_0 is the positive solution they are all satisfied (see [27] for (1.5), [1] for (1.4) and (1.7)). More in general, we can only prove that assumptions (1.4) and (1.5) are satisfied for most domains Ω (see Theorem 1.2) when u_0 is the least energy positive solution to (1.3). It would be interesting to prove that (1.7) also holds for generic domains.

Let us state our generic result. Let Ω_0 be a bounded and smooth domain in $\mathbb{R}^6$ and let D be an open neighbourhood of $\overline{\Omega_0}$. Set $\Omega_\theta := \Theta(\Omega_0)$ where $\Theta = I + \theta$, $\theta \in C^{3,\alpha}(\overline{D}, \mathbb{R}^6)$ with $\|\theta\|_{2,\alpha} \le \rho$, with $\alpha \in (0,1)$ and ρ small enough. Let us consider the problem on the perturbed domain Ω_θ

$$\Delta u + \lambda u + |u|u = 0 \qquad \text{in } \Omega_\theta, \tag{1.9a}$$

$$u = 0 \qquad \text{on } \partial\Omega_\theta. \tag{1.9b}$$

Theorem 1.2. *The set*

$$\Xi := \{\theta \in C^{3,\alpha}(\overline{D}, \mathbb{R}^6) : \textit{if } \lambda > 0 \textit{ and } u \in H_0^1(\Omega_\theta) \textit{ solves } (1.9) \textit{ then } u \textit{ is non-degenerate}\}$$

is a residual subset in $C^{3,\alpha}(\overline{D}, \mathbb{R}^6)$, *i.e.,* $C^{3,\alpha}(\overline{D}, \mathbb{R}^6) \setminus \Xi$ *is a countable union of close subsets without interior points.*

Moreover, if $\lambda \in (0, \lambda_1(\Omega_\theta))$ *and* u_λ *denotes the least energy positive solution of* (1.9), *for any* $\theta \in \Xi$ *there exists* $\lambda_\theta \in (0, \lambda_1(\Omega_\theta))$ *such that*

$$\lambda_\theta = 2 \max_{\Omega_\theta} u_{\lambda_\theta}.$$

The proof of Theorem 1.1 is based upon the well-known Ljapunov-Schmidt reduction. In Section 2 we describe the main steps of the proof by omitting many details which can be found up to minor changes in the quoted papers. We only prove what cannot be immediately deduced by known results. In particular, we point out the careful construction of the ansatz (2.3) which has to be refined up to a second order and the delicate estimate of the reduced energy (2.7) given in Proposition 2.2 whose leading term (2.8) arises from the interaction between the bubble and the second order term in the ansatz.

The proof of Theorem 1.2 relies on a classical transversality argument and it is carried out in Section 3.

2 The existence of a sign-changing solution

2.1 Setting of the problem and the choice of the ansatz

In what follows we denote by

$$(u,v) := \int_\Omega \nabla u \nabla v dx, \quad \|u\| := \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}} \quad \text{and} \quad |u|_r := \left(\int_\Omega |u|^r\, dx\right)^{\frac{1}{r}},$$

the inner product and its correspond norm in $H^1_0(\Omega)$ and the standard norm in $L^r(\Omega)$, respectively. When $A \neq \Omega$ is any Lebesgue measurable set we specify the domain of integration by using $\|u\|_A, |u|_{r,A}$.

Let $(-\Delta)^{-1} : L^{\frac{3}{2}}(\Omega) \to H^1_0(\Omega)$ be the operator defined as the unique solution of the equation

$$\begin{aligned} -\Delta u = v \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

By the Holder inequality it follows that

$$\|(-\Delta)^{-1}(v)\| \leq C|v|_{\frac{3}{2}}, \quad \forall v \in L^{\frac{3}{2}}(\Omega),$$

for some positive constant C, which does not depend on v. Hence we can rewrite problem (1.8) as

$$u = (-\Delta)^{-1}[f(u) + (\lambda_0 + \epsilon)u], \quad u \in H^1_0(\Omega), \tag{2.1}$$

with $f(u) = |u|u$.

Next we remind the expansion of the projection of the bubble. We denote by $G(x,y)$ the Green's function of the Laplace operator given by

$$G(x,y) = \frac{1}{4\omega_6}\left(\frac{1}{|x-y|^4} - H(x,y)\right),$$

where ω_6 denotes the surface area of the unit sphere in $\mathbb{R}^6$ and H is the regular part of the Green's function, namely for all $y \in \Omega$, $H(x,y)$ satisfies

$$\begin{aligned} &\Delta H(x,y) = 0 && \text{in } \Omega, \\ &H(x,y) = \frac{1}{|x-y|^4}, && x \in \partial\Omega. \end{aligned}$$

It is known that the following expansion holds (see [24])

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \alpha_6 \delta^2 H(x,\xi) + \mathcal{O}(\delta^4) \quad \text{as } \delta \to 0 \tag{2.2}$$

uniformly with respect to ξ in compact sets of Ω.

Moreover we recall (see [5]) that every solution to the linear equation

$$-\Delta\psi = 2U_{\delta,\xi}\psi \quad \text{in } \mathbb{R}^6$$

is a linear combination of the functions $Z^j_{\delta,\xi}$, $j = 0,\cdots,6$ given by

$$\begin{aligned} &Z^0_{\delta,\xi}(x) = \partial_\delta U_{\delta,\xi}(x) = 2\alpha_6\delta \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^3}, \\ &Z^j_{\delta,\xi}(x) = \partial_{\xi_j} U_{\delta,\xi}(x) = 4\alpha_6\delta^2 \frac{x_j - \xi_j}{(\delta^2 + |x-\xi|^2)^3}, \qquad j = 1,\cdots,6. \end{aligned}$$

If we denote by $PZ^j_{\delta,\xi}$ the projection of $Z^j_{\delta,\xi}$ onto $H^1_0(\Omega)$, i.e.,

$$\begin{aligned} &-\Delta PZ^j_{\delta,\xi} = 2U_{\delta,\xi} Z^j_{\delta,\xi} && \text{in } \Omega, \\ &PZ^j_{\delta,\xi} = 0 && \text{on } \partial\Omega, \end{aligned}$$

elliptic estimates give

$$\begin{aligned} &PZ^0_{\delta,\xi}(x) = Z^0_{\delta,\xi} - 2\delta\alpha_6 H(x,\xi) + \mathcal{O}(\delta^3) && \text{as } \delta \to 0, \\ &PZ^j_{\delta,\xi}(x) = Z^j_{\delta,\xi} - \delta^2\alpha_6 \partial_{\xi_j} H(x,\xi) + \mathcal{O}(\delta^4), && j = 1,\cdots,6 \quad \text{as } \delta \to 0, \end{aligned}$$

uniformly with respect to ξ in compact sets of Ω.

We look for a solution of (1.8) of the form

$$u_\epsilon(x) = \underbrace{u_0(x) + \epsilon v_0 - PU_{\delta,\xi}(x)}_{:=W_{\delta,\xi}} + \phi_\epsilon(x), \tag{2.3}$$

where δ, ξ are chosen so that

$$\delta = |\epsilon| d \quad \text{with } d \in \left(\sigma, \frac{1}{\sigma}\right) \text{ and } \xi = \xi_0 + \sqrt{\delta}\eta \text{ with } |\eta| \le \frac{1}{\sigma}, \text{ where } \sigma > 0 \text{ is small,} \tag{2.4}$$

and ϕ_ϵ is a remainder term, which is small as $\epsilon \to 0$ which belongs to the space $\mathcal{K}^\perp_{\delta,\xi}$ defined as follows.

Now let us define

$$\mathcal{K}_{\delta,\xi} := \text{span}\{PZ^j_{\delta,\xi} : j = 0, \cdots, 6\},$$
$$\mathcal{K}^\perp_{\delta,\xi} := \{\phi \in H^1_0(\Omega) : (\phi, PZ^j_{\delta,\xi}) = 0,\ j = 0, \cdots, 6\}.$$

Let us denote by $\Pi_{\delta,\xi}$ and $\Pi^\perp_{\delta,\xi}$ the projection of $H^1_0(\Omega)$ on $\mathcal{K}_{\delta,\xi}$ and $\mathcal{K}^\perp_{\delta,\xi}$ respectively.

Then solving problem (2.1) is equivalent to solve the system

$$\Pi^\perp_{\delta,\xi}\left\{u_\epsilon(x) - (-\Delta)^{-1}\left[f(u_\epsilon) + \lambda u_\epsilon\right]\right\} = 0, \tag{2.5a}$$
$$\Pi_{\delta,\xi}\left\{u_\epsilon(x) - (-\Delta)^{-1}\left[f(u_\epsilon) + \lambda u_\epsilon\right]\right\} = 0. \tag{2.5b}$$

2.2 The remainder term: solving Eq. (2.5a)

Eq. (2.5a) can be written as

$$\mathcal{L}_{\delta,\xi}(\phi_\epsilon) + \mathcal{R}_{\delta,\xi} + \mathcal{N}_{\delta,\xi}(\phi_\epsilon) = 0,$$

where

$$\mathcal{L}_{\delta,\xi}(\phi_\epsilon) = \Pi^\perp_{\delta,\xi}\left\{\phi_\epsilon(x) - (-\Delta)^{-1}\left[f'(W_{\delta,\xi})\phi_\epsilon + \lambda\phi_\epsilon\right]\right\}$$

is the linearized operator at the approximate solution,

$$\mathcal{R}_{\delta,\xi} = \Pi^\perp_{\delta,\xi}\left\{W_{\delta,\xi}(x) - (-\Delta)^{-1}\left[f(W_{\delta,\xi}) + \lambda W_{\delta,\xi}\right]\right\}$$

is the error term and

$$\mathcal{N}_{\delta,\xi}(\phi_\epsilon) = \Pi^\perp_{\delta,\xi}\left\{-(-\Delta)^{-1}\left[f(W_{\delta,\xi} + \phi_\epsilon) - f(W_{\delta,\xi}) - f'(W_{\delta,\xi})\phi_\epsilon\right]\right\}$$

is a quadratic term in ϕ_ϵ.

First of all, we estimate the size of the error term $\mathcal{R}_{\delta,\xi}$.

Lemma 2.1. *For any $\sigma > 0$ there exist $c > 0$ and $\varepsilon_0 > 0$ such that for any $d > 0$ and $\eta \in \mathbb{R}^6$ satisfying* (2.4) *and for any $\epsilon \in (-\varepsilon_0, \varepsilon_0)$*

$$\|\mathcal{R}_{\delta,\xi}\| \le c\epsilon^2 |\ln|\epsilon||^{\frac{2}{3}}.$$

Proof. First we remark that

$$\begin{aligned}
&-\Delta W_{\delta,\xi} - |W_{\delta,\xi}|W_{\delta,\xi} - (\lambda_0 + \epsilon)W_{\delta,\xi}\\
=&-\Delta u_0 - \epsilon\Delta v_0 - U^2_{\delta,\xi} - |u_0 + \epsilon v_0 - PU_{\delta,\xi}|(u_0 + \epsilon v_0 - PU_{\delta,\xi})\\
&-\lambda_0 u_0 - \lambda_0\epsilon v_0 + (\lambda_0 + \epsilon)PU_{\delta,\xi} - \epsilon u_0 - \epsilon^2 v_0\\
=&-|u_0 + \epsilon v_0 - PU_{\delta,\xi}|(u_0 + \epsilon v_0 - PU_{\delta,\xi}) - U^2_{\delta,\xi} + |u_0|u_0\\
&+\epsilon\underbrace{(-\Delta v_0 - \lambda_0 v_0 - u_0)}_{=2|u_0|v_0 \text{ because of (1.6)}} + (\lambda_0 + \epsilon)PU_{\delta,\xi} - \epsilon^2 v_0.
\end{aligned}$$

By the continuity of $\Pi^{\perp}_{\delta,\xi}$ we get that

$$\begin{aligned}\|\mathcal{R}_{\delta,\xi}\| \le & c\left|-\Delta W_{\delta,\xi}-f(W_{\delta,\xi})-\lambda W_{\delta,\xi}\right|_{\frac{3}{2}}\\ \le & c\underbrace{\left|-|u_0+\epsilon v_0-PU_{\delta,\xi}|(u_0+\epsilon v_0-PU_{\delta,\xi})-PU^2_{\delta,\xi}+|u_0|u_0+2\epsilon|u_0|v_0\right|_{\frac{3}{2}}}_{(I)}\\ &+c\underbrace{\left|PU^2_{\delta,\xi}-U^2_{\delta,\xi}\right|_{\frac{3}{2}}}_{(II)}+(\lambda_0+\epsilon)\left|PU_{\delta,\xi}\right|_{\frac{3}{2}}+\underbrace{\epsilon^2\left|v_0\right|_{\frac{3}{2}}}_{:=\mathcal{O}(\epsilon^2)}.\end{aligned}$$

First of all, we point out that

$$|PU_{\delta,\xi}|_{\frac{3}{2}} \le c|U_{\delta,\xi}|_{\frac{3}{2}} \le c\delta^2|\ln\delta|^{\frac{2}{3}},$$

and by (2.2)

$$(II)\le c\left(\int_\Omega \underbrace{|PU_{\delta,\xi}-U_{\delta,\xi}|^{\frac{3}{2}}}_{=O(\delta^2)}\underbrace{|PU_{\delta,\xi}+U_{\delta,\xi}|^{\frac{3}{2}}}_{\le cU_{\delta,\xi}}\right)^{\frac{2}{3}} \le c\delta^2\left(\int_\Omega |U_{\delta,\xi}|^{\frac{3}{2}}\,dx\right)^{\frac{2}{3}} = \mathcal{O}\left(\delta^4|\ln\delta|^{\frac{2}{3}}\right).$$

First let us estimate (I) in $B(\xi,\sqrt{\delta})$ and $\Omega\setminus B(\xi,\sqrt{\delta})$:

$$\begin{aligned}(I)\le & c\left(\int_{B(\xi,\sqrt{\delta})}\left||u_0+\epsilon v_0-PU_{\delta,\xi}|(u_0+\epsilon v_0-PU_{\delta,\xi})|+(PU_{\delta,\xi})^2\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\\ &+c\underbrace{\left(\int_{B(\xi,\sqrt{\delta})}\left||u_0|u_0+2\epsilon|u_0|v_0\right|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}}_{=\mathcal{O}(\delta^2)}\\ &+c\left(\int_{\Omega\setminus B(\xi,\sqrt{\delta})}\left||u_0+\epsilon v_0-PU_{\delta,\xi}|(u_0+\epsilon v_0-PU_{\delta,\xi})-|u_0|u_0-2|u_0|(\epsilon v_0-PU_{\delta,\xi})\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\\ &+c\left(\int_{\Omega\setminus B(\xi,\sqrt{\delta})}\left|(PU_{\delta,\xi})^2+2|u_0|PU_{\delta,\xi}\right|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\\ =&\mathcal{O}\left(\delta^2|\ln\delta|^{\frac{2}{3}}\right),\end{aligned}$$

since by mean value Theorem (here $\theta\in[0,1]$)

$$\begin{aligned}&\int_{B(\xi,\sqrt{\delta})}\left||u_0+\epsilon v_0-PU_{\delta,\xi}|(u_0+\epsilon v_0-PU_{\delta,\xi})+(PU_{\delta,\xi})^2\right|^{\frac{3}{2}}\\ =&2\int_{B(\xi,\sqrt{\delta})}|(\theta(u_0+\epsilon v_0)-PU_{\delta,\xi})(u_0+\epsilon v_0)|^{\frac{3}{2}}\,dx\\ \le & \underbrace{c\int_{B(\xi,\sqrt{\delta})}|PU_{\delta,\xi}|^{\frac{3}{2}}\,dx}_{=\mathcal{O}(\delta^3|\log\delta|)}+\underbrace{c\int_{B(\xi,\sqrt{\delta})}|u_0+\epsilon v_0|^3\,dx}_{=\mathcal{O}(\delta^3)},\end{aligned}$$

and by the inequality

$$\big||a+b|(a+b)-|a|a-2|a|b\big|\le 7b^2 \quad \text{for any } \ a,b\in\mathbb{R}, \tag{2.6}$$

and

$$\begin{aligned}
&\int_{\Omega\setminus B(\xi,\sqrt{\delta})}\Big||u_0+\epsilon v_0-PU_{\delta,\xi}|(u_0+\epsilon v_0-PU_{\delta,\xi})-|u_0|u_0-2|u_0|(\epsilon v_0-PU_{\delta,\xi})\Big|^{\frac32}\\
&\le c\int_{\Omega\setminus B(\xi,\sqrt{\delta})}|\epsilon v_0-PU_{\delta,\xi}|^3dx\\
&\le c\underbrace{\int_{\Omega\setminus B(\xi,\sqrt{\delta})}|\epsilon v_0|^3dx}_{=\mathcal{O}(\epsilon^3)}+c\underbrace{\int_{\Omega\setminus B(\xi,\sqrt{\delta})}|U_{\delta,\xi}|^3dx}_{=\mathcal{O}(\delta^3)},\\
&\left(\int_{\Omega\setminus B(\xi,\sqrt{\delta})}\Big||u_0+\epsilon v_0-PU_{\delta,\xi}|(u_0+\epsilon v_0-PU_{\delta,\xi})-|u_0|u_0-2|u_0|(\epsilon v_0-PU_{\delta,\xi})\Big|^{\frac32}\right)^{\frac23}\\
&=\mathcal{O}(\epsilon^2),\\
&\int_{\Omega\setminus B(\xi,\sqrt{\delta})}\Big||PU_{\delta,\xi}|(PU_{\delta,\xi})+2|u_0|PU_{\delta,\xi}\Big|^{\frac32}\\
&\le c\underbrace{\int_{\Omega\setminus B(\xi,\sqrt{\delta})}|U_{\delta,\xi}|^3\,dx}_{=\mathcal{O}(\delta^3)}+\underbrace{\int_{\Omega\setminus B(\xi,\sqrt{\delta})}|U_{\delta,\xi}|^{\frac32}\,dx}_{=\mathcal{O}(\delta^3|\log\delta|)},
\end{aligned}$$

which ends the proof. □

Next we analyze the invertibility of the linear operator $\mathcal{L}_{\delta,\xi}$ (see for example [30], Lemma 2.4 or [25], Lemma 4.2).

Lemma 2.2. *For any $\sigma>0$ there exist $c>0$ and $\varepsilon_0>0$ such that for any $d>0$ and $\eta\in\mathbb{R}^6$ satisfying* (2.4) *and for any $\epsilon\in(-\varepsilon_0,\varepsilon_0)$*

$$\|\mathcal{L}_{\delta,\xi}(\phi)\|\ge c\|\phi\| \quad \textit{for any} \ \ \phi\in\mathcal{K}^{\perp}_{\delta,\xi}.$$

Moreover, $\mathcal{L}_{\delta,\xi}$ is invertible and $\|\mathcal{L}^{-1}_{\delta,\xi}\|\le\frac{1}{c}$.

We are in position now to find a solution of Eq. (2.5a) whose proof relies on a standard contraction mapping argument (see for example [19, Proposition 1.8] and [18, Proposition 2.1])

Proposition 2.1. *For any $\sigma>0$ there exist $c>0$ and $\varepsilon_0>0$ such that for any $d>0$ and $\eta\in\mathbb{R}^6$ satisfying* (2.4) *and for any $\epsilon\in(-\varepsilon_0,\varepsilon_0)$, there exists a unique $\phi_\epsilon=\phi_\epsilon(d,\eta)\in\mathcal{K}^{\perp}_{\delta,\xi}$ solution to* (2.5a) *which is continuously differentiable with respect to d and η and such that*

$$\|\phi_\epsilon\|\le c\epsilon^2|\ln|\epsilon||^{\frac23}.$$

2.3 The reduced problem: solving Eq. (2.5b)

To solve Eq. (2.5b), we shall find the parameter δ and the point $\xi \in \Omega$ as in (2.4), i.e., $d > 0$ and $\eta \in \mathbb{R}^6$, so that (2.5b) is satisfied. It is well known that this problem has a variational structure, in the sense that solutions of (2.5b) reduces to find critical points to some given explicit finite dimensional functional. Indeed, let $J_\epsilon : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{3}\int_\Omega |u|^3 dx$$

and let $\tilde{J}_\epsilon : \mathbb{R}_+ \times \mathbb{R}^6 \to \mathbb{R}$ be the reduced energy which is defined by

$$\tilde{J}_\epsilon(d,\eta) = J_\epsilon(W_{\delta,\xi} + \phi_\epsilon).$$

Proposition 2.2. *For any $\sigma > 0$ there exists $\varepsilon_0 > 0$ such that for any $\epsilon \in (-\varepsilon_0, \varepsilon_0)$*

$$\tilde{J}_\epsilon(d,\eta) = \mathfrak{c}_0(\epsilon) + |\epsilon|^3 \Upsilon(d,\eta) + o\left(|\epsilon|^3\right) \tag{2.7}$$

with

$$\Upsilon(d,\eta) := \mathtt{sgn}(\epsilon)\left(1 - 2v_0(\xi_0)\right) d^2 \mathfrak{a}_1 + d^3\left(\mathfrak{a}_2 \langle D^2 u_0(\xi_0)\eta, \eta\rangle - \mathfrak{a}_3\right), \tag{2.8}$$

uniformly with respect to (d,η) which satisfies (2.4)*, where the $\mathfrak{c}_0(\epsilon)$ depends only on ϵ and the $\mathfrak{a}_i$'s are positive constants. Moreover, if (d,η) is a critical point of $\tilde{J}_\epsilon$, then $W_{\delta,\xi} + \phi_\epsilon$ is a solution of* (1.8).

Proof. It is quite standard to prove that if (d,η) satisfies (2.4) and is a critical point of $\tilde{J}_\epsilon$, then $W_{\delta,\xi} + \phi_\epsilon$ is a solution of (1.8) (see for example [18, Proposition 2.2]). Moreover, it is not difficult to check that

$$\tilde{J}_\epsilon(d,\eta) = J_\epsilon(W_{\delta,\xi}) + o\left(|\epsilon|^3\right)$$

uniformly with respect to (d,η) which satisfies (2.4) (see for example [18, Proposition 2.2]).

We need only to estimate the main term of the reduced energy $J_\epsilon(W_{\delta,\xi})$, i.e.,

$$\begin{aligned}
&J_\epsilon(u_0 + \epsilon v_0 - PU_{\delta,\xi})\\
=&\frac{1}{2}\int_\Omega |\nabla(u_0 + \epsilon v_0 - PU_{\delta,\xi})|^2 - \frac{\lambda_0 + \epsilon}{2}\int_\Omega (u_0 + \epsilon v_0 - PU_{\delta,\xi})^2 - \frac{1}{3}\int_\Omega |u_0 + \epsilon v_0 - PU_{\delta,\xi}|^3\\
=&\frac{1}{2}\int_\Omega |\nabla(u_0 + \epsilon v_0)|^2 + \frac{1}{2}\int_\Omega |\nabla PU_{\delta,\xi}|^2 - \frac{\lambda_0 + \epsilon}{2}\int_\Omega (u_0 + \epsilon v_0)^2 - \frac{\lambda_0 + \epsilon}{2}\int_\Omega (PU_{\delta,\xi})^2\\
&- \underbrace{\left(\int_\Omega \nabla u_0 \nabla PU_{\delta,\xi} - \lambda_0 \int_\Omega u_0 PU_{\delta,\xi}\right)}_{=\int_\Omega |u_0| u_0 PU_{\delta,\xi}} - \epsilon\underbrace{\left(\int_\Omega \nabla v_0 \nabla PU_{\delta,\xi} - \lambda_0 \int_\Omega v_0 PU_{\delta,\xi} - \int_\Omega u_0 PU_{\delta,\xi}\right)}_{=\int_\Omega 2|u_0| v_0 PU_{\delta,\xi}}\\
&+ \epsilon^2 \int_\Omega v_0 PU_{\delta,\xi} - \frac{1}{3}\int_\Omega |u_0 + \epsilon v_0 - PU_{\delta,\xi}|^3
\end{aligned}$$

$$
\begin{aligned}
=&\underbrace{\frac{1}{2}\int_\Omega |\nabla(u_0+\epsilon v_0)|^2-\frac{\lambda_0+\epsilon}{2}\int_\Omega (u_0+\epsilon v_0)^2-\frac{1}{3}\int_\Omega |u_0+\epsilon v_0|^3}_{=:I_1}\\
&+\underbrace{\frac{1}{2}\int_\Omega |\nabla PU_{\delta,\xi}|^2-\frac{1}{3}\int_\Omega PU^3_{\delta,\xi}}_{=:I_2}\underbrace{-\frac{\lambda_0}{2}\int_\Omega PU^2_{\delta,\xi}+\int_\Omega u_0PU^2_{\delta,\xi}}_{=:I_3}\underbrace{-\frac{\epsilon}{2}\int_\Omega PU^2_{\delta,\xi}+\epsilon\int v_0PU^2_{\delta,\xi}}_{=:I_4}\\
&\underbrace{-\frac{1}{3}\int_\Omega\Big(|u_0+\epsilon v_0-PU_{\delta,\xi}|^3-|u_0+\epsilon v_0|^3-PU^3_{\delta,\xi}+3(u_0+\epsilon v_0)PU^2_{\delta,\xi}+3|u_0+\epsilon u_0|(u_0+\epsilon v_0)PU_{\delta,\xi}\Big)}_{=:I_5}\\
&\underbrace{+\int_\Omega\Big[|u_0+\epsilon v_0|(u_0+\epsilon v_0)-(|u_0|u_0+2\epsilon|u_0|v_0)\Big]PU_{\delta,\xi}}_{=:I_6}+\underbrace{\epsilon^2\int_\Omega v_0PU_{\delta,\xi}}_{=:I_7}.
\end{aligned}
$$

It is clear that

$$I_7=\mathcal{O}\left(\epsilon^2\int_\Omega \frac{\delta^2}{|x-\xi|^4}dx\right)=\mathcal{O}\left(\epsilon^2\delta^2\right)=\mathcal{O}(\epsilon^4).$$

To estimate I_6 by (2.6) it follows that

$$I_6=\mathcal{O}\left(\epsilon^2\int_\Omega PU_{\delta,\xi}\right)=\mathcal{O}\left(\epsilon^2\delta^2\right)=\mathcal{O}(\epsilon^4).$$

Now, I_1 does not depend neither on d nor on η and it will be included in the constant $\mathfrak{c}_0$ in (2.7). By (2.2)

$$
\begin{aligned}
I_2=&\frac{1}{2}\int_\Omega U^3_{\delta,\xi}-\frac{1}{3}\int_\Omega PU^3_{\delta,\xi}\\
=&\frac{1}{2}\int_\Omega U^3_{\delta,\xi}-\frac{1}{3}\int_\Omega\Big(U_{\delta,\xi}(x)-\alpha_6\delta^2H(x,\xi)+\mathcal{O}(\delta^4)\Big)^3\\
=&\frac{1}{6}\int_{\mathbb{R}^6}U^3+\mathcal{O}\left(\delta^2\int_\Omega U^2_{\delta,\xi}\right)+O(\delta^4)\\
=&\frac{1}{6}\int_{\mathbb{R}^6}U^3_{\delta,\xi}+O(\delta^4).
\end{aligned}
$$

Now, setting

$$\varphi_{\delta,\xi}:=PU_{\delta,\xi}-U_{\delta,\xi}=\mathcal{O}(\delta^2),$$

by (2.2) and (2.4)

$$
\begin{aligned}
I_3=&\int_\Omega\left(u_0(x)-\frac{\lambda_0}{2}\right)(U_{\delta,\xi}+\varphi_{\delta,\xi})^2\\
=&\int_\Omega\left(u_0(x)-u_0(\xi_0)\right)U^2_{\delta,\xi}+\mathcal{O}(\delta^4)\\
=&\int_\Omega\left[\frac{1}{2}\langle D^2u_0(\xi_0)(x-\xi_0),(x-\xi_0)\rangle+\mathcal{O}(|x-\xi_0|^3)\right]\alpha_6^2\frac{\delta^4}{(\delta^2+|x-\xi|^2)^4}dx+\mathcal{O}(\delta^4)\\
=&\alpha_6^2\int_\Omega\frac{1}{2}\langle D^2u_0(\xi_0)(x-\xi_0),(x-\xi_0)\rangle\frac{\delta^4}{(\delta^2+|x-\xi|^2)^4}dx+\mathcal{O}(\delta^4)
\end{aligned}
$$

$$
\begin{aligned}
&=\alpha_6^2\delta^2\int_{\frac{\Omega-\xi}{\delta}}\frac{1}{2}\langle D^2u_0(\xi_0)(\delta y+\sqrt{\delta}\eta),(\delta y+\sqrt{\delta}\eta)\rangle\frac{1}{(1+|y|^2)^4}dy+\mathcal{O}(\delta^4)\\
&=\frac{\alpha_6^2}{2}\delta^3\left(\int_{\mathbb{R}^6}\frac{1}{(1+|y|^2)^4}dy\right)\langle D^2u_0(\xi_0)\eta,\eta\rangle+\mathcal{O}(\delta^4|\ln\delta|)\\
&=\frac{\alpha_6^2}{2}d^3|\epsilon|^3\left(\int_{\mathbb{R}^6}\frac{1}{(1+|y|^2)^4}dy\right)\langle D^2u_0(\xi_0)\eta,\eta\rangle+\mathcal{O}(\epsilon^4|\ln|\epsilon||),
\end{aligned}
$$

and analogously

$$
\begin{aligned}
I_4=&\epsilon\int_\Omega\left(v_0(x)-\frac{1}{2}\right)PU_{\delta,\xi}^2\\
=&\epsilon\left[\alpha_6^2\delta^2\left(\int_{\mathbb{R}^6}\frac{1}{(1+|y|^2)^4}dy\right)\left(v_0(\xi_0)-\frac{1}{2}\right)+o(1)\right]\\
=&\epsilon^3d^2\left[\alpha_6^2\left(\int_{\mathbb{R}^6}\frac{1}{(1+|y|^2)^4}dy\right)\left(v_0(\xi_0)-\frac{1}{2}\right)+o(1)\right].
\end{aligned}
$$

Finally, we have to estimate I_5.

We point out that

$$
\begin{aligned}
&|u_0+\epsilon v_0-PU_{\delta,\xi}|^3-|u_0+\epsilon v_0|^3-PU_{\delta,\xi}^3\\
&\quad+3(u_0+\epsilon v_0)PU_{\delta,\xi}^2+3|u_0+\epsilon u_0|(u_0+\epsilon v_0)PU_{\delta,\xi}=0\quad\text{if }\ u_0+\epsilon v_0\le 0,
\end{aligned}
$$

and so

$$
\begin{aligned}
I_5=&-\frac{1}{3}\int_{\{u_0+\epsilon v_0\ge 0\}}\Big(|u_0+\epsilon v_0-PU_{\delta,\xi}|^3-(u_0+\epsilon v_0)^3-PU_{\delta,\xi}^3\\
&+3(u_0+\epsilon v_0)PU_{\delta,\xi}^2+3(u_0+\epsilon v_0)^2PU_{\delta,\xi}\Big)\,dx\\
=&-\frac{1}{3}\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}}\left(-2PU_{\delta,\xi}^3+6(u_0+\epsilon v_0)PU_{\delta,\xi}^2\right)\\
&-\frac{1}{3}\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right).
\end{aligned}
$$

First of all we claim that for any $\sigma>0$ there exists $\varepsilon_0>0$ such that for any $\epsilon\in(-\varepsilon_0,\varepsilon_0)$ and (d,ξ) satisfying (2.4)

$$
B\left(\xi,R_\delta^1\sqrt{\delta}\right)\subset\{x\in\Omega:0<u_0(x)+\epsilon v_0(x)<PU_{\delta,\xi}(x)\}\cap B(\xi,\delta^{\frac{1}{4}})\subset B\left(\xi,R_\delta^2\sqrt{\delta}\right), \tag{2.9}
$$

where

$$
R_\delta^1,R_\delta^2=R_0+o(1)\quad\text{with }\ R_0:=\left(\frac{\alpha_6}{u_0(\xi_0)}\right)^{\frac{1}{4}}. \tag{2.10}
$$

We remind that $\delta=\mathcal{O}(\epsilon)$ and also that

$$
PU_{\delta,\xi}(x)=\alpha_6\frac{\delta^2}{(\delta^2+|x-\xi|^2)^2}+\mathcal{O}(\epsilon^2)
$$

uniformly in Ω. If $|x-\xi| < R^1_\delta\sqrt{\delta}$ is small enough then by mean value theorem $u_0(x)+\epsilon v_0(x) = u_0(\xi_0)+\mathcal{O}_1(\epsilon)$ and

$$\begin{aligned} u_0(x)+\epsilon v_0(x) < PU_{\delta,\xi}(x) \Leftrightarrow \frac{u_0(\xi_0)}{\alpha_6}+\mathcal{O}_1(\epsilon) < \frac{\delta^2}{(\delta^2+|x-\xi|^2)^2} \\ \Leftrightarrow |x-\xi| \le \sqrt{\delta}\underbrace{\left(\frac{1}{\left(\frac{u_0(\xi_0)}{\alpha_6}+\mathcal{O}_1(\epsilon)\right)^{\frac12}}-\delta\right)^{\frac12}}_{R^1_\delta}, \end{aligned}$$

and the first inclusion in (2.9) together with (2.10) follow. On the other hand, again by mean value theorem we have

$$u_0(x)+\epsilon v_0(x) = u_0(\xi_0)+\mathcal{O}_2(\sqrt{\delta})$$

for any $x\in B(\xi,\delta^{\frac14})$ and arguing as above we get the second inclusion in (2.9) and (2.10).

It is useful to point out that by (2.9) we immediately get

$$B^c\left(\xi, R^1_\delta\sqrt{\delta}\right) \supset \{x\in\Omega : u_0(x)+\epsilon v_0(x) \ge PU_{\delta,\xi}(x)\}\cup B^c(\xi,\delta^{\frac14}) \supset B^c\left(\xi,R^2_\delta\sqrt{\delta}\right). \tag{2.11}$$

Now by (2.9) and (2.11) we deduce

$$\begin{aligned} I_5 =& -\frac13\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}}\left(-2PU^3_{\delta,\xi}+6(u_0+\epsilon v_0)PU^2_{\delta,\xi}\right) \\ &-\frac13\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right) \\ =& -\frac13\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6(u_0+\epsilon v_0)PU^2_{\delta,\xi}\right) \\ &+\frac13\int_{B^c\left(\xi,\delta^{\frac14}\right)\setminus\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cap B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6(u_0+\epsilon v_0)PU^2_{\delta,\xi}\right) \\ &-\frac13\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac14}\right)}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right) \\ &-\frac13\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B^c\left(\xi,\delta^{\frac14}\right)}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right) \\ =& -\frac13\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6(u_0+\epsilon v_0)PU^2_{\delta,\xi}\right) \\ &-\frac13\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac14}\right)}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right)+o(\delta^3), \end{aligned}$$

because

$$\begin{aligned}
&\int_{B^c\left(\xi,\delta^{\frac14}\right)\setminus\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cap B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6(u_0+\epsilon v_0)PU^2_{\delta,\xi}\right)\\
&=\mathcal{O}\left(\int_{B^c\left(\xi,\delta^{\frac14}\right)}\left(U^3_{\delta,\xi}+U^2_{\delta,\xi}\right)\right)=\mathcal{O}(\delta^{\frac72}),\\
&\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B^c\left(\xi,\delta^{\frac14}\right)}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right)\\
&=\mathcal{O}\left(\delta^3\mathtt{meas}\{0<u_0(x)<2\delta\}\right)=o(\delta^3),
\end{aligned}$$

since

$$\begin{aligned}
&PU_{\delta,\xi}(x)=\mathcal{O}(\delta) && \text{if } \ |x-\xi|\ge\delta^{\frac14},\\
&\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B^c(\xi,\delta^{\frac14})\subset\{0<u_0(x)<2\delta\} && \text{if } \ \delta \text{ is small enough.}
\end{aligned}$$

Next we claim that

$$\begin{aligned}
&-\frac13\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6(u_0+\epsilon v_0)PU^2_{\delta,\xi}\right)\\
&\quad-\frac13\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac14}\right)}\left(-2(u_0+\epsilon v_0)^3+6(u_0+\epsilon v_0)^2PU_{\delta,\xi}\right)+o(\delta^3)\\
=&-\frac13\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6u_0PU^2_{\delta,\xi}\right)\\
&\quad-\frac13\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac14}\right)}\left(-2u_0^3+6u_0^2PU_{\delta,\xi}\right)+o(\delta^3).
\end{aligned}$$

Indeed using (2.11) and (2.9) we get

$$\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B^c\left(\xi,\delta^{\frac14}\right)}PU^2_{\delta,\xi}=\mathcal{O}\left(\int_{B^c\left(\xi,\delta^{\frac12}\right)}U^2_{\delta,\xi}\right)=\mathcal{O}\left(\delta^3\right),$$

$\mathtt{meas}B\left(\xi,\delta^{\frac12}\right)=\mathcal{O}(\delta^3)$ and

$$\int_{\{u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac14}\right)}PU_{\delta,\xi}=\mathcal{O}\left(\int_{B\left(\xi,\delta^{\frac12}\right)}U_{\delta,\xi}\right)=\mathcal{O}\left(\delta^3\right).$$

We estimate the last two terms in the expansion of I_5. By (2.11)

$$B^c\left(\xi,R^2_\delta\sqrt{\delta}\right)\subset\{x\in\Omega\,:\,u_0(x)+\epsilon v_0(x)\ge PU_{\delta,\xi}\}\cup B^c(\xi,\delta^{\frac14})\subset B^c\left(\xi,R^1_\delta\sqrt{\delta}\right).$$

Hence

$$\begin{aligned}
&\int_{|x-\xi|>R^2_\delta\sqrt{\delta}}\left(-2PU^3_{\delta,\xi}+6u_0PU^2_{\delta,\xi}\right)\\
\le&\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6u_0PU^2_{\delta,\xi}\right)\\
\le&\int_{|x-\xi|>R^1_\delta\sqrt{\delta}}\left(-2PU^3_{\delta,\xi}+6u_0PU^2_{\delta,\xi}\right).
\end{aligned}$$

Now if R_δ denotes either R^1_δ or R^2_δ we get

$$\begin{aligned}
&\int_{|x-\xi|>R_\delta\sqrt{\delta}}\left(-2PU^3_{\delta,\xi}+6u_0PU^2_{\delta,\xi}\right)\\
=&-2\int_{|x-\xi|>R_\delta\sqrt{\delta}}U^3_{\delta,\xi}+6\int_{|x-\xi|>R_\delta\sqrt{\delta}}u_0U^2_{\delta,\xi}+\mathcal{O}\big(\delta^4\big)\\
=&-2\int_{|y|>\frac{R_\delta}{\sqrt{\delta}}}\frac{\alpha_6^3}{(1+|y|^2)^6}+6\delta^2\int_{|y|>\frac{R_\delta}{\sqrt{\delta}}}u_0(\delta y+\xi)\frac{\alpha_6^2}{(1+|y|^2)^4}+\mathcal{O}\big(\delta^4\big)\\
=&-2\omega_6\alpha_6^3\int_{\frac{R_\delta}{\sqrt{\delta}}}^{+\infty}\frac{r^5}{(1+r^2)^6}+6\delta^2\omega_6\alpha_6^2u_0(\xi_0)\int_{\frac{R_\delta}{\sqrt{\delta}}}^{+\infty}\frac{r^5}{(1+r^2)^4}\\
&+\mathcal{O}\left(\delta^4\int_{\frac{R_\delta}{\sqrt{\delta}}}^{+\infty}\frac{r^7}{(1+r^2)^4}\right)+\mathcal{O}\big(\delta^4\big)\\
=&-\frac13\omega_6\alpha_6^3R_\delta^{-6}\delta^3+3\delta^3\omega_6\alpha_6^2R_\delta^{-2}u_0(\xi_0)+\mathcal{O}\left(\delta^4|\log\delta|\right)\\
=&-\frac13\omega_6\alpha_6^3R_0^{-6}\delta^3+3\delta^3\omega_6\alpha_6^2R_0^{-2}u_0(\xi_0)+o\big(\delta^3\big)\quad\text{because of (2.10)}
\end{aligned}$$

and by comparison

$$\begin{aligned}
&\int_{\{u_0+\epsilon v_0\ge PU_{\delta,\xi}\}\cup B^c\left(\xi,\delta^{\frac14}\right)}\left(-2PU^3_{\delta,\xi}+6u_0PU^2_{\delta,\xi}\right)\\
=&-\frac13\omega_6\alpha_6^3(R_0)^{-6}\delta^3+3\delta^3\omega_6\alpha_6^2(R_0)^{-2}u_0(\xi_0)+o\big(\delta^3\big).
\end{aligned}\tag{2.12}$$

In a similar way, by (2.9)

$$\begin{aligned}
&\int_{|x-\xi|<R^1_\delta\sqrt{\delta}}\left(-2u_0^3+6u_0^2PU_{\delta,\xi}\right)\\
\le&\int_{\{0<u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac14}\right)}\left(-2u_0^3+6u_0^2PU_{\delta,\xi}\right)\\
\le&\int_{|x-\xi|<R^2_\delta\sqrt{\delta}}\left(-2u_0^3+6u_0^2PU_{\delta,\xi}\right),
\end{aligned}$$

and if R_δ denotes either R^1_δ or R^2_δ we get

$$\begin{aligned}
&\int_{|x-\xi|<R_\delta\sqrt{\delta}} \left(-2u_0^3+6u_0^2 PU_{\delta,\xi}\right)\\
=&-2\delta^6\int_{|y|<\frac{R_\delta}{\sqrt{\delta}}} u_0^3(\delta y+\xi)+6\delta^4\int_{|y|<\frac{R_\delta}{\sqrt{\delta}}} u_0^2(\delta y+\xi)\frac{\alpha_6}{(1+|y|^2)^2}+\mathcal{O}\left(\delta^5\right)\\
=&\left(-2u_0^3(\xi_0)+\mathcal{O}\left(\sqrt{\delta}\right)\right)\delta^6\omega_6\int_0^{\frac{R_\delta}{\sqrt{\delta}}} r^5\\
&\quad+6\alpha_6\left(u_0^2(\xi_0)+\mathcal{O}\left(\sqrt{\delta}\right)\right)\delta^4\omega_6\int_0^{\frac{R_\delta}{\sqrt{\delta}}}\frac{r^5}{(1+r^2)^2}+\mathcal{O}\left(\delta^5\right)\\
=&-2\delta^3u_0^3(\xi_0)\omega_6R_\delta^6+3\alpha_6\delta^3u_0^2(\xi_0)\omega_6R_\delta^2+\mathcal{O}\left(\delta^{\frac{7}{2}}\right)\\
=&-2\delta^3u_0^3(\xi_0)\omega_6R_0^6+3\alpha_6\delta^3u_0^2(\xi_0)\omega_6R_0^2+o\left(\delta^3\right)\quad\text{because of (2.10),}
\end{aligned}$$

and by comparison

$$\begin{aligned}
&\int_{\{u_0+\epsilon v_0<PU_{\delta,\xi}\}\cap B\left(\xi,\delta^{\frac{1}{4}}\right)} \left(-2u_0^3+6u_0^2PU_{\delta,\xi}\right)\\
=&-2\delta^3u_0^3(\xi_0)\omega_6R_0^6+3\alpha_6\delta^3u_0^2(\xi_0)\omega_6R_0^2+o\left(\delta^3\right).
\end{aligned}\tag{2.13}$$

Finally, by (2.13) and (2.12)

$$I_5=|\epsilon|^3d^3\left(-\frac{11}{9}\omega_6\alpha_6^{\frac{3}{2}}(u_0(\xi_0))^{\frac{3}{2}}+o(1)\right).$$

Collecting all the previous estimates we get

$$\tilde{J}_\epsilon(d,\eta)=\mathfrak{c}_0(\epsilon)+|\epsilon|^3\underbrace{\left\{\mathtt{sgn}(\epsilon)\left(1-2v_0(\xi_0)\right)d^2\mathfrak{a}_1+d^3\left(\mathfrak{a}_2\langle D^2u_0(\xi_0)\eta,\eta\rangle-\mathfrak{a}_3\right)\right\}}_{=:\Upsilon(d,\eta)}+o\left(|\epsilon|^3\right)$$

with

$$\begin{aligned}
\mathfrak{a}_1&=\alpha_6^2\left(\int_{\mathbb{R}^6}\frac{1}{(1+|y|^2)^4}dy\right)=96\omega_6,\\
\mathfrak{a}_2&=\frac{\alpha_6^2}{2}\int_{\mathbb{R}^6}\frac{dy}{(1+|y|^2)^4},\\
\mathfrak{a}_3&=\frac{11}{9}\omega_6\alpha_6^{\frac{3}{2}}(u_0(\xi_0))^{\frac{3}{2}},
\end{aligned}$$

and that concludes the proof. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. The claim follows by Proposition 2.2 taking into account that if

$$\mathtt{sgn}(\epsilon)\left(1-2v_0(\xi_0)\right)>0$$

the function Υ has always an isolated maximum point $(d_0,0)$, with

$$d_0:=\frac{2\mathfrak{a}_1}{3\mathfrak{a}_3}\mathtt{sgn}(\epsilon)\left(1-2v_0(\xi_0)\right),$$

which is stable under uniform perturbations. □

3 A generic result

Let Ω_0 be a bounded and smooth domain in $\mathbb{R}^n$, we let D be and open neighbourhood of $\overline{\Omega_0}$ and $\alpha\in(0,1)$. There exists $\epsilon>0$ such that if $\theta\in C^{3,\alpha}(\overline{D},\mathbb{R}^n)$ with $\|\theta\|_{2,\alpha}\le\epsilon$ then $\Theta=I+\theta$ maps Ω_0 in a one-to-one way onto the smooth domain $\Omega_\theta:=\Theta(\Omega_0)$ with boundary $\partial\Omega_\theta=\Theta(\partial\Omega_0)$. If $x\in\Omega_0$ we agree that $\hat{x}=\Theta x=(I+\theta)x\in\Omega_\theta$. If $\hat{u}\in H^1_0(\Omega_\theta)\cap H^2(\Omega_\theta)$ then it is clear that $u=\hat{u}\circ\Theta\in H^1_0(\Omega_0)\cap H^2(\Omega_0)$.

Our result reads as follows.

Theorem 3.1. *The set*

$$\begin{aligned}\Xi:=\Big\{\theta\in C^{3,\alpha}(\overline{D},\mathbb{R}^n):\ & \textit{if } \lambda>0 \textit{ and } u\in H^1_0(\Omega_\theta) \textit{ solve}\\ & \Delta u+\lambda u+|u|^{\frac{4}{n-2}}u=0 \ \textit{in}\ \Omega_\theta,\ u=0 \textit{ on } \partial\Omega_\theta,\\ & \textit{then } u \textit{ is non-degenerate}\Big\}\end{aligned}\tag{3.1}$$

is a residual subset in $C^{3,\alpha}(\overline{D},\mathbb{R}^n)$, *i.e.*, $C^{3,\alpha}(\overline{D},\mathbb{R}^n)\setminus\Xi$ *is a countable union of close subsets without interior points.*

The proof relies on the following abstract transversality theorem (see [23,26,29]).

Theorem 3.2. *Let* X,Y,Z *be three Banach spaces and* $U\subset X$, $V\subset Y$ *open subsets. Let* $F:U\times V\to Z$ *be a* $C^\alpha-$*map with* $\alpha\ge1$. *Assume that*

i) for any $y\in V$, $F(\cdot,y):U\to Z$ *is a Fredholm map of index* l *with* $l\le\alpha$;

ii) 0 *is a regular value of* F, *i.e., the operator* $F'(x_0,y_0):X\times Y\to Z$ *is onto at any point* (x_0,y_0) *such that* $F(x_0,y_0)=0$;

iii) the map $\pi\circ i:F^{-1}(0)\to Y$ *is* $\sigma-$*proper, i.e.,* $F^{-1}(0)=\cup_{s=1}^{+\infty}C_s$ *where* C_s *is a closed set and the restriction* $\pi\circ i_{|C_s}$ *is proper for any* s; *here* $i:F^{-1}(0)\to Y$ *is the canonical embedding and* $\pi:X\times Y\to Y$ *is the projection.*

Then the set $\mathcal{V}:=\{y\in V:\ 0 \textit{ is a regular value of } F(\cdot,y)\}$ *is a residual subset of* V, *i.e.,* $V\setminus\mathcal{V}$ *is a countable union of close subsets without interior points.*

Indeed, in our case we choose

$$
\begin{aligned}
&X=\mathbb{R}\times\left(H_0^1(\Omega_0)\cap H^2(\Omega_0)\right),\\
&U=(0,\infty)\times\left(H_0^1(\Omega_0)\cap H^2(\Omega_0)\setminus\{0\}\right),\\
&Y=C^{3,\alpha}\left(\overline{D},\mathbb{R}^n\right),\\
&V=\mathcal{B}_\epsilon:=\left\{\theta\in C^{3,\alpha}(\overline{D},\mathbb{R}^n)\ :\ \|\theta\|_{3,\alpha}<\epsilon\right\},\\
&Z=\mathbb{R}\times L^2(\Omega_0).
\end{aligned}
$$

X and Z are Banach spaces equipped with the norms $\|(a,u)\|_X := |a| + \|u\|_{H_0^1\cap H^2(\Omega_0)}$, and $\|(a,u)\|_Z := |a| + \|u\|_{L^2(\Omega_0)}$, respectively. Moreover, the function $F : U\times V \to Z$ is defined by

$$F(\lambda,u,\theta):=\left(Q(\lambda,\hat{u},\theta),\Delta_{\hat{x}}\hat{u}+|\hat{u}|^{p-1}\hat{u}+\lambda\hat{u}\right),$$

where

$$Q(\lambda,\hat{u},\theta):=\int_{\Omega_\theta}\left(|\nabla_{\hat{x}}\hat{u}|^2-|\hat{u}|^{p+1}-\lambda\hat{u}^2\right)d\hat{x}.$$

It is clear that

$$F(\lambda,u,\theta)=(0,0)\ \Leftrightarrow\ \Delta_{\hat{x}}\hat{u}+|\hat{u}|^{p-1}\hat{u}+\lambda\hat{u}=0\quad\text{in }\ \Omega_\theta,\quad \hat{u}=0\quad\text{on }\partial\Omega_\theta.$$

Theorem 3.1 will follow by Theorem 3.2 as soon as we prove that F satisfies the assumptions and this is done below.

First of all, we rewrite F in terms of the $x-$variable (see [21,26])

Lemma 3.1. *We have*

$$Q(\lambda,\hat{u},\theta):=\int_{\Omega_0}\left\{\nabla u\cdot\left[(\det\Theta')(\Theta')^{-1}({}^t\Theta')^{-1}\nabla u\right]-\left(|u|^{p+1}+\lambda u^2\right)(\det\Theta')\right\}dx, \tag{3.2a}$$

$$\Delta_{\hat{x}}\hat{u}+|\hat{u}|^{p-1}\hat{u}+\lambda\hat{u}=\operatorname{div}\left[(\det\Theta')(\Theta')^{-1}({}^t\Theta')^{-1}\nabla u\right]+\left(|u|^{p-1}u+\lambda u\right)(\det\Theta'). \tag{3.2b}$$

At this point it is useful to point out the following fact.

Remark 3.1. We can choose $\epsilon>0$ small enough so that for any $\theta\in\mathcal{B}_\epsilon$

$$\left(\int_{\Omega_0}\left(\left|\left\langle(\det\Theta')(\Theta')^{-1}({}^t\Theta')^{-1}\nabla u,\nabla u\right\rangle\right|^2+\left|\operatorname{div}\left[(\det\Theta')(\Theta')^{-1}({}^t\Theta')^{-1}\nabla u\right]\right|^2\right)dx\right)^{1/2}$$

defines on $H_0^1(\Omega_0)\cap H^2(\Omega_0)$ a norm which is equivalent to the standard one

$$\|u\|_{H_0^1\cap H^2(\Omega_0)}=\left(\int_{\Omega_0}\left(|\nabla u|^2+|\Delta u|^2\right)dx\right)^{1/2}.$$

Next, we check the differentiability of F (see [21,26]).

Lemma 3.2. *The function F is differentiable at any $(\lambda_0, u_0, \theta_0) \in U \times V$ such that $F(\lambda_0, u_0, \theta_0) = (0,0)$. Moreover if $\Theta_0 = I + \theta_0$,*

$$\begin{aligned} F'(\lambda_0, u_0, \theta_0)[\lambda, u] = \Big(\int_{\Omega_0} \Big\{ & 2\nabla u_0 \cdot \left[(\det \Theta_0')(\Theta_0')^{-1}({}^t\Theta_0')^{-1} \nabla u \right] \\ & - \big((p+1)|u_0|^{p-1}u_0 + 2\lambda_0 u_0 \big) u (\det \Theta_0') \Big\} dx - \lambda \int_{\Omega_0} u_0^2 (\det \Theta_0') dx, \\ & \operatorname{div} \Big[(\det \Theta_0')(\Theta_0')^{-1}({}^t\Theta_0')^{-1} \nabla u \Big] \\ & + (p|u_0|^{p-1} + \lambda_0) u (\det \Theta_0') + \lambda u_0 (\det \Theta_0') \Big), \end{aligned} \tag{3.3}$$

and if $\theta_0 = 0$,

$$\begin{aligned} & F'(\lambda_0, u_0, \theta_0)[\theta] \\ = & \bigg(\int_{\Omega_0} \Big\{ \nabla u_0 \cdot \left[(\operatorname{div}\theta) \nabla u_0 - (\theta' + {}^t\theta') \nabla u_0 \right] - \Big(|u_0|^{p+1} + \lambda_0 u_0^2 \Big) (\operatorname{div}\theta) \Big\} dx \, , \\ & \operatorname{div} \left[(\operatorname{div}\theta) \nabla u_0 - (\theta' + {}^t\theta') \nabla u_0 \right] + \Big(|u_0|^{p-1} u_0 + \lambda_0 u_0 \Big) (\operatorname{div}\theta) \bigg) \, . \end{aligned} \tag{3.4}$$

Let us check assumption i) of Theorem 3.2.

Lemma 3.3. *For any $\theta \in V$ the function $F(\cdot, \cdot, \theta)$ is a Fredholm map from U into Z of index 0.*

Proof. The partial derivative $F'_{\lambda,u}(\lambda_0, u_0, \theta_0) : X \to Z$ is the sum of an isomorphism $\mathcal{I}$ and a compact perturbation $\mathcal{K}$, namely

$$\begin{aligned} \mathcal{I}(\lambda, u) &:= \bigg(-\lambda \int_{\Omega_0} u_0^2 (\det \Theta_0') dx, \operatorname{div} \Big[(\det \Theta_0')(\Theta_0')^{-1}({}^t\Theta_0')^{-1} \nabla u \Big] \bigg), \\ \mathcal{K}(\lambda, u) &:= \bigg(\int_{\Omega_0} \Big\{ 2\nabla u_0 \cdot \Big[(\det \Theta_0')(\Theta_0')^{-1}({}^t\Theta_0')^{-1} \nabla u \Big] \\ & \qquad - \Big((p+1)|u_0|^{p-1}u_0 + 2\lambda_0 u_0 \Big) u (\det \Theta_0') \Big\} dx \, , \\ & \qquad \Big(p|u_0|^{p-1} + \lambda_0 \Big) u (\det \Theta_0') + \lambda u_0 (\det \Theta_0') \bigg) \, . \end{aligned}$$

Thus, we completed the proof. □

Let us check assumption iii) of Theorem 3.2.

Lemma 3.4. *The map $\pi \circ i : F^{-1}(0) \to Y$ is $\sigma-$proper.*

Proof. Let us write

$$F^{-1}(0,0) = \cup_{m=1}^{\infty} \mathcal{C}_m, \quad \mathcal{C}_m = (A_m \times B_m \times C_m) \cap F^{-1}(0,0),$$

where

$$
\begin{aligned}
&A_m := \left\{ \frac{1}{m} \le \lambda \le m \right\}, \\
&B_m := \left\{ u \in H_0^1(\Omega_0) \cap H^2(\Omega_0) \ : \ \frac{1}{m} \le \|u\| := \left(\int_{\Omega_0} \left(|\nabla u|^2 + (\Delta u)^2 \right) dx \right)^{\frac{1}{2}} \le m \right\}, \\
&C_m := \left\{ \theta \in C^{3,\alpha}(\Omega_0) \ : \ \|\theta\|_{3,\alpha} \le \epsilon \left(1 - \frac{1}{m} \right) \right\}.
\end{aligned}
$$

Let us fix m. We have to prove that if $(\theta_k)_{k\ge 1} \subset C_m$ with $\theta_k \to \theta$ and $(\lambda_k, u_k)_{k\ge 1} \subset A_m \times B_m$ is such that $F(\lambda_k, u_k, \theta_k) = 0$ then, up to a subsequence, $(\lambda_k, u_k) \to (\lambda, u) \in A_m \times B_m$ and $F(\lambda, u, \theta) = 0$. First of all, up to a subsequence, we have $\lambda_k \to \lambda \in A_m$ and $u_k \to u$ weakly in $H_0^1(\Omega_0) \cap H^2(\Omega_0)$ and strongly in $L^q(\Omega_0)$ for any $q > 1$ if $n = 3, 4$ and $1 < q < \frac{2n}{n-4}$ if $n \ge 5$. If $\Theta_k = I + \theta_k$ we know that $\Theta_k \to \Theta := I + \theta$ in $C^{1,\alpha}(\Omega_0, \mathbb{R}^n)$. Now, condition $F(\lambda_k, u_k, \theta_k) = 0$ reads as

$$
\begin{aligned}
&\operatorname{div}\left(\underbrace{(\det \Theta_k')(\Theta_k')^{-1}({}^t\Theta_k')^{-1}}_{=A_k} \nabla u_k \right) + \underbrace{\left(|u_k|^{p-1}u_k + \lambda_k u_k \right)(\det \Theta_k')}_{=f_k} = 0 \quad &&\text{in } \Omega_0, \\
&u = 0 &&\text{on } \partial\Omega_0.
\end{aligned}
$$

In particular, for any $\varphi \in H_0^1(\Omega_0)$

$$
\int_{\Omega_0} \left[\langle A_k \nabla u_k, \nabla \varphi \rangle + f_k \varphi \right] dx = 0 \tag{3.5}
$$

and so passing to the limit

$$
\int_{\Omega_0} \left[\left\langle \underbrace{(\det \Theta')(\Theta')^{-1}({}^t\Theta')^{-1}}_{=A} \nabla u, \nabla \varphi \right\rangle - \underbrace{\left(|u|^{p-1}u + \lambda u \right)(\det \Theta')}_{=f} \varphi \right] dx = 0, \tag{3.6}
$$

namely

$$
\operatorname{div}(A\nabla u) + f = 0 \quad \text{in } \Omega_0, \quad u = 0 \quad \text{on } \partial\Omega_0,
$$

i.e., $F(\lambda, u, \theta) = 0$.

Now, let us prove that $u_k \to u$ strongly in $H_0^1(\Omega_0) \cap H^2(\Omega_0)$. By (3.5) and (3.6) we deduce

$$
\begin{aligned}
&\int_{\Omega_0} \langle A\nabla(u_k - u), \nabla(u_k - u) \rangle \\
=& \int_{\Omega_0} \langle A\nabla u_k, \nabla u_k \rangle + \int_{\Omega_0} \langle A\nabla u, \nabla u \rangle - 2\int_{\Omega_0} \langle A\nabla u, \nabla u_k \rangle \\
=& \int_{\Omega_0} \langle (A - A_k)\nabla u_k, \nabla u_k \rangle + \int_{\Omega_0} (-f_k u_k - f u + 2 f u_k) \\
=& o(1),
\end{aligned}
$$

because $A_k \to A$ in $C^0(\Omega_0)$ and $u_k \to u$ strongly in $L^{\frac{2n}{n-2}}(\Omega_0)$. Moreover, we also have

$$\begin{aligned}&\int_{\Omega_0} (\text{div}\,(A\nabla(u_k-u)))^2\\ =&\int_{\Omega_0} (\text{div}\,((A-A_k)\nabla u_k)-f_k+f)^2\\ \le& 2\int_{\Omega_0} (\text{div}\,((A-A_k)\nabla u_k))^2+2\int_{\Omega_0}(f_k-f)^2\\ =&o(1),\end{aligned}$$

because $A_k \to A$ in $C^1(\Omega_0)$ and $u_k \to u$ strongly in $L^{\frac{2(n+2)}{n-2}}(\Omega_0)$. Then the claim follows directly from Remark 3.1. □

Let us check assumption ii) of Theorem 3.2.

Proposition 3.1. $(0,0)$ *is a regular value of* F.

Proof. Let $(\lambda_0,u_0,\theta_0)\in U\times V$ such that $F(\lambda_0,u_0,\theta_0)=(0,0)$. We shall prove that if $(\lambda,u)\in X$ is such that

$$\begin{cases} F'(\lambda_0,u_0,\theta_0)[\lambda,u]=0\\ \langle F'(\lambda_0,u_0,\theta_0)[\theta],(\lambda,u)\rangle_Z=0 \quad \text{for any }\ \theta\in Y\end{cases} \Rightarrow \lambda=0 \quad\text{and}\quad u\equiv 0. \tag{3.7}$$

Without loss of generality we can assume $\theta_0=0$. Then $\Theta_0=I$ and by (3.2a) and (3.2b) condition $F(\lambda_0,u_0,\theta_0)=(0,0)$ reads as

$$\begin{cases} \int_{\Omega_0}\left(|\nabla u_0|^2-|u_0|^{p+1}-\lambda_0u_0^2\right)dx=0,\\ \Delta u_0+|u_0|^{p-1}u_0+\lambda_0u_0=0 \quad \text{in }\ \Omega_0,\quad u=0\quad \text{on }\ \partial\Omega_0.\end{cases} \tag{3.8}$$

Moreover by (3.3) and (3.4) condition (3.7) can be rephrased as

$$\begin{cases} \int_{\Omega_0}\left\{2\nabla u_0\nabla u-\left((p+1)|u_0|^{p-1}u_0+2\lambda_0u_0\right)u-\lambda u_0^2\right\}dx=0,\\ \Delta u+\left(p|u_0|^{p-1}+\lambda_0\right)u+\lambda u_0=0 \quad \text{in }\ \Omega_0,\quad u=0\quad \text{on }\ \partial\Omega_0,\end{cases} \tag{3.9}$$

and

$$\begin{aligned}&\lambda\int_{\Omega_0}\left\{\nabla u_0\cdot\left[(\text{div}\theta)\nabla u_0-(\theta'+{}^t\theta')\nabla u_0\right]-\left(|u_0|^{p+1}+\lambda_0u_0^2\right)(\text{div}\theta)\right\}dx\\ &\quad+\int_{\Omega_0}\left\{\text{div}\left[(\text{div}\theta)\nabla u_0-(\theta'+{}^t\theta')\nabla u_0\right]+\left(|u_0|^{p-1}u_0+\lambda_0u_0\right)(\text{div}\theta)\right\}udx\\ =&0,\quad \forall\theta\in Y.\end{aligned} \tag{3.10}$$

We can simplify expression (3.10). Indeed, taking into account that

$$\Delta u_0 + \underbrace{|u_0|^{p-1}u_0 + \lambda_0 u_0}_{=g(u_0)} = 0 \quad \text{in } \Omega_0, \quad u = 0 \quad \text{on } \partial\Omega_0, \tag{3.11}$$

we have

$$\begin{aligned} &\operatorname{div}\left[(\operatorname{div}\theta)\nabla u_0 - (\theta' + {}^t\theta')\nabla u_0\right] \\ =&\operatorname{div}(\theta\Delta u_0) - \Delta(\theta\nabla u_0) \\ =&-\operatorname{div}(g(u_0)\theta) - \Delta(\theta\nabla u_0) \\ =&-g(u_0)(\operatorname{div}\theta) - g'(u_0)\nabla u_0\theta - \Delta(\theta\nabla u_0). \end{aligned}$$

Moreover,

$$\int_{\Omega_0} \Delta(\theta\nabla u_0)u dx = -\int_{\partial\Omega_0} \theta\nabla u_0 \partial_\nu u dx + \int_{\Omega_0} \theta\nabla u_0 \Delta u dx.$$

Therefore, (3.10) reads as

$$\begin{aligned} 0 =&\lambda \int_{\Omega_0} \left\{ \left[g(u_0)u_0(\operatorname{div}\theta) + g'(u_0)u_0\nabla u_0\theta + \theta\nabla u_0 \underbrace{\Delta u_0}_{=-g(u_0)} \right] - \underbrace{\left(|u_0|^{p+1} + \lambda_0 u_0^2\right)}_{=g(u_0)u_0}(\operatorname{div}\theta) \right\} dx \\ &- \lambda \int_{\partial\Omega_0} \theta\nabla u_0 \partial_\nu u_0 dx + \int_{\Omega_0} \left\{ \left[-g(u_0)u(\operatorname{div}\theta) - g'(u_0)u\nabla u_0\theta - \theta\nabla u_0 \underbrace{\Delta u}_{=-g'(u_0)u-\lambda u_0} \right] \right. \\ &\left. + \underbrace{\left(|u_0|^{p-1}u_0 + \lambda_0 u_0\right)}_{=g(u_0)}(\operatorname{div}\theta)u \right\} dx + \int_{\partial\Omega_0} \theta\nabla u_0 \partial_\nu u dx \\ =&\lambda \int_{\Omega_0} \underbrace{\left(g'(u_0)u_0 - g(u_0) + u_0\right)}_{=(p-1)|u_0|^{p-1}u_0+u_0} \theta\nabla u_0 dx + \int_{\partial\Omega_0} \theta\nabla u_0 \left(\partial_\nu u - \lambda\partial_\nu u_0\right) dx. \end{aligned} \tag{3.12}$$

Now, we prove that $\lambda = 0$. Indeed by taking deformations θ which take fix the boundary of Ω_0 by (3.12) we get

$$\lambda \int_{\Omega_0} \left[(p-1)|u_0|^{p-1}u_0 + u_0\right] \theta\nabla u_0 dx = 0 \quad \text{for any } \theta \in V, \quad \theta = 0 \quad \text{on } \partial\Omega_0.$$

If $\lambda \neq 0$ then we necessarily have

$$u_0 \left[(p-1)|u_0|^{p-1} + 1\right] \nabla u_0 = 0 \quad \text{a.e. in } \Omega_0,$$

and so $u_0\nabla u_0 = 0$ a.e. in Ω. This is not possible because u_0 solves (3.11) and by the unique continuation theorem in [2] we know that $\text{meas}\{x \in \Omega_0 : u_0(x) = 0\} = \text{meas}\{x \in \Omega_0 : \nabla u_0(x) = 0\} = 0$.

Since $\lambda = 0$ by (3.12) we deduce that

$$\int_{\partial\Omega_0} \theta \nabla u_0 \partial_\nu u dx = 0 \quad \text{for any } \theta \in Y$$

and arguing exactly as in [26, pp. 313–314], we deduce that $u = 0$. That concludes the proof. □

Proposition 3.2. *For any $\theta \in \Xi$ as in* (3.1) *there exists $\lambda_\theta \in (0, \lambda_1(\Omega_\theta))$ such that*

$$\lambda_\theta = 2 \max_{\Omega_\theta} u_{\lambda_\theta}. \tag{3.13}$$

Proof. Let $\theta \in \Xi$ as in (3.1) and let us consider the perturbed domain Ω_θ. For any $\lambda \in (0, \lambda_1(\Omega_\theta))$ let u_λ be the least energy positive solution on the domain Ω_θ, which is non-degenerate because of Theorem 3.1. Therefore, by the Implicit function Theorem we deduce that the map $\lambda \to u_\lambda$ is continuous. Let us consider the continuous function

$$f(\lambda) := \lambda - 2\|u_\lambda\|_{L^\infty(\Omega_\theta)}, \quad \lambda \in (0, \lambda_1(\Omega_\theta)).$$

Since

$$\lim_{\lambda \to 0} \|u_\lambda\|_{L^\infty(\Omega_\theta)} = +\infty \quad \text{and} \quad \lim_{\lambda \to \lambda_1(\Omega_\theta)} \|u_\lambda\|_{L^\infty(\Omega_\theta)} = 0,$$

(see [14] and the classical bifurcation theory, respectively), there exists λ_θ such that $f(\lambda_\theta) = 0$ and the claim (3.13) follows. □

Proof of Theorem 1.2. It follows immediately by Theorem 3.1 and Proposition 3.2. □

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