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$\sup \times \inf$ Inequalities for the Scalar Curvature Equation in Dimensions 4 and 5

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Abstract. We consider the following problem on bounded open set Ω of $\mathbb{R}^n$:

$$\begin{cases} -\Delta u = V u^{\frac{n+2}{n-2}} & \text{in } \Omega \subset \mathbb{R}^n, \quad n = 4, 5, \\ u > 0 & \text{in } \Omega. \end{cases}$$

We assume that :

$$\begin{aligned} & V \in C^{1,\beta}(\Omega), && 0 < \beta \le 1, \\ & 0 < a \le V \le b < +\infty, && \\ & |\nabla V| \le A, \quad |\nabla^{1+\beta} V| \le B && \text{in } \Omega. \end{aligned}$$

Then, we have a $\sup \times \inf$ inequality for the solutions of the previous equation, namely:

$$\begin{aligned} & \Big(\sup_K u\Big)^{\beta} \times \inf_{\Omega} u \le c = c(a, b, A, B, \beta, K, \Omega) && \text{for } n = 4, \\ & \Big(\sup_K u\Big)^{1/3} \times \inf_{\Omega} u \le c = c(a, b, A, B, K, \Omega) && \text{for } n = 5 \quad \text{and} \quad \beta = 1. \end{aligned}$$

Key Words: $\sup \times \inf$, dimension 4 and 5, blow-up, moving-plane method.

AMS Subject Classifications: 35J61, 35B44, 35B45, 35B50

1 Introduction and main result

We work on $\Omega \subset\subset \mathbb{R}^4$ and we consider the following equation:

$$\begin{cases} -\Delta u = V u^{\frac{n+2}{n-2}} & \text{in } \Omega \subset \mathbb{R}^n, \quad n = 4, 5, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{E}$$

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with

$$\begin{cases} V \in C^{1,\beta}(\Omega), \\ 0 < a \le V \le b < +\infty & \text{in } \Omega, \\ |\nabla V| \le A & \text{in } \Omega, \\ |\nabla^{1+\beta} V| \le B & \text{in } \Omega. \end{cases} \qquad (C_\beta)$$

Without loss of generality, we suppose $\Omega = B_1(0)$ the unit ball of $\mathbb{R}^n$.

The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$ is:

$$-\Delta u = V(x)e^u. \qquad (E')$$

Eq. (E') was studied by many authors and we can find very important result about a priori estimates in [8, 9, 12, 16, 19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \le c = c\Big(\inf_\Omega V,\ ||V||_{L^\infty(\Omega)},\ \inf_\Omega u,\ K,\ \Omega\Big).$$

And, precisely, in [8,12,16,19], we have:

$$C \sup_K u + \inf_\Omega u \le c = c\Big(\inf_\Omega V, ||V||_{L^\infty(\Omega)}, K, \Omega\Big),$$
$$\sup_K u + \inf_\Omega u \le c = c\Big(\inf_\Omega V, ||V||_{C^\alpha(\Omega)}, K, \Omega\Big),$$

where K is a compact subset of Ω, C is a positive constant which depends on $\frac{\inf_\Omega V}{\sup_\Omega V}$, and, $\alpha \in (0, 1]$.

For $n \ge 3$ we have the following general equation on a Riemannian manifold:

$$-\Delta u + hu = V(x)u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad (E_n)$$

where h, V are two continuous functions. In the case $c_n h = R_g$ the scalar curvature, we call V the prescribed scalar curvature. Here c_n is a universal constant.

Eq. (E_n) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2–4,11,15]. In this case we have a sup $\times$ inf inequality.

In the case $V \equiv 1$ and M compact, Eq. (E_n) is Yamabe equation. T. Aubin and R. Schoen proved the existence of solution in this case, see for example [1,14] for a complete and detailed summary.

When M is a compact Riemannian manifold, there exist some compactness result for Eq. (E_n) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose M Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [17] proved that the product sup $\times$ inf is bounded. Also, see [3,5,6] for other

Harnack type inequalities, and, see [3,7] about some caracterization of the solutions of this equation (E_n) in this case ($V \equiv 1$).

Here we extend a result of [11] on an open set of $\mathbb{R}^n$, $n=4,5$. In fact we consider the prescribed scalar curvature equation on an open set of $\mathbb{R}^n$, $n=4,5$, and, we prove a $\sup \times \inf$ inequality on compact set of the domain when the derivative of the prescribed scalar curvature is β-holderian, $\beta > 0$.

Our proof is an extension of Chen-Lin result in dimension 4 and 5, see [11], and the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [10], one of the application of this method.

We have the following result in dimension 4, which is the consequence of the work of Chen-Lin.

Theorem 1.1. *For all $a,b,A,B>0$, and for all compact K of Ω, there exists a positive constant $c=c(a,b,A,B,K,\Omega)$ such that:*

$$\sup_K u \times \inf_\Omega \leq c,$$

where u is solution of (E) with V, C^2 satisfying (C_β) for $\beta = 1$.

Here, we give an inequality of type $\sup \times \inf$ for Eq. (E) in dimension 4 and with general conditions on the prescribed scalar curvature, exactly we take a $C^{1,\beta}$ condition. In fact we extend the result of Chen-Lin in dimension 4.

Here we prove:

Theorem 1.2. *For all $a,b,A,B>0$, $1 \geq \beta > 0$, and for all compact K of Ω, there exists a positive constant $c=c(a,b,A,B,\beta,K,\Omega)$ such that:*

$$\Big(\sup_K u\Big)^\beta \times \inf_\Omega u \leq c,$$

where u is solution of (E) with V satisfying (C_β).

We have the following result in dimension 5, which is the consequence of the work of Chen-Lin.

Theorem 1.3. *For all $a,b,m,A,B>0$, and for all compact K of Ω, there exists a positive constant $c=c(a,b,m,A,B,K,\Omega)$ such that:*

$$\sup_K u \leq c, \quad \text{if} \ \inf_\Omega u \geq m,$$

where u is solution of Eq. (E) with V satisfying $(C_\beta)=(C_1)$ for $\beta=1$.

Here, we give an inequality of type $\sup \times \inf$ for Eq. (E) in dimension 5 and with general conditions on the prescribed scalar curvature, exactly we take a C^2 condition ($\beta = 1$ in (C_β)). In fact we extend the result of Chen-Lin in dimension 5.

Here we prove:

Theorem 1.4. *For all $a,b,A,B>0$, and for all compact K of Ω, there exists a positive constant $c=c(a,b,A,B,K,\Omega)$ such that:*

$$\Big(\sup_K u\Big)^{1/3}\times\inf_\Omega u\le c,$$

where u is solution of (E) with V satisfying (C_β) for $\beta=1$.

2 The method of moving-plane

In this section we will formulate a modified version of the method of moving-plane for use later. Let Ω an open set and Ω^c the complement of Ω. We consider a solution u of the following equation:

$$\begin{cases}\Delta u+f(x,u)=0,\\ u>0,\end{cases}\qquad (E'')$$

where $f(x,u)$ is nonegative, Holder continuous in x, C^1 in u, and defined on $\bar\Omega\times(0,+\infty)$. Let e be a unit vector in $\mathbb{R}^n$. For $\lambda<0$, we let $T_\lambda=\{x\in\mathbb{R}^n,\langle x,e\rangle=\lambda\}$, $\Sigma_\lambda=\{x\in\mathbb{R}^n,\langle x,e\rangle>\lambda\}$, and $x^\lambda=x+(2\lambda-2\langle x,e\rangle)e$ to denote the reflexion point of x with respect to T_λ, where $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{R}^n$. Define:

$$\lambda_1\equiv\sup\{\lambda<0,\ \Omega^c\subset\Sigma_\lambda\},$$

$\Sigma'_\lambda=\Sigma_\lambda-\Omega^c$ for $\lambda\le\lambda_1$, and $\bar\Sigma'_\lambda$ the closure of Σ'_λ. Let $u^\lambda(x)=u(x^\lambda)$ and $w_\lambda(x)=u(x)-u^\lambda(x)$ for $x\in\Sigma'_\lambda$. Then we have, for any arbitrary function $b_\lambda(x)$,

$$\Delta w_\lambda(x)+b_\lambda(x)w_\lambda(x)=Q(x,b_\lambda(x)),$$

where

$$Q(x,b_\lambda(x))=f(x^\lambda,u^\lambda)-f(x,u)+b_\lambda(x)w_\lambda(x).$$

The hypothesis $(*)$ is said to be satisfied if there are two families of functions $b_\lambda(x)$ and $h^\lambda(x)$ defined in Σ'_λ, for $\lambda\in(-\infty,\lambda_1)$ such that, the following assertions holds:

$$0\le b_\lambda(x)\le c(x)|x|^{-2},$$

where $c(x)$ is independent of λ and tends to zero as $|x|$ tends to $+\infty$,

$$h^\lambda(x)\in C^1(\Sigma_\lambda\cap\Omega),$$

and satisfies:

$$\begin{cases}\Delta h^\lambda(x)\ge Q(x,b_\lambda(x)) & \text{in } \Sigma_\lambda\cap\Omega,\\ h^\lambda(x)>0 & \text{in } \Sigma_\lambda\cap\Omega,\end{cases}$$

in the distributional sense and,

$$h^\lambda(x) = 0 \quad \text{on } T_\lambda \quad \text{and} \quad h^\lambda(x) = \mathcal{O}(|x|^{-t_1}),$$

as $|x| \to +\infty$ for some constant $t_1 > 0$,

$$h^\lambda(x) + \epsilon < w_\lambda(x),$$

in a neighborhood of $\partial\Omega$, where ϵ is a positive constant independent of x.

$$\begin{cases} h^\lambda(x) \quad \text{and} \quad \nabla_x h^\lambda \text{ are continuous with respect to both variables,} \\ x \text{ and } \lambda, \text{ and for any compact set of } \Omega, \quad w_\lambda(x) > h^\lambda(x), \\ \text{holds when } -\lambda \text{ is sufficiently large.} \end{cases}$$

We have the following lemma:

Lemma 2.1. *Let u be a solution of (E''). Suppose that $u(x) \geq C > 0$ in a neighborhood of $\partial\Omega$ and $u(x) = \mathcal{O}(|x|^{-t_2})$ at $+\infty$ for some positive t_2. Assume there exist $b_\lambda(x)$ and $h^\lambda(x)$ such that the hypothesis $(*)$ is satisfied for $\lambda \leq \lambda_1$. Then $w_\lambda(x) > 0$ in Σ'_λ, and $\langle \nabla u, e\rangle > 0$ on T_λ for $\lambda \in (-\infty, \lambda_1)$.*

For the proof see Chen and Lin, [11].

Remark 2.1. If we know that $w_\lambda - h^\lambda > 0$ for some $\lambda = \lambda_0 < \lambda_1$ and b_λ and h^λ satisfy the hypothesis $(*)$ for $\lambda_0 \leq \lambda \leq \lambda_1$, then the conclusion of the Lemma 2.1 holds.

3 Proof of the result

Proof of the Theorem 1.2. When $n = 4:$ to prove the theorem, we argue by contradiction and we assume that the $(\sup)^\beta \times \inf$ tends to infinity.

Step 1: blow-up analysis. We want to prove that:

$$\tilde{R}^2 \Big(\sup_{B_{\tilde{R}}(0)} u \Big)^\beta \times \inf_{B_{3\tilde{R}}(0)} u \leq c = c(a, b, A, B, \beta).$$

If it is not the case, we have:

$$\tilde{R}_i^2 \Big(\sup_{B_{\tilde{R}_i}(0)} u_i \Big)^\beta \times \inf_{B_{3\tilde{R}_i}(0)} u_i = i^6 \to +\infty,$$

for positive solutions $u_i > 0$ of Eq. (E) and $\tilde{R}_i \to 0$. Thus,

$$\frac{1}{i} \tilde{R}_i \Big(\sup_{B_{\tilde{R}_i}(0)} u_i \Big)^{(1+\beta)/2} \to +\infty.$$

Let a_i such that:

$$u_i(a_i) = \max_{B_{\tilde{R}_i}(0)} u_i.$$

We set

$$s_i(x) = (\tilde{R}_i - |x - a_i|)^{2/(1+\beta)} u_i(x),$$

we have

$$s_i(\bar{x}_i) = \max_{B_{\tilde{R}_i}(a_i)} s_i \geq s_i(a_i) = \tilde{R}_i^{2/(1+\beta)} \sup_{B_{\tilde{R}_i}(0)} u_i \to +\infty,$$

we set

$$R_i = \frac{1}{2}(\tilde{R}_i - |\bar{x}_i - a_i|).$$

We have, for $|x - \bar{x}_i| \leq \frac{R_i}{i}$,

$$\tilde{R}_i - |x - a_i| \geq \tilde{R}_i - |\bar{x}_i - a_i| - |x - \bar{x}_i| \geq 2R_i - R_i = R_i.$$

Thus

$$\frac{u_i(x)}{u_i(\bar{x}_i)} \leq \beta_i \leq 2^{2/(1+\beta)}$$

with $\beta_i \to 1$. We set

$$M_i = u_i(\bar{x}_i), \qquad v_i^*(y) = \frac{u_i(\bar{x}_i + M_i^{-1} y)}{u_i(\bar{x}_i)},$$

$$|y| \leq \frac{1}{i} R_i M_i^{(1+\beta)/2} = 2\tilde{L}_i, \qquad \frac{1}{i^2} \tilde{R}_i^2 M_i^{\beta} \times \inf_{B_{3\tilde{R}_i}(0)} u_i \to +\infty.$$

Without loss of generality, we can assume $\bar{x}_i$ a local maximum of u_i.

By the elliptic estimates, v_i^* converge on each compact set of $\mathbb{R}^4$ to a function $U_0^* > 0$ solution of :

$$\begin{cases} -\Delta U_0^* = V(0) U_0^{*3} & \text{in } \mathbb{R}^4, \\ U_0^*(0) = 1 = \max_{\mathbb{R}^4} U_0^*. \end{cases}$$

For simplicity, we assume that $0 < V(0) = n(n-2) = 8$. By a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$U_0^*(y) = (1 + |y|^2)^{-1}.$$

We set

$$v_i(y) = v_i^*(y + e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$

$$U_0(y) = U_0^*(y + e).$$

We want to prove that:

$$\min_{\{0\le|y|\le r\}} v_i^* \le (1+\epsilon)U_0^*(r)$$

for $0\le r\le L_i$, with $L_i=\frac{1}{2i}\tilde{R}_iM_i^{(1+\beta)/2}$.

We assume that it is not true, then, there is a sequence of number $r_i\in(0,L_i)$ and $\epsilon>0$, such that:

$$\min_{\{0\le|y|\le r_i\}} v_i^* \ge (1+\epsilon)U_0^*(r_i).$$

We have:

$$r_i\to+\infty.$$

Thus, we have for $r_i\in(0,L_i)$:

$$\min_{\{0\le|y|\le r_i\}} v_i \ge (1+\epsilon)U_0(r_i).$$

Also, we can find a sequence of number $l_i\to+\infty$ such that:

$$l_i{}^{n-2}||v_i^*-U_0||_{C^2(B_{l_i}(0))}\to 0.$$

Thus,

$$\min_{\{0\le|y|\le l_i\}} v_i \ge (1-\epsilon/2)U_0(l_i).$$

Step 2 : The Kelvin transform and the Moving-plane method.

1. a linear equation perturbed by a term, and, the auxiliary function $D_i=|\nabla V_i(x_i)|\to 0$. We have the same estimate as in the paper of Chen-Lin. We argue by contradiction. We consider $r_i\in(0,L_i)$ where L_i is the number of the blow-up analysis

$$L_i=\frac{1}{2i}\tilde{R}_iM_i^{(1+\beta)/2}.$$

We use the assumption that the sup times inf is not bounded to prove $w_\lambda>h_\lambda$ in $\Sigma_\lambda=\{y,y_1>\lambda\}$, and on the boundary.

The function v_i has a local maximum near $-e$ and converge to $U_0(y)=U_0^*(y+e)$ on each compact set of $\mathbb{R}^5$. U_0 has a maximum at $-e$. We argue by contradiction and we suppose that:

$$D_i=|\nabla V_i(x_i)|\not\to 0.$$

Then, without loss of generality we can assume that:

$$\nabla V_i(x_i)\to e=(1,0,\cdots,0).$$

Where x_i is :

$$x_i=\bar{x}_i+M_i^{-1}e,$$

with $\bar{x}_i$ is the local maximum in the blow-up analysis.

As in the paper of Chen-Lin, we use the Kelvin transform twice and we set (we take the same notations):

$$I_\delta(y) = \frac{\frac{|y|}{|y|^2} - \delta e}{\left(\left|\frac{|y|}{|y|^2} - \delta e\right|\right)^2}, \qquad v_i^\delta(y) = \frac{v_i(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}},$$

$$V_\delta(y) = V_i(x_i + M_i^{-1} I_\delta(y)), \qquad U_\delta(y) = \frac{U_0(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}}.$$

Then, U_δ has a local maximum near $e_\delta \to -e$ when $\delta \to 0$. The function v_i^δ has a local maximum near $-e$.

We want to prove by the application of the maximum principle and the Hopf lemma that near e_δ we have not a local maximum, which is a contradiction.

We set on

$$\Sigma'_\lambda = \Sigma_\lambda - \left\{y, \left|y - \frac{e}{\delta}\right| \leq \frac{c_0}{r_i}\right\} \simeq \Sigma_\lambda - \{y, |I_\delta(y)| \geq r_i\},$$
$$h_\lambda(y) = -\int_{\Sigma_\lambda} G_\lambda(y,\eta) Q_\lambda(\eta) d\eta,$$

with

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^3.$$

And, by the same estimates, we have for $\eta \in A_1 = \{\eta, |\eta| \leq R = \epsilon_0/\delta\}$,

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-1}(\eta_1 - \lambda) + o(1) M_i^{-1} |\eta^\lambda|,$$

and we have for $\eta \in A_2 = \Sigma_\lambda - A_1$:

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq C M_i^{-1}(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|).$$

And we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$w_\lambda(y) = v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1+|y|)^n}$$

for $y_1 > \lambda_0$.

Because, by the maximum principle:

$$\min_{\{l_i \leq |I_\delta(y)| \leq r_i\}} v_i = \min\left\{ \min_{\{|I_\delta(y)| = l_i\}}, v_i \min_{\{|I_\delta(y)| = r_i\}} v_i \right\} \geq (1-\epsilon) U_\delta\left(\frac{e}{\delta}\right)$$
$$\geq (1 + c_1\delta - \epsilon) U_\delta\left(\left(\frac{e}{\delta}\right)^\lambda\right) \geq (1 + c_1\delta - 2\epsilon) v_i^\delta(y^\lambda),$$

and for $|I_\delta(y)| \leq l_i$ we use the C^2 convergence of v_i^δ to U_δ.

Thus,

$$w_\lambda(y) > 2\epsilon > 0,$$

By the same estimates as in Chen-Lin paper (we apply the Lemma 2.1 of the second section), and by our hypothesis on v_i, we have:

$$0 < h_\lambda(y) = \mathcal{O}(1) M_i^{-2/3}(y_1 - \lambda)(1 + |y|)^{-n} < 2\epsilon < w_\lambda(y),$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_2 r_i^{-1}$.

For

$$|\nabla V_i(x_i)|^{1/\beta}[u_i(x_i)] \leq C.$$

Here, also, we argue by contradiction. We use the same computation as in Chen-Lin paper, we choose the same h_λ, except the fact that here we use the computation with $M_i^{-(1+\beta)}$ in front the regular part of h_λ. Here also, we consider $r_i \in (0, L_i)$, where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} \tilde{R}_i M_i^{(1+\beta)/2}.$$

We argue by contradiction and we suppose that:

$$M_i^\beta D_i \to +\infty.$$

Then, without loss of generality we can assume that:

$$\frac{\nabla V_i(x_i)}{|\nabla V_i(x_i)|} \to e = (1, 0, \cdots, 0).$$

We use the Kelvin transform twice and around this point and around 0.

$$h_\lambda(y) = \epsilon r_i^{-2} G_\lambda\Big(y, \frac{e}{\delta}\Big) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta$$

with

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^3.$$

And, by the same estimates, we have for $\eta \in A_1$

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-1} D_i((\eta_1 - \lambda) + o(1)|\eta^\lambda|),$$

and, we have for $\eta \in A_2, |I_\delta(\eta)| \leq c_2 M_i D_i^{1/\beta}$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq C M_i^{-1} D_i(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),$$

and for $M_i D_i^{1/\beta} \leq |I_\delta(\eta)| \leq r_i$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq M_i^{-1} D_i |I_\delta(\eta)| + M_i^{-(1+\beta)} |I_\delta(\eta)|^{(1+\beta)}.$$

By the same estimates, we have for $|I_\delta(\eta)| \leq r_i$ or $|y - e/\delta| \geq c_3 r_i^{-1}$:

$$\begin{aligned} h_\lambda(y) \simeq & \epsilon r_i^{-2} G_\lambda\Big(y, \frac{e}{\delta}\Big) + c_4 M_i^{-1} D_i \frac{(y_1-\lambda)}{|y|^n} + o(1) M_i^{-1} D_i \frac{(y_1-\lambda)}{|y|^n} \\ & + o(1) M_i^{-(1+\beta)} G_\lambda\Big(y, \frac{e}{\delta}\Big) \end{aligned}$$

with $c_4 > 0$.

And, we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1+|y|)^n}$$

for $y_1 > \lambda_0$.

By the same estimates as in Chen-Lin paper (we apply the Lemma 2.1 of the second section), and by our hypothesis on v_i, we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y),$$

also, we have the same etimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_5 r_i^{-1}$

2. Conclusions : a linear equation perturbed by a term, and, the auxiliary function. Here also, we use the computations of Chen-Lin, and, we take the same auxiliary function h_λ (which correspond to this step), except the fact that here in front the regular part of this function we have $M_i^{-(1+\beta)}$. Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} \tilde{R}_i M_i^{(1+\beta)/2}.$$

We set

$$v_i(z) = v_i^*(z+e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$

$$U_0(z) = U_0^*(z+e).$$

We have, for $|y| \geq L_i'^{-1}$, $L_i' = \frac{1}{2} \tilde{R}_i M_i^{(1+\beta)/2}$,

$$\begin{aligned} & \bar{v}_i(y) = \frac{1}{|y|^{n-2}} v_i\left(\frac{y}{|y|^2}\right), \\ & \Big| V_i\Big(\bar{x}_i + M_i^{-1} \frac{y}{|y|^2}\Big) - V_i(\bar{x}_i) \Big| \leq M_i^{-(1+\beta)} (1 + |y|^{-1}), \\ & x_i = \bar{x}_i + M_i^{-1} e. \end{aligned}$$

Then, for simplicity, we can assume that, $\bar{v}_i$ has a local maximum near $e^* = (-\frac{1}{2}, 0, \cdots, 0)$. Also, we have:

$$\Big|V_i\Big(x_i + M_i^{-1}\frac{y}{|y|^2}\Big) - V_i(x_i + M_i^{-1}\frac{y^\lambda}{|y^\lambda|^2}\Big)\Big| \leq M_i^{-(1+\beta)}(1+|y|^{-1}),$$
$$h_\lambda(y) \simeq \epsilon r_i^{-2} G_\lambda(y,0) - \int_{\Sigma'_\lambda} G_\lambda(y,\eta) Q_\lambda(\eta) d\eta,$$

where, $\Sigma'_\lambda = \Sigma_\lambda - \{\eta, |\eta| \leq r_i^{-1}\}$, and

$$Q_\lambda(\eta) = \Big(V_i\Big(x_i + M_i^{-1}\frac{y}{|y|^2}\Big) - V_i\Big(x_i + M_i^{-1}\frac{y^\lambda}{|y^\lambda|^2}\Big)\Big)(v_i(y^\lambda))^3,$$

we have by the same computations that:

$$\int_{\Sigma'_\lambda} G_\lambda(y,\eta) Q_\lambda(\eta) d\eta \leq C M_i^{-(1+\beta)} G_\lambda(y,0) \ll \epsilon r_i^{-2} G_\lambda(y,0).$$

By the same estimates as in Chen-Lin paper (we apply the Lemma 2.1 of the second section), and by our hypothesis on v_i, we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y),$$

also, we have the same estimate on the boundary, $|y| = \frac{1}{r_i}$. □

Proof of the Theorem 1.4. When $n = 5$: to prove the theorem, we argue by contradiction and we assume that the $(\sup)^{1/3} \times \inf$ tends to infinity.

Step 1: blow-up analysis. We want to prove that:

$$\tilde{R}^3\Big(\sup_{B_{\tilde{R}}(0)} u\Big)^{1/3} \times \inf_{B_{3\tilde{R}}(0)} u \leq c = c(a,b,A,B).$$

If it is not the case, we have:

$$\tilde{R}_i^3\Big(\sup_{B_{\tilde{R}_i}(0)} u_i\Big)^{1/3} \times \inf_{B_{3\tilde{R}_i}(0)} u_i = i^6 \to +\infty.$$

For positive solutions $u_i > 0$ of Eq. (E) and $\tilde{R}_i \to 0$. Thus,

$$\frac{1}{i}\tilde{R}_i\Big(\sup_{B_{\tilde{R}_i}(0)} u_i\Big)^{2/3} \to +\infty.$$

Let a_i such that:

$$u_i(a_i) = \max_{B_{\tilde{R}_i}(0)} u_i.$$

We set

$$s_i(x) = (\tilde{R}_i - |x - a_i|)^{9/4} u_i(x),$$

we have

$$s_i(\bar{x}_i) = \max_{B_{\tilde{R}_i}(a_i)} s_i \geq s_i(a_i) = \tilde{R}_i^{9/4} \sup_{B_{\tilde{R}_i}(0)} u_i \to +\infty,$$

we set

$$R_i = \frac{1}{2}(\tilde{R}_i - |\bar{x}_i - a_i|).$$

We have, for $|x - \bar{x}_i| \leq \frac{R_i}{i}$,

$$\tilde{R}_i - |x - a_i| \geq \tilde{R}_i - |\bar{x}_i - a_i| - |x - \bar{x}_i| \geq 2R_i - R_i = R_i.$$

Thus

$$\frac{u_i(x)}{u_i(\bar{x}_i)} \leq \beta_i \leq 2^{9/4}$$

with $\beta_i \to 1$. We set

$$M_i = u_i(\bar{x}_i), \quad v_i^*(y) = \frac{u_i(\bar{x}_i + M_i^{-2/3} y)}{u_i(\bar{x}_i)}, \quad |y| \leq \frac{1}{i} R_i M_i^{4/9} = 2\tilde{L}_i.$$

And

$$\frac{1}{i^3} \tilde{R}_i^3 M_i^{1/3} \times \inf_{B_{3\tilde{R}_i}(0)} u_i \to +\infty.$$

Without loss of generality one can assume $\bar{x}_i$ a local maximum of u_i.

By the elliptic estimates, v_i^* converge on each compact set of $\mathbb{R}^5$ to a function $U_0^* > 0$ solution of :

$$\begin{cases} -\Delta U_0^* = V(0) U_0^{*7/3} & \text{in } \mathbb{R}^5, \\ U_0^*(0) = 1 = \max_{\mathbb{R}^5} U_0^*. \end{cases}$$

For simplicity, we assume that $0 < V(0) = n(n-2) = 15$. By a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$U_0^*(y) = (1 + |y|^2)^{-3/2}.$$

We set

$$v_i(y) = v_i^*(y + e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$

$$U_0(y) = U_0^*(y + e).$$

We want to prove that:

$$\min_{\{0 \leq |y| \leq r\}} v_i^* \leq (1 + \epsilon) U_0^*(r)$$

for $0 \leq r \leq L_i$, with $L_i = \frac{1}{2i}\tilde{R}_i M_i^{4/9}$.

We assume that it is not true, then, there is a sequence of number $r_i \in (0, L_i)$ and $\epsilon > 0$, such that:

$$\min_{\{0 \leq |y| \leq r_i\}} v_i^* \geq (1+\epsilon) U_0^*(r_i).$$

We have:

$$r_i \to +\infty.$$

Thus, we have for $r_i \in (0, L_i)$

$$\min_{\{0 \leq |y| \leq r_i\}} v_i \geq (1+\epsilon) U_0(r_i).$$

Also, we can find a sequence of number $l_i \to +\infty$ such that:

$$l_i^{n-2} ||v_i^* - U_0||_{C^2(B_{l_i}(0))} \to 0.$$

Thus,

$$\min_{\{0 \leq |y| \leq l_i\}} v_i \geq (1-\epsilon/2) U_0(l_i).$$

Step 2 : The Kelvin transform and the Moving-plane method.

1. A linear equation perturbed by a term, and the auxiliary function: $D_i = |\nabla V_i(x_i)| \to 0$.

We have the same estimate as in the paper of Chen-Lin. We argue by contradiction. We consider $r_i \in (0, L_i)$, where L_i is the number of the blow-up analysis

$$L_i = \frac{1}{2i} \tilde{R}_i M_i^{4/9}.$$

We use the assumption that the sup times inf is not bounded to prove $w_\lambda > h_\lambda$ in $\Sigma_\lambda = \{y, y_1 > \lambda\}$, and on the boundary.

The function v_i has a local maximum near $-e$ and converge to $U_0(y) = U_0^*(y+e)$ on each compact set of $\mathbb{R}^5$. U_0 has a maximum at $-e$.

We argue by contradiction and we suppose that:

$$D_i = |\nabla V_i(x_i)| \not\to 0.$$

Then, without loss of generality we can assume that:

$$\nabla V_i(x_i) \to e = (1, 0, \cdots, 0).$$

Where x_i is :

$$x_i = \bar{x}_i + M_i^{-2/3} e,$$

with $\bar{x}_i$ is the local maximum in the blow-up analysis.

As in the paper of Chen-Lin, we use the Kelvin transform twice and we set (we take the same notations):

$$I_\delta(y)=\frac{\frac{|y|}{|y|^2}-\delta e}{\big(\big|\frac{|y|}{|y|^2}-\delta e\big|\big)^2},\qquad v_i^\delta(y)=\frac{v_i(I_\delta(y))}{|y|^{n-2}|y-e/\delta|^{n-2}},$$
$$V_\delta(y)=V_i(x_i+M_i^{-2/3}I_\delta(y)),\qquad U_\delta(y)=\frac{U_0(I_\delta(y))}{|y|^{n-2}|y-e/\delta|^{n-2}}.$$

Then, U_δ has a local maximum near $e_\delta\to -e$ when $\delta\to 0$. The function v_i^δ has a local maximum near $-e$.

We want to prove by the application of the maximum principle and the Hopf lemma that near e_δ we have not a local maximum, which is a contradiction.

We set on

$$\Sigma'_\lambda=\Sigma_\lambda-\Big\{y,\Big|y-\frac{e}{\delta}\Big|\le\frac{c_0}{r_i}\Big\}\simeq\Sigma_\lambda-\{y,|I_\delta(y)|\ge r_i\},$$
$$h_\lambda(y)=-\int_{\Sigma_\lambda}G_\lambda(y,\eta)Q_\lambda(\eta)d\eta,$$

with

$$Q_\lambda(\eta)=(V_\delta(\eta)-V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^{(n+2)/(n-2)}.$$

And, by the same estimates, we have for $\eta\in A_1=\{\eta,|\eta|\le R=\epsilon_0/\delta\}$,

$$V_\delta(\eta)-V_\delta(\eta^\lambda)\ge M_i^{-2/3}(\eta_1-\lambda)+o(1)M_i^{-2/3}|\eta^\lambda|,$$

and we have for $\eta\in A_2=\Sigma_\lambda-A_1$:

$$|V_\delta(\eta)-V_\delta(\eta^\lambda)|\le CM_i^{-2/3}(|I_\delta(\eta)|+|I_\delta(\eta^\lambda)|).$$

And we have for some $\lambda_0\le -2$ and $C_0>0$:

$$v_i^\delta(y)-v_i^\delta(y^{\lambda_0})\ge C_0\frac{y_1-\lambda_0}{(1+|y|)^n}$$

for $y_1>\lambda_0$.

By the same estimates, and by our hypothesis on v_i, we have, for $c_1>0$:

$$0<h_\lambda(y)<2\epsilon<w_\lambda(y),$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)|=r_i$ or $|y-e/\delta|=c_2r_i^{-1}$.

For $|\nabla V_i(x_i)|[u_i(x_i)]^{2/3}\le C$. Here, also, we argue by contradiction. We use the same computation as in Chen-Lin paper, we take $\alpha=2$ and we choose the same h_λ, except the fact that here we use the computation with $M_i^{-4/3}$ in front the regular part of h_λ.

Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis

$$L_i = \frac{1}{2i} \tilde{R}_i M_i^{4/9}.$$

We argue by contradiction and we suppose that:

$$M_i^{2/3} D_i \to +\infty.$$

Then, without loss of generality we can assume that:

$$\frac{\nabla V_i(x_i)}{|\nabla V_i(x_i)|} \to e = (1, 0, \cdots, 0).$$

We use the Kelvin transform twice and around this point and around 0

$$h_\lambda(y) = \epsilon r_i^{-3} G_\lambda\Big(y, \frac{e}{\delta}\Big) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta$$

with

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^{(n+2)/(n-2)}.$$

And by the same estimates, we have for $\eta \in A_1$

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-2/3} D_i((\eta - \lambda) + o(1)|\eta^\lambda|),$$

and, we have for $\eta \in A_2$, $|I_\delta(\eta)| \leq c_2 M_i^{2/3} D_i$

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq C M_i^{-2/3} D_i(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),$$

and for $M_i^{2/3} D_i \leq |I_\delta(\eta)| \leq r_i$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq M_i^{-2/3} D_i |I_\delta(\eta)| + M_i^{-4/3} |I_\delta(\eta)|^2.$$

By the same estimates, we have for $|I_\delta(\eta)| \leq r_i$ or $|y - e/\delta| \geq c_3 r_i^{-1}$:

$$\begin{aligned} h_\lambda(y) \simeq & \epsilon r_i^{-3} G_\lambda\Big(y, \frac{e}{\delta}\Big) + c_4 M_i^{-2/3} D_i \frac{(y_1 - \lambda)}{|y|^n} \\ & + o(1) M_i^{-2/3} D_i \frac{(y_1 - \lambda)}{|y|^n} + o(1) M_i^{-4/3} G_\lambda\Big(y, \frac{e}{\delta}\Big) \end{aligned}$$

with $c_4 > 0$.

And, we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n}$$

for $y_1 > \lambda_0$.

By the same estimates as in Chen-Lin paper (we apply the Lemma 2.1 of the second section), and by our hypothesis on v_i, we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y),$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_5 r_i^{-1}$:

Step 2. conclusions : a linear equation perturbed by a term, and, the auxiliary function. Here also, we use the computations of Chen-Lin, and, we take the same auxiliary function h_λ (which correspond to this step), except the fact that here in front the regular part of this function we have $M_i^{-4/3}$.

Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis

$$L_i = \frac{1}{2i}\tilde{R}_i M_i^{4/9}.$$

We set

$$v_i(z) = v_i^*(z+e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$

$$U_0(z) = U_0^*(z+e).$$

We have, for $|y| \geq L_i'^{-1}$, $L_i' = \frac{1}{2}\tilde{R}_i M_i^{4/9}$,

$$\begin{aligned}
&\bar{v}_i(y) = \frac{1}{|y|^{n-2}} v_i\left(\frac{y}{|y|^2}\right),\\
&\left|V_i\Big(\bar{x}_i + M_i^{-2/3}\frac{y}{|y|^2}\Big) - V_i(\bar{x}_i)\right| \leq M_i^{-4/3}(1+|y|^{-2}),\\
&x_i = \bar{x}_i + M_i^{-2/3} e.
\end{aligned}$$

Then, for simplicity, we can assume that, $\bar{v}_i$ has a local maximum near $e^* = (-1/2, 0, \cdots, 0)$.

Also, we have:

$$\begin{aligned}
&\left|V_i\Big(x_i + M_i^{-2/3}\frac{y}{|y|^2}\Big) - V_i\Big(x_i + M_i^{-2/3}\frac{y^\lambda}{|y^\lambda|^2}\Big)\right| \leq M_i^{-4/3}(1+|y|^{-2}),\\
&h_\lambda(y) \simeq \epsilon r_i^{-3} G_\lambda(y,0) - \int_{\Sigma'_\lambda} G_\lambda(y,\eta) Q_\lambda(\eta) d\eta,
\end{aligned}$$

where, $\Sigma'_\lambda = \Sigma_\lambda - \{\eta, |\eta| \leq r_i^{-1}\}$, and

$$Q_\lambda(\eta) = \left(V_i\Big(x_i + M_i^{-2/3}\frac{y}{|y|^2}\Big) - V_i\Big(x_i + M_i^{-2/3}\frac{y^\lambda}{|y^\lambda|^2}\Big)\right)(v_i(y^\lambda))^{\frac{n+2}{n-2}},$$

we have by the same computations that:

$$\int_{\Sigma'_\lambda} G_\lambda(y,\eta)Q_\lambda(\eta)d\eta \leq CM_i^{-4/3}G_\lambda(y,0) \ll \epsilon r_i^{-3}G_\lambda(y,0).$$

By the same estimates as in Chen-Lin paper (we apply the Lemma 2.1 of the second section), and by our hypothesis on v_i, we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y),$$

also, we have the same estimate on the boundary, $|\eta| = \frac{1}{r_i}$. □

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