Analysis in Theory and Applications
Anal. Theory Appl., Vol. **38**, No. 2 (2022), pp. 121-127

DOI: 10.4208/ata.OA-0023
June 2022

Singular Solutions to Monge-Ampère Equation

Luis A. Caffarelli[1] and Yu Yuan[2,*]

[1] *Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA*
[2] *University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195, USA*

Received 27 July 2021; Accepted (in revised version) 25 November 2021

Abstract. We construct merely Lipschitz and $C^{1,\alpha}$ with rational $\alpha \in (0, 1-2/n]$ viscosity solutions to the Monge-Ampère equation with constant right hand side.

Key Words: Monge-Ampère equation.

AMS Subject Classifications: 35J96

1 Introduction

In this note, we construct (convex) Lipschitz and $C^{1,\alpha}$ viscosity solutions to the Monge-Ampère equation with constant right hand side via Cauchy-Kovalevskaya, after integerizing fractional powers in the corresponding equation for those singular profiles from [8] and [3,5].

Theorem 1.1. *There exists a merely Lipschitz viscosity solution to* $\det D^2u = 1$ *in* $B_1 \subset \mathbb{R}^n$ *for* $n \geq 3$. *There also exist merely* $C^{1,\alpha-1}$ *with rational* $\alpha = \frac{q}{p} \in (1, 2-\frac{2}{n}]$ *viscosity solutions to* $\det D^2u = 1$ *in* $B_1 \subset \mathbb{R}^n$ *for* $n \geq 3$.

These $C^{1,\alpha}$ solutions to the Monge-Ampère equation $\det D^2u = 1$ illustrate a regularity wall phenomena: merely $C^{1,\alpha}$ with rational $\alpha \in (0, 1-2/n]$ solutions can never become better. This is in contrast with the regularity theory for Monge-Ampère equations [9] and [4]: once solutions are $C^{1,(1-2/n)+}$, they self-improve to smoothness.

Note that our singular solutions via Cauchy-Kovalevskaya to the Monge-Ampère equation $\det D^2u = 1$ are singular precisely along a segment of one axis, where the convex solutions are linear, or zero, to be precise. If one tries to produce higher dimensional

*Corresponding author. *Email addresses:* `caffarel@math.utexas.edu` (L. Caffarelli), `yuan@math.washington.edu` (Y. Yuan)

subspace singular set, where the dimension S must be less than $n/2$ by the theorem in [3], a good start is the Pogorelov type profile there,

$$u(x) = |x'|^{2-2S/n} f(|x''|)$$

with $x = (x'_1, \cdots, x'_{n-S}, x''_1, \cdots, x''_S)$. The profile with

$$f(|x''|) = 1 + |x''|^2$$

satisfies the Monge-Ampère with the right hand side being a polynomial of $|x''|^2$, positive near the origin. The ODE for $f(|x''|)$ with singular term $f'(|x''|)/|x''|$ corresponding to $\det D^2u = 1$ can be solved by the method in [2] and [1].

Alternatively, relying on the existence of solutions to the Dirichlet problem for Monge-Ampère equations, with S dimensional singular set profile $|x'|^{2-2S/n}(1+|x''|^2)$ as boundary value in a small ball, one obtains the following

Proposition 1.1. *There exist local merely $C^{1,1-S/n}$ viscosity solutions to* $\det D^2u = 1$ *in $\mathbb{R}^n$ for $n \geq 3$ such that singular set of the solutions is the S dimensional set*

$$\mathbb{S} = \{(x', x'') : |x'| = 0\}$$

in a small ball for $1 \leq S < n/2$.

Let us sketch a proof for this proposition. Case $S = 1$ is also noted in the above. The Lipschitz limit of a family of (convex) smooth solutions to $\det D^2u = 1$ with smooth boundary value approximations of subsolution

$$u_- = \gamma |x'|^{2-2S/n} \left(1 + |x''|^2\right)$$

for $\gamma = (1-2S/n)^{-1/n}$ on the boundary of a small ball is our viscosity solution. The convex solution $u(x)$ vanishes in subspace x'' with $|x'| = 0$, because it is between the convex combinations of zero boundary value and the subsolution u_- there. Surely $u(x)$ is singular in the S dimensional subspace $(0', x'')$.

We show that u is regular everywhere else. By [4,5], the other possible singular set of u outside $\mathbb{S}$, must contain a line segment, where u is linear. This singular segment intersects the boundary of the small ball or the set $\mathbb{S}$. The barrier argument in [9] and [4,5] shows the two ends of the segment cannot be both on the boundary of the small ball, where u is smooth. The only other scenario that the segment has one end on $\mathbb{S}$, and the other end on the boundary of the small ball is not possible either. This is because the linear function, the restriction of u on the segment, equaling 0 and $u_- > 0$ respectively on the two ends, cannot be less than the supersolution

$$u^+ = 2\gamma |x'|^{2-2S/n}$$

with sublinear growth near $|x'| = 0$.

Note that the solution u is trapped between the supersolution u^+ and the subsolution

$$u_- = \gamma \left|x'\right|^{2-2S/n} \left(1 + \left|x''\right|^2\right).$$

We see that u is exactly $C^{1,1-2S/n}$. This finishes the sketch of the proof for Proposition 1.1.

In closing, we remark that Mooney [6] recently showed that the $n-1$ dimensional Hausdorff measure of the singular set of every subsolution to $\det D^2 u = 1$ is zero, and the collection of S-dimensional affine singular sets, on each of which the subsolution is linear, also has zero $n-S$ dimensional Hausdorff measure. In particular, the affine dimension S is less than $n/2$. This provides a new proof for the theorem in [3]. The no better than $C^{1,\beta}$ with $\beta \in [0,1/3]$ solutions in [6,7] have almost $n-1$ and exactly $n-1$ respectively Hausdorff dimensional singular sets, where each of the solutions is not a single linear function.

2 Proof of Theorem 1.1

Proof. **Lipschitz case.** We seek for solutions in the Lipschitz profile from [5]

$$u\left(x', x_n\right) = \rho + \rho^{n/2} f\left(\rho, x_n\right)$$

with

$$\rho = \left|x'\right| = \left|\left(x'_1, \cdots, x'_{n-1}\right)\right|.$$

The upper half Hessian $D^2 u$ is

$$\begin{bmatrix} \frac{1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho}{\rho} & & & & \\ & \cdots & & & \\ & & \frac{1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho}{\rho} & & \\ & & & \begin{array}{c} \frac{n}{2}\left(\frac{n}{2}-1\right)\rho^{\frac{n}{2}-2}f \\ +2\frac{n}{2}\rho^{\frac{n}{2}-1}f_\rho + \rho^{\frac{n}{2}}f_{\rho\rho} \end{array} & \begin{array}{c} \frac{n}{2}\rho^{\frac{n}{2}-1}f_n \\ +\rho^{\frac{n}{2}}f_{\rho n} \end{array} \\ & & & & \rho^{\frac{n}{2}}f_{nn} \end{bmatrix},$$

and its determinant

$$\begin{aligned} \det D^2 u &= \left[\frac{1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho}{\rho}\right]^{n-2} \left\{ \begin{array}{c} \left[\begin{array}{c} \frac{n}{2}\left(\frac{n}{2}-1\right)\rho^{\frac{n}{2}-2}f \\ +2\frac{n}{2}\rho^{\frac{n}{2}-1}f_\rho + \rho^{\frac{n}{2}}f_{\rho\rho} \end{array}\right] \rho^{\frac{n}{2}} f_{nn} \\ -\left(\frac{n}{2}\rho^{\frac{n}{2}-1}f_n + \rho^{\frac{n}{2}}f_{\rho n}\right)^2 \end{array} \right\} \\ &= \left(1+\frac{n}{2}\rho^{\frac{n}{2}-1}f+\rho^{\frac{n}{2}}f_\rho\right)^{n-2} \left\{ \begin{array}{c} \left[\frac{n}{2}\left(\frac{n}{2}-1\right)f + n\rho f_\rho + \rho^2 f_{\rho\rho}\right] f_{nn} \\ -\left(\frac{n}{2}f_n + \rho f_{\rho n}\right)^2 \end{array} \right\}. \end{aligned}$$

We make the following change of variable to move to an analytic equation.

Set

$$s=\rho^{1/2} \quad \text{and} \quad h\left(s,x_n\right)=f\left(s^2,x_n\right),$$

then

$$\partial_s=2s\partial_\rho \quad \text{or} \quad \partial_\rho=\frac{1}{2s}\partial_s, \quad \text{and} \quad \partial_\rho^2=\frac{1}{4}\left(\frac{-1}{s^3}\partial_s+\frac{1}{s^2}\partial_s^2\right).$$

The determinant becomes

$$\det D^2u=\begin{pmatrix} 1+\frac{n}{2}s^{n-2}h \\ +\frac{1}{2}s^{n-1}h_s \end{pmatrix}^{n-2}\left\{\begin{array}{c} \left[\begin{array}{c} \frac{n}{2}\left(\frac{n}{2}-1\right)h \\ +\frac{n}{2}sh_s+\frac{1}{4}\left(-sh_s+s^2h_{ss}\right) \end{array}\right]h_{nn} \\ -\left(\frac{n}{2}h_n+\frac{1}{2}sh_{sn}\right)^2 \end{array}\right\}.$$

Now we solve the reduced Monge-Ampère equation

$$\begin{cases} h_{nn}=\dfrac{\left(1+\frac{n}{2}s^{n-2}h+\frac{1}{2}s^{n-1}f_s\right)^{2-n}+\left(\frac{n}{2}h_n+\frac{1}{2}sh_{sn}\right)^2}{\frac{n}{2}\left(\frac{n}{2}-1\right)h+\frac{(2n-1)}{4}sh_s+\frac{1}{4}s^2h_{ss}}, \\ h_n\left(s,0\right)=0, \\ h\left(s,0\right)=1. \end{cases}$$

Cauchy-Kovalevskaya gives the analytic solution in $B_{r_1}(0)\subset\mathbb{R}^2$

$$h\left(s,x_n\right)=1+\frac{2}{n\left(n-2\right)}x_n^2+\cdots.$$

Thus we have a Lipschitz solution to $\det D^2u=1$

$$u\left(x\right)=\left|x'\right|+\left|x'\right|^{\frac{n}{2}}h\left(\left|x'\right|^{\frac{1}{2}},x_n\right)$$

in $B_1(0)\subset\mathbb{R}^n$ by scaling.

Lastly, let us check our u is a viscosity to $\det D^2u=1$. For any convex quadratic $Q(x)$ touching $u(x)$ from above, observe that the touching point can never be a singular Lipschitz point of $u(x)$, and in turn, $\det D^2Q\ge 1$ at the smooth touching point of $u(x)$. On the lower side, for any quadratic $Q(x)$ touching $u(x)$ from below, when the touching point is at $x'=0$, observe that convex $Q(x)$ must vanish along $x'=0$ as $u(x',x_n)$ vanishes, then $\det D^2Q=0<1$; when the touching point is at $x'\neq 0$, immediately $\det D^2Q\le 1$ as $u(x)$ is smooth nearby.

$C^{1,\frac{q}{p}-1}$ **case.** We search for solutions in the form

$$u\left(x',x_n\right)=\rho^\alpha f\left(\rho,x_n\right)=\rho^\alpha\left[1+\rho^\beta g\left(\rho,x_n\right)\right] \quad \text{with} \quad \beta=2\left(n-1\right)-n\alpha.$$

The upper half Hessian D^2u is

$$\begin{bmatrix} \frac{\alpha\rho^{\alpha-1}f+\rho^{\alpha}f_\rho}{\rho} & & & & \\ & \cdots & & & \\ & & \frac{\alpha\rho^{\alpha-1}f+\rho^{\alpha}f_\rho}{\rho} & & \\ & & & \begin{matrix}\alpha\left(\alpha-1\right)\rho^{\alpha-2}f\\ +2\alpha\rho^{\alpha-1}f_\rho+\rho^{\alpha}f_{\rho\rho}\end{matrix} & \alpha\rho^{\alpha-1}f_n+\rho^{\alpha}f_{\rho n} \\ & & & & \rho^{\alpha}f_{nn}\end{bmatrix},$$

and its determinant

$$\begin{aligned}\det D^2u &= \left[\rho^{\alpha-2}\left(\alpha f+\rho f_\rho\right)\right]^{n-2}\rho^{2\alpha-2}\left\{\begin{matrix}\left[\alpha\left(\alpha-1\right)f+2\alpha\rho f_\rho+\rho^2 f_{\rho\rho}\right]f_{nn}\\ -\left(\alpha f_n+\rho f_{\rho n}\right)^2\end{matrix}\right\}\\ &=\left(\alpha f+\rho f_\rho\right)^{n-2}\left\{\begin{matrix}\left[\alpha\left(\alpha-1\right)f+2\alpha\rho f_\rho+\rho^2 f_{\rho\rho}\right]f_{nn}\\ -\left(\alpha f_n+\rho f_{\rho n}\right)^2\end{matrix}\right\}\rho^{n\alpha-2(n-1)}.\end{aligned}$$

Note that

$$\begin{aligned}&f=1+\rho^{\beta}g\left(\rho,x_n\right),\\ &f_\rho=\beta\rho^{\beta-1}g+\rho^{\beta}g_\rho,\\ &f_{\rho\rho}=\beta\left(\beta-1\right)\rho^{\beta-2}g+2\beta\rho^{\beta-1}g_\rho+\rho^{\beta}g_{\rho\rho},\end{aligned}$$

then

$$\begin{aligned}\det D^2u &= \left[\alpha+\alpha\rho^{\beta}g+\beta\rho^{\beta}g+\rho^{\beta+1}g_\rho\right]^{n-2}\rho^{n\alpha-2(n-1)}\\ &\quad\left\{\begin{matrix}\left[\begin{matrix}\alpha\left(\alpha-1\right)\left(1+\rho^{\beta}g\right)+2\alpha\left(\beta\rho^{\beta}g+\rho^{\beta+1}g_\rho\right)\\ +\beta\left(\beta-1\right)\rho^{\beta}g+2\beta\rho^{\beta+1}g_\rho+\rho^{\beta+2}g_{\rho\rho}\end{matrix}\right]\rho^{\beta}g_{nn}\\ -\left(\alpha\rho^{\beta}g_n+\beta\rho^{\beta}g_n+\rho^{\beta+1}g_{\rho_n}\right)^2\end{matrix}\right\}\\ &=\left[\alpha+\left(\alpha+\beta\right)\rho^{\beta}g+\rho^{\beta+1}g_\rho\right]^{n-2}\rho^{n\alpha-2(n-1)+\beta}\\ &\quad\left\{\begin{matrix}\left[\begin{matrix}\alpha\left(\alpha-1\right)+\left(\alpha+\beta\right)\left(\alpha+\beta-1\right)\rho^{\beta}g\\ +2\left(\alpha+\beta\right)\rho^{\beta+1}g_\rho+\rho^{\beta+2}g_{\rho\rho}\end{matrix}\right]g_{nn}\\ -\rho^{\beta}\left[\left(\alpha+\beta\right)g_n+\rho g_{\rho_n}\right]^2\end{matrix}\right\}\\ &=\left[\alpha+\left(\alpha+\beta\right)\rho^{\beta}g+\rho^{\beta+1}g_\rho\right]^{n-2}\\ &\quad\left\{\begin{matrix}\left[\begin{matrix}\alpha\left(\alpha-1\right)+\left(\alpha+\beta\right)\left(\alpha+\beta-1\right)\rho^{\beta}g\\ +2\left(\alpha+\beta\right)\rho^{\beta+1}g_\rho+\rho^{\beta+2}g_{\rho\rho}\end{matrix}\right]g_{nn}\\ -\rho^{\beta}\left[\left(\alpha+\beta\right)g_n+\rho g_{\rho_n}\right]^2\end{matrix}\right\},\end{aligned}$$

where we used $n\alpha-2\left(n-1\right)+\beta=0$.

We make the following change of variable for $\alpha = q/p$ to move to an analytic equation.

Set

$$s = \rho^{1/p} \quad \text{and} \quad h(s, x_n) = g(s^p, x_n),$$

then

$$\partial_s = ps^{p-1}\partial_\rho \quad \text{or} \quad \partial_\rho = \frac{1}{ps^{p-1}}\partial_s, \quad \text{and} \quad \partial_\rho^2 = \frac{1}{p^2}\left(\frac{1-p}{s^{2p-1}}\partial_s + \frac{1}{s^{2p-2}}\partial_s^2\right).$$

The determinant becomes

$$\det D^2 u = \left[\alpha + (\alpha+\beta)s^{p\beta}h + s^{p\beta}\frac{1}{p}sh_s\right]^{n-2}$$
$$\left\{\begin{array}{c} \left\{\begin{array}{c} \alpha(\alpha-1) + (\alpha+\beta)(\alpha+\beta-1)s^{p\beta}h \\ +2(\alpha+\beta)s^{p\beta}\frac{1}{p}sh_s + s^{p\beta}\frac{1}{p^2}\left[(1-p)sh_s + s^2 h_{ss}\right] \end{array}\right\} h_{nn} \\ -s^{p\beta}\left[(\alpha+\beta)h_n + \frac{1}{p}sh_{sn}\right]^2 \end{array}\right\}.$$

Now we solve the reduced Monge-Ampère equation

$$\begin{cases} h_{nn} = \dfrac{\left[\alpha + (\alpha+\beta)s^{p\beta}h + s^{p\beta}\frac{1}{p}sh_s\right]^{2-n} + s^{p\beta}\left[(\alpha+\beta)h_n + \frac{1}{p}sh_{sn}\right]^2}{\alpha(\alpha-1) + (\alpha+\beta)(\alpha+\beta-1)s^{p\beta}h + 2(\alpha+\beta)s^{p\beta}\frac{1}{p}sh_s + s^{p\beta}\frac{1}{p^2}\left[(1-p)sh_s + s^2h_{ss}\right]}, \\ h_n(s,0) = 0, \\ h(s,0) = 1, \end{cases}$$

where integer $p\beta = 2p(n-1) - nq \geq 0$. Cauchy-Kovalevskaya gives the analytic solution in $B_{r_{q/p}}(0) \subset \mathbb{R}^2$

$$h(s, x_n) = 1 + \frac{1}{2}\frac{\alpha^{2-n}}{\alpha(\alpha-1)}x_n^2 + \cdots.$$

Thus we have a $C^{1,\frac{q}{p}-1}$ solution to $\det D^2 u = 1$

$$u(x', x_n) = |x'|^{\frac{q}{p}}\left[1 + |x'|^{2(n-1)-n\frac{q}{p}} h\left(|x'|^{\frac{1}{p}}, x_n\right)\right]$$

in $B_1(0) \subset \mathbb{R}^n$ by scaling.

Exactly as in the Lipschitz case, we verify that our $u(x)$ is a viscosity solution to the Monge-Ampère equation $\det D^2 u = 1$. □

Acknowledgements

Both authors are partially supported by NSF grants. This work was done while both authors were visiting IAS, Princeton in the spring of 2009. We are grateful to the referee for the optimal improvement of Proposition 1.1.

References

[1] D. B. Bacani, J. E. C. Lope, and H. Tahara, On the unique solvability of nonlinear Fuchsian partial differential equations, Tokyo J. Math., 41 (2018), 225–239.

[2] M. S. Baouendi, and C. Goulaouic, Singular nonlinear Cauchy problems, J. Differential Equations, 22 (1976), 268–291.

[3] L. A. Caffarelli, A note on the degeneracy of convex solutions to Monge Ampère equation, Commun. Partial Differential Equations, 18 (1993), 1213–1217.

[4] L. A. Caffarelli, A priori estimates and the geometry of the Monge Ampère equation, Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), 5–63, IAS/Park City Math. Ser., 2, Amer. Math. Soc., Providence, RI, 1996.

[5] L. A. Caffarelli, Monge Ampère equation div-curl theorems in Lagrangian coordinates compression and rotation, NSF-CBMS Lecture, Florida Atlantic University, 1997.

[6] C. Mooney, Partial regularity for singular solutions to the Monge-Ampère equation, Commun. Pure Appl. Math., 68 (2015), 1066–1084.

[7] C. Mooney, $W^{2,1}$ estimate for singular solutions to the Monge-Ampère equation, Ann. Sc. Norm. Super. Pisa Cl. Sci., 14 (2015), 1283–1303.

[8] A. V. Pogorelov, The Minkowski Multidimensional Problem, Translated from the Russian by Vladimir Oliker. Introduction by Louis Nirenberg. Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D. C.; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1978.

[9] J. I. E. Urbas, Regularity of generalized solutions of Monge-Ampère equations, Math. Z., 197 (1988), 365–393.