Fixed Point Theorems for Weakly Contractive Mappings in Ordered Metric Spaces with an Application

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Abstract. In this paper, we prove fixed point theorem for weakly contractive mappings using locally $T$-transitivity of binary relation and presenting an analogous version of Harjani and Sadarangani theorem involving more general relation theoretic metrical notions. Our fixed point results under universal relation reduces to Harjani and Sadarangani [Nonlinear Anal., 71 (2009), 3403–3410] fixed point theorems. In this way we also generalize some of the recent fixed point theorems for weak contraction in the existing literature.

Key Words: $R$-continuity, locally $T$-transitive binary relation, weakly contractive map.

AMS Subject Classifications: 47H10, 54H25

1 Introduction

In 1997, Alber and Guerre-Delabrere [5] introduced the concept of weak contraction in Hilbert spaces and proved the corresponding fixed result. Later Rhoades [17] showed that the result is also valid in complete metric spaces. Further results in this direction were obtained by Dutta and Choudhury in [9]. Results on weakly contractive mappings in ordered metric spaces, together with applications to differential equations, were given by Harjani and Sadarangani in [10]. On the other hand, extension of classical Banach contraction principle [6] to the field of (partially) ordered metric spaces can be traced back to Turinici [21, 24] which was later undertaken by several researchers [1, 4, 8, 11, 12, 14, 15, 17, 19, 20, 22, 23].

In all these extensions, we must notify the one due to Alam and Imdad [2], where some relation theoretic analogues of standard metric notions (such as continuity and completeness) were used. Further, Alam and Imdad [3] extended the above setting by

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using $T$-transitivity of the ambient relation $\mathcal{R}$, and obtained an extension of the Boyd-Wong Fixed Point Theorem [7] to such spaces. It is our aim in this paper is to give an extension of these results to weakly contractive maps.

## 2 Preliminaries

Throughout this paper, $\mathcal{R}$ stands for a non-empty binary relation $N_0$ stands for the set of whole numbers (i.e., $N_0 = N \cup \{0\}$) and $\mathcal{R}$ for the set of all real numbers.

In this section, we present some basic definitions, propositions and relevant relation-theoretic variants of metrical notions such as completeness and continuity.

**Definition 2.1** ([16]). A binary relation on a non-empty set $X$ is defined as a subset of $X \times X$, which will be denoted by $\mathcal{R}$. We say that $x$ relates to $y$ under $\mathcal{R}$ iff $\langle x, y \rangle \in \mathcal{R}$.

**Definition 2.2.** Let $\mathcal{R}$ be a binary relation on a nonempty set $X$ and $x, y \in X$. We say that $x$ and $y$ are $\mathcal{R}$-comparative if either $\langle x, y \rangle \in \mathcal{R}$ or $\langle y, x \rangle \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

**Definition 2.3** ([16]). Let $X$ be a nonempty set and $\mathcal{R}$ a binary relation on $X$.

1. The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(y, x) : \langle x, y \rangle \in \mathcal{R}\}$.
2. The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s =: \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on $X$ containing $\mathcal{R}$.

**Proposition 2.1** ([2]). For a binary relation $\mathcal{R}$ defined on a nonempty set $X$, $\langle x, y \rangle \in \mathcal{R} \iff [x, y] \in \mathcal{R}$.

**Definition 2.4** ([2]). Let $X$ be a nonempty set and $\mathcal{R}$ a binary relation on $X$. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if $\langle x_n, x_{n+1} \rangle \in \mathcal{R} \forall n \in N_0$.

**Definition 2.5** ([2]). Let $X$ be a nonempty set and $T$ a self-mapping on $X$. A binary relation $\mathcal{R}$ on $X$ is called $T$-closed if for any $x, y \in X$, $\langle x, y \rangle \in \mathcal{R} \Rightarrow \langle Tx, Ty \rangle \in \mathcal{R}$.

**Proposition 2.2** ([2]). Let $X$ be a nonempty set, $\mathcal{R}$ a binary relation on $X$ and $T$ a self-mapping on $X$. If $\mathcal{R}$ is $T$-closed, then $\mathcal{R}^s$ is also $T$-closed.

**Proposition 2.3** ([2]). Let $X$ be a nonempty set, $\mathcal{R}$ a binary relation on $X$ and $T$ a self-mapping on $X$. If $\mathcal{R}$ is $T$-closed, then, for all $n \in N_0$, $\mathcal{R}$ is also $T^n$-closed, where $T^n$ denotes $n$th iterate of $T$.

**Definition 2.6** ([3]). Let $(X, d)$ be a metric space and $\mathcal{R}$ a binary relation on $X$. We say that $(X, d)$ is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in $X$ converges.

Clearly, every complete metric space is $\mathcal{R}$-complete, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with usual completeness.
Definition 2.7 ([3]). Let \((X,d)\) be a metric space, \(\mathcal{R}\) a binary relation on \(X\) and \(x \in X\). A self-mapping \(T\) on \(X\) is called \(\mathcal{R}\)-continuous at \(x\) if for any \(\mathcal{R}\)-preserving sequence \(\{x_n\}\) such that \(x_n \xrightarrow{d} x\), we have \(T(x_n) \xrightarrow{d} T(x)\). Moreover, \(T\) is called \(\mathcal{R}\)-continuous if it is \(\mathcal{R}\)-continuous at each point of \(X\).

Clearly, every continuous mapping is \(\mathcal{R}\)-continuous, for any binary relation \(\mathcal{R}\). Particularly, under the universal relation the notion of \(\mathcal{R}\)-continuity coincides with usual continuity.

Definition 2.8 ([20]). Let \((X,d)\) be a metric space and \(\mathcal{R}\) a binary relation on \(X\). We say that \((X,d,\mathcal{R})\) is regular if for any \(\mathcal{R}\)-preserving sequence \(\{x_n\}\) such that \(x_n \xrightarrow{d} x\), there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) with \([x_{n_k}, x] \in \mathcal{R}\), \(\forall k \in \mathbb{N}_0\).

Definition 2.9 ([3]). Let \(X\) be a nonempty set and \(\mathcal{R}\) a binary relation on \(X\). A subset \(E\) of \(X\) is called \(\mathcal{R}\)-connected if for each pair \(x, y \in E\), there exists a path in \(\mathcal{R}\) from \(x\) to \(y\).

Definition 2.10 ([19]). Let \(X\) be a nonempty set and \(\mathcal{R}\) a binary relation on \(X\). A subset \(E\) of \(X\) is called \(\mathcal{R}\)-directed if for each pair \(x, y \in E\), there exists \(z \in X\) such that \((x,z) \in \mathcal{R}\) and \((y,z) \in \mathcal{R}\).

Definition 2.11 ([3]). Let \(X\) be a nonempty set and \(T\) a self-mapping on \(X\). A binary relation \(\mathcal{R}\) on \(X\) is called \(T\)-transitive if for any \(x, y, z \in X\), \((Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}\).

Inspired by Turinici [22, 23], Alam and Imdad [3] introduced the following notion by localizing the notion of transitivity.

Definition 2.12 ([3]). A binary relation \(\mathcal{R}\) on a nonempty set \(X\) is called locally transitive if for each (effectively) \(\mathcal{R}\)-preserving sequence \(\{x_n\} \subset X\) (with range \(E =: \{x_n : n \in \mathbb{N}_0\}\)), the binary relation \(\mathcal{R}|_E\) is transitive. Where \(\mathcal{R}|_E\) denote the restriction of \(\mathcal{R}\) to \(E\).

Thus, the notions of \(T\)-transitivity and locally transitivity both are relatively weaker than the notion of transitivity but they are independent of each others. In order to make them compatible, Alam and Imdad introduce the following notion of transitivity.

Definition 2.13 ([3]). Let \(X\) be a nonempty set and \(T\) a self-mapping on \(X\). A binary relation \(\mathcal{R}\) on \(X\) is called locally \(T\)-transitive if for each (effectively) \(\mathcal{R}\)-preserving sequence \(\{x_n\} \subset T(X)\) (with range \(E =: \{x_n : n \in \mathbb{N}_0\}\)), the binary relation \(\mathcal{R}|_E\) is transitive.

Definition 2.14 ([3]). Let \(X\) be a nonempty set and \(\mathcal{R}\) a binary relation on. For \(x, y \in X\), a path of length \(k\) (where \(k\) is a natural number) in \(\mathcal{R}\) from \(x\) to \(y\) is a finite sequence \(\{z_0, z_1, \cdots, z_k\} \subset X\) satisfying the following conditions:

(i) \(z_0 = x\) and \(z_k = y\),

(ii) \((z_i, z_{i+1}) \in \mathcal{R}\) for each \(i\) \(0 \leq i \leq k - 1\).

Notice that a path of length \(k\) involves \(k + 1\) elements of \(X\), although they are not necessarily distinct.
Proposition 2.4. If \((X, d)\) is a metric space, \(\mathcal{R}\) is a binary relation on \(X\), \(T\) is a self-mapping on \(X\) and \(\psi : [0, \infty) \to [0, \infty)\) is a continuous and non-decreasing then the following contractive conditions are equivalent:

(I) \(d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))\), \(\forall x, y \in X\) with \((x, y) \in \mathcal{R}\),

(II) \(d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))\), \(\forall x, y \in X\) with \([x, y] \in \mathcal{R}\).

We skip the proof of above proposition as it is similar to that of Proposition 2.3 [2].

Given a binary relation \(\mathcal{R}\) and a self-mapping \(T\) on a nonempty set \(X\), we use the following notations:

(i) \(F(T) = \) the set of all fixed points of \(T\),

(ii) \(X(T, \mathcal{R}) = \{x \in X : (x, Tx) \in \mathcal{R}\}\).

3 Fixed point theorems

Theorem 3.1. Let \((X, d)\) be a metric space, \(\mathcal{R}\) a binary relation on \(X\) and \(T\) a self-mapping on \(X\). Suppose that the following conditions hold:

(a) \((X, d)\) is \(\mathcal{R}\)-complete,

(b) \(\mathcal{R}\) is \(T\)-closed and locally \(T\)-transitive,

(c) either \(T\) is \(\mathcal{R}\)-continuous or \((X, d, \mathcal{R})\) is regular,

(d) \(X(T, \mathcal{R})\) is non-empty,

(e) there exists \(\psi : [0, \infty) \to [0, \infty)\) is continuous and non-decreasing function such that it is positive in \((0, \infty)\), \(\psi(0) = 0\) and

\[d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))\]

for all \(x, y \in X\) such that \((x, y) \in \mathcal{R}\). Then \(T\) has a fixed point.

Proof. In view of the assumption (d), let \(x_0\) be an arbitrary element of \(X(T, \mathcal{R})\). Define the sequence \(\{x_n\}\) of Picard iterates with initial point \(x_0\), i.e.,

\[x_n = T^n(x_0), \quad \forall n \in \mathbb{N}_0.\]  \hspace{1cm} (3.1)

As \((x_0, Tx_0) \in \mathcal{R}\), using \(T\)-closedness of \(\mathcal{R}\) and Proposition 2.3, we obtain

\[(T^n x_0, T^{n+1} x_0) \in \mathcal{R},\]

so that

\[(x_n, x_{n+1}) \in \mathcal{R}, \quad \forall n \in \mathbb{N}_0.\]  \hspace{1cm} (3.2)
Thus the sequence \( \{x_n\} \) is \( \mathcal{R} \)-preserving. Applying the contractive condition (e) to (3.2), we have

\[
d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) - \psi(d(x_n, x_{n-1})), \quad \forall n \in N_0.
\]

If there exists \( n_0 \in N_0 \) such that \( d(x_{n_0}, x_{n_0} - 1) = 0 \) then \( x_{n_0} = Tx_{n_0 - 1} = x_{n_0 - 1} \) is a fixed point and the proof is finished.

In the other case, suppose that \( d(x_n, x_{n-1}) \neq 0, \forall n \in N_0 \). Then using contractive condition in view of the assumption (e) about \( \psi \)

\[
d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) - \psi(d(x_n, x_{n-1})) < d(x_n, x_{n-1}).
\]

Put \( t_n = d(x_{n+1}, x_n) \). Then we have

\[
t_n \leq t_{n-1} - \psi(t_{n-1}) < t_{n-1}. \tag{3.3}
\]

Therefore \( \{t_n\} \) is a non-negative non-increasing sequence and hence possesses a limit \( t \). From (3.3), taking limit \( n \to \infty \), we get

\[
t \leq t - \psi(t) < t,
\]

which implies that \( \psi(t) = 0 \). Thus, by our assumption about \( \psi \), \( t = 0 \).

Now we shall show that \( \{x_n\} \) is a Cauchy sequence.

Fix \( \epsilon > 0 \). As \( t_n = d(x_{n+1}, x_n) \to 0 \), there exists \( n_0 \in N_0 \) such that

\[
d(x_{n_0+1}, x_{n_0}) \leq \min \left\{ \frac{\epsilon}{2}, \frac{\psi(\epsilon)}{2} \right\}. \tag{3.4}
\]

We claim that

\[
T(B(x_{n_0}, \epsilon) \cap \{y \in X : (x_{n_0}, y) \in \mathcal{R}\}) \subset B(x_{n_0}, \epsilon).
\]

Let \( z \in B(x_{n_0}, \epsilon) \cap \{y \in X : (x_{n_0}, y) \in \mathcal{R}\} \). Then in view of the locally \( T \)-transitivity of \( \mathcal{R} \), \( (Tx_{n_0}, Tz) \in \mathcal{R} \) and the following two cases arise here:

Case 1. \( d(z, x_{n_0}) \leq \frac{\epsilon}{2} \). In this case, as \( (x_{n_0}, Tx_{n_0}), (Tx_{n_0}, Tz) \in \mathcal{R} \), we have

\[
d(Tz, x_{n_0}) \leq d(Tz, Tx_{n_0}) + d(Tx_{n_0}, x_{n_0}) = d(Tz, Tx_{n_0}) + d(x_{n_0+1}, x_{n_0}) \leq d(z, x_{n_0}) - \psi(d(z, x_{n_0}) + d(x_{n_0+1}, x_{n_0}) \leq d(z, x_{n_0}) + d(x_{n_0+1}, x_{n_0}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Case 2. \( \frac{\epsilon}{2} < d(z, x_{n_0}) \leq \epsilon \). In this case, owing to assumption (e) about \( \psi \) we have,

\[
\psi \left( \frac{\epsilon}{2} \right) \leq \psi(d(z, x_{n_0}) \leq \psi(d(z, x_{n_0}).
\]
Therefore from (3.4), we obtain
\[ d(Tz, x_{n_0}) \leq d(Tz, Tx_{n_0}) + d(Tx_{n_0}, x_{n_0}) \]
\[ = d(Tz, Tx_{n_0}) + d(x_{n_0+1}, x_{n_0}) \]
\[ \leq d(z, x_{n_0}) - \psi(d(z, x_{n_0}) + d(x_{n_0+1}, x_{n_0})) \]
\[ \leq d(z, x_{n_0}) - \psi\left(\frac{\epsilon}{2}\right) + d(x_{n_0+1}, x_{n_0}) \]
\[ \leq d(z, x_{n_0}) - \psi\left(\frac{\epsilon}{2}\right) + \psi\left(\frac{\epsilon}{2}\right) \]
\[ \leq d(z, x_{n_0}) \leq \epsilon. \]

This proves the claim.

As \( x_{n_0+1} \in B(x_{n_0}, \epsilon) \cap \{ y \in X : (x_{n_0}, y) \in R \} \), the claim gives us that
\[ Tx_{n_0+1} = x_{n_0+2} \in B(x_{n_0}, \epsilon) \cap \{ y \in X : (x_{n_0}, y) \in R \}. \]

Continuing this process it follows that \( x_n \in B(x_{n_0}, \epsilon) \) for \( n \geq n_0 \). Since \( \epsilon \) is arbitrary, \( \{ x_n \} \) is a Cauchy sequence. Thus, \( \{ x_n \} \) is an \( R \)-preserving Cauchy sequence. By \( R \)-completeness of \( X \), there exists \( z \in X \) such that \( x_n \to z \).

Finally, owing to assumption (c), i.e., \( T \) is \( R \)-continuous. As \( \{ x_n \} \) is \( R \)-preserving with \( x_n \to z \), \( R \)-continuity of \( T \) implies that \( x_{n+1} = Tx_n \to Tz \). Using the uniqueness of limit, we obtain \( Tz = z \), i.e., \( z \) is a fixed point of \( T \).

Alternately, let us assume that \( (X, d, R) \) is regular. Again as \( \{ x_n \} \) is \( R \)-preserving and
\[ x_n \to z, \]
there exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) with \( [x_{n_k}, z] \in R \forall k \in N_0 \). By using the fact that \( [x_{n_k}, z] \in R \), contractive condition (e) and Proposition 2.4, we have
\[ d(x_{n_k+1}, Tz) = d(Tx_{n_k}, Tz) \leq d(x_{n_k}, z) - \psi(d(x_{n_k}, z) \leq d(x_{n_k}, z) \]
and taking limit \( k \to \infty \), and using \( x_{n_k} \to z \), we obtain \( x_{n_k+1} \to Tz \). Owing to the uniqueness of limit, we have \( Tz = z \) so that \( z \) is a fixed point of \( T \).

Remark 3.1. Theorem 3.1 remains true if we replace the locally \( T \)-transitivity of \( R \) by any one of the following conditions (besides retaining rest of the hypotheses):

(i) \( R \) is transitive,
(ii) \( R \) is \( T \)-transitive,
(iii) \( R \) is locally transitive.
4 Uniqueness results

Theorem 4.1. In addition to the hypotheses of Theorem 3.1, suppose that the following condition holds:

(u) $T(X)$ is $\mathcal{R}^g$-connected.

Then $T$ has a unique fixed point.

Proof. Let $x$ and $y$ be two fixed points of $T$, i.e., $F(T) \neq \emptyset$ and $x, y \in F(T)$, then for all $n \in N_0$, we have

$$T^nx = x, \quad T^ny = y.$$  (4.1)

Clearly $x, y \in T(X)$. By assumption (u), there exists a path (say $z_0, z_1, \ldots, z_k$) of some finite length $k$ in $\mathcal{R}^g$ from $x$ to $y$ so that

$$z_0 = x, \quad z_k = y \quad \text{and} \quad [z_i, z_{i+1}] \in \mathcal{R} \quad \text{for each} \quad i \ (0 \leq i \leq k - 1).$$  (4.2)

As $\mathcal{R}$ is $T$-closed, using Propositions 2.2 and 2.3, we have

$$[T^nz_i, T^nz_{i+1}] \in \mathcal{R} \quad \text{for each} \quad i \ (0 \leq i \leq k - 1) \quad \text{and for each} \quad n \in N_0. \quad (4.3)$$

Now suppose that $(x, y) \in \mathcal{R}^g$ then by Proposition 2.1 either $(T^nx, T^ny) \in \mathcal{R}$ or $(T^ny, T^nx) \in \mathcal{R}$. Using $T$-closedness of $\mathcal{R}$ in view of the Proposition 2.3, we have $(T^nx, T^ny) \in \mathcal{R}$ for $n = 0, 1, \ldots$, and

$$d(x, y) = d(T^nx, T^ny) \leq d(T^{n-1}x, T^{n-1}y) - \psi(d(T^{n-1}x, T^{n-1}y)) \leq d(x, y).$$

Consequently, $\psi(d(x, y)) = 0$, which implies that $d(x, y) = 0$.

Also, if $(x, y) \notin \mathcal{R}^g$ then there exists a path of length $k > 1$ in $\mathcal{R}^g$. In light of (4.2), (4.3), we define $t_n^i = d(T^nz_i, T^nz_{i+1})$. Also, $T$-closedness of $\mathcal{R}^g$ implies that $t_n^i = d(T^nz_i, T^nz_{i+1}) \in \mathcal{R}^g$ for $i \ (0 \leq i \leq k - 1)$ and $n = 0, 1, \ldots$. Moreover, for any fix $i$ we have

$$t_n^i = d(T^nz_i, T^nz_{i+1}) \leq d(T^{n-1}z_i, T^{n-1}z_{i+1}) - \psi(d(T^{n-1}z_i, T^{n-1}z_{i+1})) \leq d(T^{n-1}z_i, T^{n-1}z_{i+1}) = t_{n-1}^i.$$

Consequently, $t_n^i = d(T^nz_i, T^nz_{i+1})$ is a non-negative non-increasing sequence and hence possess limit $t$. Taking limit $n \to \infty$ in last inequality we have $t \leq t - \psi(t) \leq t$, and hence $t = 0$ for each $i \ (0 \leq i \leq k - 1)$.

Finally, making the use of triangular inequality in view of the above conclusion, we obtain

$$d(x, y) = d(T^nz_0, T^nz_k) \leq t_n^0 + t_n^1 + \cdots + t_n^{k-1} \to 0$$

as $n \to \infty$. Hence $T$ has a unique fixed point. \qed
Remark 4.1. Theorem 4.1 remains true if we replace the condition (u) by one of the following conditions (besides retaining rest of the assumption):

(u₁) \( R|_{T(X)} \) is complete,

(u₂) \( T(X) \) is \( R^s \)-directed.

Example 4.1. Let \( X = [0, 6] \) and \( d \) be the standard metric \( d(x, y) = |x - y| \) so that \((X, d)\) is a complete metric space. Define a binary relation \( R = \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (1, 3)\} \) on \( X \) and the mapping \( T : X \to X \) by

\[
T(x) = \begin{cases} 
0, & x \in [0, 1], \\
1, & x \in (1, 2], \\
2, & x \in (2, 3]. 
\end{cases}
\]

Clearly \( R \) is \( T \)-closed and locally \( T \)-transitive.

Note that \( R \) is not \( T \)-transitive as \((1, 0), (0, 2) \in R \) but \((1, 2) \notin R \). Since \( T \) is not \( R \)-continuous, we claim that \( R \) is \( d \)-self closed.

Let \( \{x_n\} \) be any \( R \)-preserving sequence with \( x_n \xrightarrow{d} x \), so that \((x_n, x_{n+1}) \in R, \forall k \in N_0 \) which implies that \( x_n \subset \{0, 1, 2\} \). As \( \{0, 1, 2\} \) is closed, we can take a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} = x \) for all \( k \in N_0 \), which implies that \([x_{n_k}, x] \in R, \forall k \in N_0 \). This proves our claim.

Now, define a continuous and non-decreasing function \( \psi : [0, \infty) \to [0, \infty) \) by \( \psi(t) = \frac{t}{2} \). By routine calculation, we can easily verify assumption (e) of the Theorem 3.1. Thus all the conditions (a)-(e) of Theorem 3.1 are satisfied and \( T \) has a fixed point in \( X \) i.e., \( x = 0 \).

5 An application

In this section, as an application we present an example, where Theorems 3.1 and 4.1 can be applied. We prove an existence and uniqueness of solution for the following first order periodic boundary value problem:

\[
x'(t) = f(t, x(t)), \quad t \in I = [0, T], \quad x(0) = x(T),
\]

where \( T > 0 \) and \( f : I \times R \to R \) is a continuous function.

Let \( C(I) \) denote the space of all continuous functions defined on \( I \). Now, we recall the following definitions.

Definition 5.1 ([14]). A function \( \alpha \in C^1(I) \) is called a lower solution of (5.1), if

\[
\alpha'(t) \leq f(t, \alpha(t)), \quad t \in I, \quad x(0) \leq x(T).
\]

Definition 5.2 ([14]). A function \( \alpha \in C^1(I) \) is called a upper solution of (5.1), if

\[
\alpha'(t) \geq f(t, \alpha(t)), \quad t \in I, \quad x(0) \geq x(T).
\]
Theorem 5.1. In addition to the problem (5.1), suppose that there exist $\lambda > 0$ such that for all $x, y \in R$ with $x \leq y$

$$0 \leq f(t, y) + \lambda y - [f(t, x) + \lambda x] \leq \lambda \ln(y - x + 1).$$  \hspace{1cm} (5.2)

Then the existence of a lower solution or an upper solution of problem (5.1) ensures the existence and uniqueness of a solution of problem (5.1).

Proof. Problem (5.1) can be rewritten as

$$x'(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t), \quad t \in I = [0, T],$$
$$x(0) = x(T).$$

This problem is equivalent to the integral equation

$$x(t) = \int_0^T G(t, s) [f(s, x(s)) + \lambda x(s)] ds,$$

where

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s < t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t < s \leq T. \end{cases}$$

Define a mapping $A : C(I) \to C(I)$ by

$$(Ax)(t) = \int_0^T G(t, s) [f(s, x(s)) + \lambda x(s)] ds,$$

and a binary relation

$$R = \{(x, y) \in C(I) \times C(I) : x(t) \leq y(t), \forall t \in I\}.$$  

(i) Note that $C(I)$ equipped with the sup-metric, i.e.,

$$d(x, y) = \sup |x(t) - y(t)| \quad \text{for} \quad t \in I \quad \text{and} \quad x, y \in C(I),$$

is complete metric space and hence $(C(I), d)$ is $R$-complete.

(ii) Choose an $R$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} z$. Then for all $t \in I$, we get

$$x_0(t) \leq x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq x_{n+1}(t) \leq \cdots$$

and convergence to $x(t)$ implies that $x_0(t) \leq z(t)$ for all $t \in I$, $n \in N_0$, which amounts to saying that $[x_n, z] \in R$, for all $n \in N_0$. Hence, $R$ is $d$-self-closed.
(iii) For any \((x, y) \in \mathcal{R}\), i.e., \(x(t) \leq y(t)\) then by (5.2), we get
\[
f(t, x(t)) + \lambda x(t) \leq f(t, y(t)) + \lambda y(t), \quad \forall t \in I,
\]
and \(G(t, s) > 0\) for \((t, s) \in I \times I\), we have
\[
(\mathcal{A}x)(t) = \int_0^T G(t, s)[f(s, x(s)) + \lambda x(s)]ds
\leq \int_0^T G(t, s)[f(s, y(s)) + \lambda y(s)]ds
= (\mathcal{A}y)(t), \quad \forall t \in I,
\]
which implies that \((\mathcal{A}x, \mathcal{A}y) \in \mathcal{R}\), i.e., \(\mathcal{R}\) is \(\mathcal{A}\)-closed.

(iv) For all \((x, y) \in \mathcal{R}\),
\[
d(\mathcal{A}x, \mathcal{A}y) = \sup |(\mathcal{A}x)(t) - (\mathcal{A}y)(t)|
= \sup((\mathcal{A}y)(t) - (\mathcal{A}x)(t)), \quad \forall t \in I,
\]
Now following the lines of the proof of Theorem 7 in [10] we obtain
\[
d(\mathcal{A}x, \mathcal{A}y) = d(x, y) - \psi(d(x, y)),
\]
and \(a(t) \leq (\mathcal{A}a)(t)\). This implies that \((a, \mathcal{A}a) \in \mathcal{R}\), i.e., \(X(\mathcal{A}, \mathcal{R})\) is non-empty. Hence, all the conditions of Theorem 3.1 are satisfied consequently \(\mathcal{A}\) has a fixed point.

(v) Finally following the proof of our earlier Theorem 4.1, \(\mathcal{A}\) has a unique fixed point, which is, in fact, a unique solution of the problem (5.1).

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References


