Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 1 (2023), pp. 1-15

DOI: 10.4208/ata.OA-2019-0023
March 2023

Multiple Integral Inequalities for Schur Convex Functions on Symmetric and Convex Bodies

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Received 18 July 2019; Accepted (in revised version) 31 January 2021

Abstract. In this paper, by making use of Divergence theorem for multiple integrals, we establish some integral inequalities for Schur convex functions defined on bodies $B \subset \mathbb{R}^n$ that are symmetric, convex and have nonempty interiors. Examples for three dimensional balls are also provided.

Key Words: Schur convex functions, multiple integral inequalities.

AMS Subject Classifications: 26D15

1 Introduction

For any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $x_{[1]} \geq \cdots \geq x_{[n]}$ denote the components of x in decreasing order, and let $x_\downarrow = \left(x_{[1]}, \cdots, x_{[n]}\right)$ denote the decreasing rearrangement of x. For $x, y \in \mathbb{R}^n$, $x \prec y$ if, by definition,

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k = 1, \cdots, n-1, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \end{cases}$$

When $x \prec y$, x is said to be majorized by y (y majorizes x). This notation and terminology was introduced by Hardy, Littlewood and Pólya in 1934.

Functions that preserve the ordering of majorization are said to be Schur-convex. Perhaps Schur-increasing would be more appropriate, but the term Schur-convex is by now well entrenched in the literature, [5, p. 80].

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A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if

$$x \prec y \quad \text{on } \mathcal{A} \Rightarrow \phi(x) \leq \phi(y). \tag{1.1}$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on A. If $A = \mathbb{R}^n$, then ϕ is simply said to be Schur-convex or strictly Schur-convex.

For fundamental properties of Schur convexity see the monograph [5] and the references therein. For some recent results, see [2–4] and [6–8].

The following result is known in the literature as Schur-Ostrowski theorem [5, p. 84]:

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an open interval and let $\phi : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ϕ to be Schur-convex on I^n are*

$$\phi \text{ is symmetric on } I^n \tag{1.2}$$

and for all $i \neq j$, with $i, j \in \{1, \cdots, n\}$,

$$(z_i - z_j)\left[\frac{\partial\phi(z)}{\partial x_i} - \frac{\partial\phi(z)}{\partial x_j}\right] \geq 0 \quad \text{for all } z \in I^n, \tag{1.3}$$

where $\frac{\partial\phi}{\partial x_k}$ denotes the partial derivative of ϕ with respect to its k-th argument.

With the aid of (1.2), condition (1.3) can be replaced by the condition

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \quad \text{for all } z \in I^n. \tag{1.4}$$

This simplified condition is sometimes more convenient to verify.

The above condition is not sufficiently general for all applications because the domain of ϕ may not be a Cartesian product.

Let $\mathcal{A} \subset \mathbb{R}^n$ be a set with the following properties:

(i) $\mathcal{A}$ is symmetric in the sense that $x \in \mathcal{A} \Rightarrow x\Pi \in \mathcal{A}$ for all permutations Π;

(ii) $\mathcal{A}$ is convex and has a nonempty interior.

We have the following result, [5, p. 85].

Theorem 1.2. *If ϕ is continuously differentiable on the interior of $\mathcal{A}$ and continuous on $\mathcal{A}$, then necessary and sufficient conditions for ϕ to be Schur-convex on $\mathcal{A}$ are*

$$\phi \text{ is symmetric on } \mathcal{A} \tag{1.5}$$

and

$$(z_1 - z_2)\left[\frac{\partial\phi(z)}{\partial x_1} - \frac{\partial\phi(z)}{\partial x_2}\right] \geq 0 \quad \text{for all } z \in \mathcal{A}. \tag{1.6}$$

It is well known that any symmetric convex function defined on a symmetric convex set $\mathcal{A}$ is Schur convex, [5, p. 97]. If the function $\phi:\mathcal{A}\to\mathbb{R}$ is symmetric and quasi-convex, namely

$$\phi\left(\alpha u+\left(1-\alpha\right)v\right)\leq\max\left\{\phi\left(u\right),\phi\left(v\right)\right\}$$

for all $\alpha\in[0,1]$ and $u,v\in\mathcal{A}$, a symmetric convex set, then ϕ is Schur convex on $\mathcal{A}$ [5, p. 98].

In the recent paper [3] we obtained the following result for Schur convex functions defined on symmetric convex domains of $\mathbb{R}^2$.

Theorem 1.3. *Let $D\subset\mathbb{R}^2$ be symmetric, convex and has a nonempty interior. If ϕ is continuously differentiable on the interior of D, continuous and Schur convex on D and ∂D is a simple, closed counterclockwise curve in the xy-plane bounding D, then*

$$\iint_D\phi\left(x,y\right)dxdy\leq\frac{1}{2}\oint_{\partial D}\left[\left(x-y\right)\phi\left(x,y\right)dx+\left(x-y\right)\phi\left(x,y\right)dy\right]. \tag{1.7}$$

If ϕ is Schur concave on D, then the sign of inequality reverses in (1.7).

Motivated by the above results, we establish in this paper a generalization of the inequality (1.7) for the case of symmetric and convex subsets in n-dimensional space $\mathbb{R}^n$. This is done by employing an identity obtained via the well known Divergence Theorem for volume and surface integrals. An example for balls in three dimensional space are also provided.

2 Some preliminary facts

Let B be a bounded open subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B. Let $F=(F_1,\cdots,F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B\cup\partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of ∂B. Then the Divergence Theorem states, see for instance [9]:

$$\int_B divFdV=\int_{\partial B}F\cdot\mathbf{n}dA, \tag{2.1}$$

where

$$divF=\nabla\cdot F=\sum_{k=1}^n\frac{\partial F_i}{\partial x_i},$$

dV is the element of volume in $\mathbb{R}^n$ and dA is the element of surface area on ∂B.

If $\mathbf{n}=(\mathbf{n}_1,\cdots,\mathbf{n}_n)$, $x=(x_1,\cdots,x_n)\in B$ and use the notation dx for dV we can write (2.1) more explicitly as

$$\sum_{k=1}^n\int_B\frac{\partial F_k\left(x\right)}{\partial x_k}dx=\sum_{k=1}^n\int_{\partial B}F_k\left(x\right)\mathbf{n}_k\left(x\right)dA. \tag{2.2}$$

By taking the real and imaginary part, we can extend the above equality for complex valued functions F_k, $k \in \{1, \cdots, n\}$ defined on B.

If $n = 2$, the normal is obtained by rotating the tangent vector through $90°$ (in the correct direction so that it points out). The quantity tds can be written (dx_1, dx_2) along the surface, so that

$$\mathbf{n}dA := \mathbf{n}ds = (dx_2, -dx_1).$$

Here t is the tangent vector along the boundary curve and ds is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

$$\begin{aligned}&\int_B \frac{\partial F_1(x_1,x_2)}{\partial x_1} dx_1 dx_2 + \int_B \frac{\partial F_2(x_1,x_2)}{\partial x_2} dx_1 dx_2 \\ &= \int_{\partial B} F_1(x_1,x_2)\, dx_2 - \int_{\partial B} F_2(x_1,x_2)\, dx_1,\end{aligned} \tag{2.3}$$

which is Green's theorem in plane.

If $n = 3$ and if ∂B is described as a level-set of a function of 3 variables i.e., $\partial B = \{x_1, x_2, x_3 \in \mathbb{R}^3 | G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $gradG$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2)$, $(x_1, x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function g on D and B corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 < g(x_1, x_2)\right\}.$$

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2}}, \quad dA = \left(1 + g_{x_1}^2 + g_{x_2}^2\right)^{1/2} dx_1 dx_2$$

and

$$\mathbf{n}dA = (-g_{x_1}, -g_{x_2}, 1)\, dx_1 dx_2.$$

From (2.2) we get

$$\begin{aligned}&\int_B \left(\frac{\partial F_1(x_1,x_2,x_3)}{\partial x_1} + \frac{\partial F_2(x_1,x_2,x_3)}{\partial x_2} + \frac{\partial F_3(x_1,x_2,x_3)}{\partial x_3}\right) dx_1 dx_2 dx_3 \\ &= -\int_D F_1(x_1, x_2, g(x_1,x_2))\, g_{x_1}(x_1,x_2)\, dx_1 dx_2 \\ &\quad - \int_D F_1(x_1, x_2, g(x_1,x_2))\, g_{x_2}(x_1,x_2) dx_1 dx_2 \\ &\quad + \int_D F_3(x_1, x_2, g(x_1,x_2))\, dx_1 dx_2,\end{aligned} \tag{2.4}$$

which is the Gauss-Ostrogradsky theorem in space.

Following Apostol [1], we can also consider a surface described by the vector equation

$$r(u,v) = x_1(u,v)\, \overrightarrow{i} + x_2(u,v)\, \overrightarrow{j} + x_3(u,v)\, \overrightarrow{k}, \tag{2.5}$$

where $(u,v) \in [a,b] \times [c,d]$.

If x_1, x_2, x_3 are differentiable on $[a,b] \times [c,d]$ we consider the two vectors

$$\begin{aligned}\frac{\partial r}{\partial u} &= \frac{\partial x_1}{\partial u}\overrightarrow{i} + \frac{\partial x_2}{\partial u}\overrightarrow{j} + \frac{\partial x_3}{\partial u}\overrightarrow{k},\\ \frac{\partial r}{\partial v} &= \frac{\partial x_1}{\partial v}\overrightarrow{i} + \frac{\partial x_2}{\partial v}\overrightarrow{j} + \frac{\partial x_3}{\partial v}\overrightarrow{k}.\end{aligned}$$

The cross product of these two vectors $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ will be referred to as the fundamental vector product of the representation r. Its components can be expressed as Jacobian determinants. In fact, we have [1, p. 420]

$$\begin{aligned}\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} &= \begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix} \overrightarrow{i} + \begin{vmatrix} \frac{\partial x_3}{\partial u} & \frac{\partial x_1}{\partial u} \\ \frac{\partial x_3}{\partial v} & \frac{\partial x_1}{\partial v} \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix} \overrightarrow{k} \\ &= \frac{\partial (x_2, x_3)}{\partial (u,v)}\overrightarrow{i} + \frac{\partial (x_3, x_1)}{\partial (u,v)}\overrightarrow{j} + \frac{\partial (x_1, x_2)}{\partial (u,v)}\overrightarrow{k}. \end{aligned} \tag{2.6}$$

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued function r defined on the box $T = [a,b] \times [c,d]$. The area of ∂B denoted $A_{\partial B}$ is defined by the double integral [1, pp. 424-425]

$$\begin{aligned} A_{\partial B} &= \int_a^b \int_c^d \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du dv \\ &= \int_a^b \int_c^d \sqrt{\left(\frac{\partial (x_2, x_3)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_3, x_1)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_1, x_2)}{\partial (u,v)}\right)^2} du dv. \end{aligned} \tag{2.7}$$

We define surface integrals in terms of a parametric representation for the surface. One can prove that under certain general conditions the value of the integral is independent of the representation.

Let $\partial B = r(T)$ be a parametric surface described by a vector-valued differentiable function r defined on the box $T = [a,b] \times [c,d]$ and let $f : \partial B \to \mathbb{C}$ defined and bounded on ∂B. The surface integral of f over ∂B is defined by [1, p. 430]

$$\begin{aligned} \iint_{\partial B} f dA &= \int_a^b \int_c^d f(x_1, x_2, x_3) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du dv \\ &= \int_a^b \int_c^d f(x_1(u,v), x_2(u,v), x_3(u,v)) \\ &\quad \times \sqrt{\left(\frac{\partial (x_2, x_3)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_3, x_1)}{\partial (u,v)}\right)^2 + \left(\frac{\partial (x_1, x_2)}{\partial (u,v)}\right)^2} du dv. \end{aligned} \tag{2.8}$$

If $\partial B = r(T)$ is a parametric surface, the fundamental vector product $N = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is normal to ∂B at each regular point of the surface. At each such point there are two unit normals, a unit normal $\mathbf{n}_1$, which has the same direction as N, and a unit normal $\mathbf{n}_2$ which has the opposite direction. Thus

$$\mathbf{n}_1 = \frac{N}{\|N\|} \quad \text{and} \quad \mathbf{n}_2 = -\mathbf{n}_1.$$

Let $\mathbf{n}$ be one of the two normals $\mathbf{n}_1$ or $\mathbf{n}_2$. Let also F be a vector field defined on ∂B and assume that the surface integral,

$$\iint_{\partial B} (F \cdot \mathbf{n})\, dA,$$

called the flux surface integral, exists. Here $F \cdot \mathbf{n}$ is the dot or inner product.

We can write [1, p. 434]

$$\iint_{\partial B} (F \cdot \mathbf{n})\, dA = \pm \int_a^b \int_c^d F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) dudv,$$

where the sign $'' + ''$ is used if $\mathbf{n} = \mathbf{n}_1$ and the $'' - ''$ sign is used if $\mathbf{n} = \mathbf{n}_2$.

If

$$\begin{aligned} &F(x_1, x_2, x_3) = F_1(x_1, x_2, x_3) \overrightarrow{i} + F_2(x_1, x_2, x_3) \overrightarrow{j} + F_3(x_1, x_2, x_3) \overrightarrow{k}, \\ &r(u,v) = x_1(u,v) \overrightarrow{i} + x_2(u,v) \overrightarrow{j} + x_3(u,v) \overrightarrow{k}, \quad \text{where} \quad (u,v) \in [a,b] \times [c,d], \end{aligned}$$

then the flux surface integral for $\mathbf{n} = \mathbf{n}_1$ can be explicitly calculated as [1, p. 435]

$$\begin{aligned} \iint_{\partial B} (F \cdot \mathbf{n})\, dA = &\int_a^b \int_c^d F_1(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_2, x_3)}{\partial(u,v)} dudv \\ &+ \int_a^b \int_c^d F_2(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_3, x_1)}{\partial(u,v)} dudv \\ &+ \int_a^b \int_c^d F_3(x_1(u,v), x_2(u,v), x_3(u,v)) \frac{\partial(x_1, x_2)}{\partial(u,v)} dudv. \end{aligned} \tag{2.9}$$

The sum of the double integrals on the right is often written more briefly as [1, p. 435]

$$\begin{aligned} &\iint_{\partial B} F_1(x_1, x_2, x_3)\, dx_2 \wedge dx_3 + \iint_{\partial B} F_2(x_1, x_2, x_3)\, dx_3 \wedge dx_1 \\ &\quad + \iint_{\partial B} F_3(x_1, x_2, x_3)\, dx_1 \wedge dx_2. \end{aligned}$$

Let $B \subset \mathbb{R}^3$ be a solid in 3-space bounded by an orientable closed surface ∂B, and let $\mathbf{n}$ be the unit outer normal to ∂B. If F is a continuously differentiable vector field defined on B, we have the Gauss-Ostrogradsky identity

$$\iiint_B (divF)\, dV = \iint_{\partial B} (F \cdot \mathbf{n})\, dA. \tag{2.10}$$

If we express

$$F\left(x_1,x_2,x_3\right)=F_1\left(x_1,x_2,x_3\right)\overrightarrow{i}+F_2\left(x_1,x_2,x_3\right)\overrightarrow{j}+F_3\left(x_1,x_2,x_3\right)\overrightarrow{k},$$

then (2.4) can be written as

$$\begin{aligned}&\iiint_B\left(\frac{\partial F_1\left(x_1,x_2,x_3\right)}{\partial x_1}+\frac{\partial F_2\left(x_1,x_2,x_3\right)}{\partial x_2}+\frac{\partial F_3\left(x_1,x_2,x_3\right)}{\partial x_3}\right)dx_1dx_2dx_3\\=&\iint_{\partial B}F_1\left(x_1,x_2,x_3\right)dx_2\wedge dx_3+\iint_{\partial B}F_2\left(x_1,x_2,x_3\right)dx_3\wedge dx_1\\&+\iint_{\partial B}F_3\left(x_1,x_2,x_3\right)dx_1\wedge dx_2.\end{aligned}\tag{2.11}$$

3 Main results

We start with the following identity that is of interest in itself:

Lemma 3.1. *Assume that $f:D\to\mathbb{C}$ has partial derivatives on the domain $D\subset\mathbb{R}^n$, $n\geq 2$. Define for $j\neq i$*

$$\Lambda_{\partial f,D}\left(x_i,x_j\right):=\left(x_i-x_j\right)\left(\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_i}-\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_j}\right),$$

where $(x_1,\cdots,x_n)\in D$. Then we have

$$\begin{aligned}&\frac{1}{n-1}\sum_{k=1}^n\frac{\partial}{\partial x_k}\left(\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)f\left(x_1,\cdots,x_n\right)\right)\\=&f\left(x_1,\cdots,x_n\right)+\frac{1}{n(n-1)}\sum_{1\leq i<j\leq n}\Lambda_{\partial f,D}\left(x_i,x_j\right).\end{aligned}\tag{3.1}$$

Proof. For $j\neq i$ we have

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(\left(x_i-x_j\right)f\left(x_1,\cdots,x_n\right)\right)&=f\left(x_1,\cdots,x_n\right)+\left(x_i-x_j\right)\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_i},\\\frac{\partial}{\partial x_j}\left(\left(x_i-x_j\right)f\left(x_1,\cdots,x_n\right)\right)&=-f\left(x_1,\cdots,x_n\right)+\left(x_i-x_j\right)\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_j},\end{aligned}$$

which gives

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left(\left(x_i-x_j\right)f\left(x_1,\cdots,x_n\right)\right)-\frac{\partial}{\partial x_j}\left(\left(x_i-x_j\right)f\left(x_1,\cdots,x_n\right)\right)\\=&2f\left(x_1,\cdots,x_n\right)+\left(x_i-x_j\right)\left(\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_i}-\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_j}\right)\end{aligned}$$

for $j \neq i$.

If we take the sum over $i, j \in \{1, \cdots, n\}$ with $j \neq i$ we get

$$\begin{aligned}
&\sum_{i,j=1,j\neq i}^{n} \left[\frac{\partial}{\partial x_i} \left(\left(x_i - x_j \right) f\left(x_1, \cdots, x_n \right) \right) - \frac{\partial}{\partial x_j} \left(\left(x_i - x_j \right) f\left(x_1, \cdots, x_n \right) \right) \right] \\
=&2 \sum_{i,j=1,j\neq i}^{n} f\left(x_1, \cdots, x_n \right) + \sum_{i,j=1,j\neq i}^{n} \left(x_i - x_j \right) \left(\frac{\partial f\left(x_1, \cdots, x_n \right)}{\partial x_i} - \frac{\partial f\left(x_1, \cdots, x_n \right)}{\partial x_j} \right). \quad (3.2)
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{i,j=1,j\neq i}^{n} f\left(x_1, \cdots, x_n \right) = n\left(n-1 \right) f\left(x_1, \cdots, x_n \right), \\
&\sum_{i,j=1,j\neq i}^{n} \left(x_i - x_j \right) \left(\frac{\partial f\left(x_1, \cdots, x_n \right)}{\partial x_i} - \frac{\partial f\left(x_1, \cdots, x_n \right)}{\partial x_j} \right) \\
&\quad = 2 \sum_{1\leq i<j\leq n} \left(x_i - x_j \right) \left(\frac{\partial f\left(x_1, \cdots, x_n \right)}{\partial x_i} - \frac{\partial f\left(x_1, \cdots, x_n \right)}{\partial x_j} \right).
\end{aligned}$$

Also

$$\begin{aligned}
&\sum_{i,j=1,j\neq i}^{n} \left[\frac{\partial}{\partial x_i} \left(\left(x_i - x_j \right) f\left(x_1, \cdots, x_n \right) \right) - \frac{\partial}{\partial x_j} \left(\left(x_i - x_j \right) f\left(x_1, \cdots, x_n \right) \right) \right] \\
=&\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1,j\neq i}^{n} \left(x_i - x_j \right) f\left(x_1, \cdots, x_n \right) \right) \\
&\quad - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\sum_{i=1,j\neq i}^{n} \left(x_i - x_j \right) f\left(x_1, \cdots, x_n \right) \right) \\
=&\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(\left(n-1 \right) x_i - \sum_{j=1,j\neq i}^{n} x_j \right) f\left(x_1, \cdots, x_n \right) \right) \\
&\quad - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\left(\sum_{i=1,j\neq i}^{n} x_i - \left(n-1 \right) x_j \right) f\left(x_1, \cdots, x_n \right) \right) \\
=&\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(\left(n-1 \right) x_i - \sum_{j=1,j\neq i}^{n} x_j \right) f\left(x_1, \cdots, x_n \right) \right) \\
&\quad + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\left(\left(n-1 \right) x_j - \sum_{i=1,j\neq i}^{n} x_i \right) f\left(x_1, \cdots, x_n \right) \right) \\
=&2 \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(\left(\left(n-1 \right) x_k - \sum_{j=1,j\neq k}^{n} x_j \right) f\left(x_1, \cdots, x_n \right) \right)
\end{aligned}$$

$$=2\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left(nx_k-\sum_{j=1}^{n}x_j\right)f\left(x_1,\cdots,x_n\right)\right).$$

By (3.2) we get

$$\begin{aligned}&2\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(\left(nx_k-\sum_{j=1}^{n}x_j\right)f\left(x_1,\cdots,x_n\right)\right)\\&=2n\left(n-1\right)f\left(x_1,\cdots,x_n\right)\\&\quad+2\sum_{1\le i<j\le n}\left(x_i-x_j\right)\left(\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_i}-\frac{\partial f\left(x_1,\cdots,x_n\right)}{\partial x_j}\right),\end{aligned}$$

which is equivalent to the desired result. □

Remark 3.1. For $n=2$ we get

$$\begin{aligned}&\frac{1}{2}\left[\frac{\partial}{\partial x_1}\left[\left(x_1-x_2\right)f\left(x_1,x_2\right)\right]+\frac{\partial}{\partial x_1}\left[\left(x_2-x_1\right)f\left(x_1,x_2\right)\right]\right]\\&=f\left(x_1,x_2\right)+\frac{1}{2}\Lambda_{\partial f,D}\left(x_1,x_2\right)\end{aligned}\tag{3.3}$$

for $(x_1,x_2)\in D$. For $n=3$ we get

$$\begin{aligned}&\frac{1}{3}\left[\frac{\partial}{\partial x_1}\left(\left(x_1-\frac{x_2+x_3}{2}\right)f\left(x_1,x_2,x_3\right)\right)+\frac{\partial}{\partial x_2}\left(\left(x_2-\frac{x_1+x_3}{2}\right)f\left(x_1,x_2,x_3\right)\right)\right.\\&\quad\left.+\frac{\partial}{\partial x_2}\left(\left(x_3-\frac{x_1+x_2}{2}\right)f\left(x_1,x_2,x_3\right)\right)\right]\\&=f\left(x_1,x_2,x_3\right)+\frac{1}{6}\left[\Lambda_{\partial f,D}\left(x_1,x_2\right)+\Lambda_{\partial f,D}\left(x_2,x_3\right)+\Lambda_{\partial f,D}\left(x_1,x_3\right)\right]\end{aligned}\tag{3.4}$$

for $(x_1,x_2,x_3)\in D$.

We have the following identity of interest:

Theorem 3.1. *Let B be a bounded closed subset of $\mathbb{R}^n$ $(n\ge 2)$ with smooth (or piecewise smooth) boundary ∂B and $\mathbf{n}=(\mathbf{n}_1,\cdots,\mathbf{n}_n)$ be the unit outward-pointing normal of ∂B. If f is a continuously differentiable function on an open neighborhood of B, then we have the representation*

$$\begin{aligned}&\frac{1}{n-1}\sum_{k=1}^{n}\int_{\partial B}\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)f\left(x\right)\mathbf{n}_k\left(x\right)dA-\int_B f\left(x\right)dx\\&=\frac{1}{n\left(n-1\right)}\sum_{1\le i<j\le n}\int_B\Lambda_{\partial f,B}\left(x_i,x_j\right)dx.\end{aligned}\tag{3.5}$$

Proof. We use the identity (3.1) on B for $x=(x_1,\cdots,x_n)$ and take the volume integral to get

$$\begin{aligned}&\frac{1}{n-1}\int_B\sum_{k=1}^n\frac{\partial}{\partial x_k}\left(\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)f(x)\right)dx\\&=\int_B f(x)\,dx+\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\int_B\Lambda_{\partial f,B}(x_i,x_j)\,dx.\end{aligned}\tag{3.6}$$

Define

$$F_k(x)=\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)f(x),\quad k\in\{1,\cdots,n\},\quad x\in B,$$

and use the Divergence theorem (2.2) to get

$$\begin{aligned}&\int_B\sum_{k=1}^n\frac{\partial}{\partial x_k}\left(\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)f(x)\right)dx\\&=\sum_{k=1}^n\int_{\partial B}\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)f(x)\,\mathbf{n}_k(x)\,dA.\end{aligned}\tag{3.7}$$

On utilising (3.6) and (3.7), we obtain

$$\begin{aligned}&\int_B f(x)\,dx+\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\int_B\Lambda_{\partial f,B}(x_i,x_j)\,dx\\=&\frac{1}{n-1}\sum_{k=1}^n\int_{\partial B}\left(x_k-\frac{1}{n}\sum_{j=1}^n x_j\right)f(x)\,\mathbf{n}_k(x)\,dA,\end{aligned}$$

that is equivalent to (3.5). □

Remark 3.2. For $n=2$ we obtain the identity

$$\begin{aligned}&\frac{1}{2}\int_{\partial B}\left[(x_1-x_2)f(x_1,x_2)\,dx_1+(x_1-x_2)f(x_1,x_2)\,dx_2\right]\\&\quad-\int_B f(x_1,x_2)\,dx_1dx_2\\=&\frac{1}{2}\int_B\Lambda_{\partial f,B}(x_1,x_2)\,dx_1dx_2,\end{aligned}\tag{3.8}$$

where B is a bounded closed subset of $\mathbb{R}^2$ with smooth (or piecewise smooth) boundary ∂B and f is a continuously differentiable function on an open neighborhood of B.

For $n=3$ we obtain the identity

$$\begin{aligned}
&\frac{1}{3}\left[\int_{\partial B}\left(x_1-\frac{x_2+x_3}{2}\right)f\left(x_1,x_2,x_3\right)dx_2\wedge dx_3\right.\\
&\quad+\int_{\partial B}\left(x_2-\frac{x_1+x_3}{2}\right)f\left(x_1,x_2,x_3\right)dx_3\wedge dx_1\\
&\quad\left.+\int_{\partial B}\left(x_3-\frac{x_1+x_2}{2}\right)f\left(x_1,x_2,x_3\right)dx_1\wedge dx_2\right]-\int_B f\left(x_1,x_2,x_3\right)dx_1dx_2dx_3\\
=&\frac{1}{6}\int_B\left[\Lambda_{\partial f,B}\left(x_1,x_2\right)+\Lambda_{\partial f,B}\left(x_2,x_3\right)+\Lambda_{\partial f,B}\left(x_1,x_3\right)\right]dx_1dx_2dx_3,
\end{aligned}$$

where B is a bounded closed subset of $\mathbb{R}^3$ with smooth (or piecewise smooth) boundary ∂B and f is a continuously differentiable function on an open neighborhood of B.

Corollary 3.1. *Let B be a bounded closed and symmetric convex subset of $\mathbb{R}^n$ $(n\geq 2)$ with smooth (or piecewise smooth) boundary ∂B and $\mathbf{n}=(\mathbf{n}_1,\cdots,\mathbf{n}_n)$ be the unit outward-pointing normal of ∂B. If f is a continuously differentiable function on an open neighborhood of B and Schur convex on B, then we have the integral inequality*

$$\frac{1}{n-1}\sum_{k=1}^{n}\int_{\partial B}\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)f\left(x\right)\mathbf{n}_k\left(x\right)dA\geq\int_B f\left(x\right)dx. \tag{3.9}$$

Proof. Since f is Schur convex on B, then by (1.3) we get $\Lambda_{\partial f,D}\left(x_i,x_j\right)\geq 0$ for all $1\leq i<j\leq n$, and by using (3.5) we get the desired inequality (3.9). □

Corollary 3.2. *With the assumptions of Corollary 3.1 and if there exists $L_{ij}>0$ for $1\leq i<j\leq n$ such that*

$$\Lambda_{\partial f,D}\left(x_i,x_j\right)\leq L_{ij}\left(x_i-x_j\right)^2\quad\text{for all }\ x=(x_1,\cdots,x_n)\in B, \tag{3.10}$$

then we also have the reverse inequality

$$\begin{aligned}
0\leq&\frac{1}{n-1}\sum_{k=1}^{n}\int_{\partial B}\left(x_k-\frac{1}{n}\sum_{j=1}^{n}x_j\right)f\left(x\right)\mathbf{n}_k\left(x\right)dA-\int_B f\left(x\right)dx\\
\leq&\frac{1}{n\left(n-1\right)}\sum_{1\leq i<j\leq n}L_{ij}\int_B\left(x_i-x_j\right)^2dx.
\end{aligned} \tag{3.11}$$

The proof follows by the equality (3.5).

Remark 3.3. For $n=2$ in (3.9) we get

$$\begin{aligned}
0\leq&\frac{1}{2}\int_{\partial B}\left[\left(x_1-x_2\right)f\left(x_1,x_2\right)dx_1+\left(x_1-x_2\right)f\left(x_1,x_2\right)dx_2\right]\\
&-\int_B f\left(x_1,x_2\right)dx_1dx_2\\
\leq&\frac{1}{2}L\int_B\left(x_1-x_2\right)^2dx_1dx_2,
\end{aligned} \tag{3.12}$$

provided that f is Schur convex on the convex and symmetric domain $B \subset \mathbb{R}^2$ and there exists $L > 0$ such that

$$\begin{aligned}\Lambda_{\partial f,D}(x_1,x_2) &= (x_1-x_2)\left(\frac{\partial f(x_1,x_2)}{\partial x_1}-\frac{\partial f(x_1,x_2)}{\partial x_2}\right)\\ &\le L(x_1-x_2)^2 \quad \text{for all } x=(x_1,x_2)\in B. \end{aligned} \tag{3.13}$$

For $n=3$ we get

$$\begin{aligned}0 \le& \frac{1}{3}\left[\int_{\partial B}\left(x_1-\frac{x_2+x_3}{2}\right)f(x_1,x_2,x_3)\,dx_2\wedge dx_3\right. &(3.14)\\ &+\int_{\partial B}\left(x_2-\frac{x_1+x_3}{2}\right)f(x_1,x_2,x_3)\,dx_3\wedge dx_1\\ &\left.+\int_{\partial B}\left(x_3-\frac{x_1+x_2}{2}\right)f(x_1,x_2,x_3)\,dx_1\wedge dx_2\right]\\ &-\int_B f(x_1,x_2,x_3)\,dx_1dx_2dx_3\\ \le&\frac{1}{6}\left[L_{12}\int_B(x_1-x_2)^2\,dx_1dx_2dx_3\right.\\ &\left.+L_{23}\int_B(x_2-x_3)^2\,dx_1dx_2dx_3+L_{13}\int_B(x_1-x_3)^2\,dx_1dx_2dx_3\right] &(3.15)\end{aligned}$$

provided that f is Schur convex on the convex and symmetric domain $B \subset \mathbb{R}^3$ and

$$\begin{aligned}\Lambda_{\partial f,D}(x_i,x_j) &= (x_i-x_j)\left(\frac{\partial f(x_1,x_2,x_3)}{\partial x_i}-\frac{\partial f(x_1,x_2,x_3)}{\partial x_j}\right)\\ &\le L_{ij}(x_i-x_j)^2 \quad \text{for all } x=(x_1,x_2,x_3)\in B, \end{aligned} \tag{3.16}$$

where $L_{ij} > 0$ for $1 \le i < j \le 3$.

4 An example for three dimensional balls

Consider the 3-dimensional ball centered in $O=(0,0,0)$ and having the radius $R>0$,

$$B(O,R) := \left\{ (x_1,x_2,x_3)\in\mathbb{R}^3 \,\middle|\, x_1^2+x_2^2+x_3^2 \le R^2 \right\}$$

and the sphere

$$S(O,R) := \left\{ (x_1,x_2,x_3)\in\mathbb{R}^3 \,\middle|\, x_1^2+x_2^2+x_3^2 = R^2 \right\}.$$

Consider the parametrization of $B(O,R)$ and $S(O,R)$ given by:

$$B(O,R):\begin{cases} x_1 = r\cos\psi\cos\varphi,\\ x_2 = r\cos\psi\sin\varphi, \qquad (r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi],\\ x_3 = r\sin\psi,\end{cases}$$

and

$$S(O,R):\begin{cases} x_1 = R\cos\psi\cos\varphi, \\ x_2 = R\cos\psi\sin\varphi, \\ x_3 = R\sin\psi, \end{cases} \qquad (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

We have

$$\begin{vmatrix} \frac{\partial x_2}{\partial\psi} & \frac{\partial x_3}{\partial\psi} \\ \frac{\partial x_2}{\partial\varphi} & \frac{\partial x_3}{\partial\varphi} \end{vmatrix} = -R^2\cos^2\psi\cos\varphi, \qquad \begin{vmatrix} \frac{\partial x_1}{\partial\psi} & \frac{\partial x_3}{\partial\psi} \\ \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_3}{\partial\varphi} \end{vmatrix} = R^2\cos^2\psi\sin\varphi,$$

$$\begin{vmatrix} \frac{\partial x_1}{\partial\psi} & \frac{\partial x_2}{\partial\psi} \\ \frac{\partial x_1}{\partial\varphi} & \frac{\partial x_2}{\partial\varphi} \end{vmatrix} = -R^2\sin\psi\cos\psi.$$

In Cartesian coordinates, we have the inequality (3.14) written as

$$\begin{aligned} 0 \le & \frac{1}{3}\left[\int_{S(O,R)}\left(x_1-\frac{x_2+x_3}{2}\right)f(x_1,x_2,x_3)\,dx_2\wedge dx_3\right. \\ & + \int_{S(O,R)}\left(x_2-\frac{x_1+x_3}{2}\right)f(x_1,x_2,x_3)\,dx_3\wedge dx_1 \\ & \left.+ \int_{S(O,R)}\left(x_3-\frac{x_1+x_2}{2}\right)f(x_1,x_2,x_3)\,dx_1\wedge dx_2\right] \\ & - \int_{B(O,R)} f(x_1,x_2,x_3)\,dx_1dx_2dx_3 \\ \le & \frac{1}{6}\left[L_{12}\int_{B(O,R)}(x_1-x_2)^2\,dx_1dx_2dx_3\right. \\ & \left.+L_{23}\int_{B(O,R)}(x_2-x_3)^2\,dx_1dx_2dx_3 + L_{13}\int_{B(O,R)}(x_1-x_3)^2\,dx_1dx_2dx_3\right] \end{aligned} \tag{4.1}$$

provided that f is a continuously differentiable function on an open neighborhood of $B(O,R)$, Schur convex on $B(O,R)$ and the condition (3.16) is fulfilled.

Now, observe that

$$\begin{aligned} & \int_{B(O,R)}(x_1-x_2)^2\,dx_1dx_2dx_3 \\ = & \int_0^R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}(r\cos\psi\cos\varphi - r\cos\psi\sin\varphi)^2 r^2\cos\psi\,drd\psi d\varphi \\ = & \int_0^R r^4dr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\psi d\psi\int_0^{2\pi}(\cos\varphi-\sin\varphi)^2\,d\varphi = \frac{R^5}{5}\left(\frac{4}{3}\right)2\pi \\ = & \frac{8}{15}\pi R^5 \end{aligned}$$

and, similarly

$$\int_{B(O,R)} (x_2 - x_3)^2 \, dx_1 dx_2 dx_3 = \int_{B(O,R)} (x_1 - x_3)^2 \, dx_1 dx_2 dx_3 = \frac{8}{15}\pi R^5.$$

In polar coordinates, (4.1) becomes

$$\begin{aligned}
0 \leq & \frac{1}{3}R^3 \left[- \int_{S(O,R)} \left(\cos\psi\cos\varphi - \frac{\cos\psi\sin\varphi + \sin\psi}{2} \right) \right. \\
& \times f\left(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi\right)\cos^2\psi\cos\varphi d\psi d\varphi \\
& + \int_{S(O,R)} \left(\cos\psi\sin\varphi - \frac{\cos\psi\cos\varphi + \sin\psi}{2} \right) \\
& \times f\left(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi\right)\cos^2\psi\sin\varphi d\psi d\varphi \\
& - \int_{S(O,R)} \left(\sin\psi - \frac{\cos\psi\cos\varphi + \cos\psi\sin\varphi}{2} \right) \\
& \left. \times f\left(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi\right)\sin\psi\cos\psi d\psi d\varphi \right] \\
& - \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f\left(r\cos\psi\cos\varphi, r\cos\psi\sin\varphi, r\sin\psi\right) r^2\cos\psi dr d\psi d\varphi \\
\leq & \frac{4}{45}\pi R^5 \left(L_{12} + L_{23} + L_{13}\right),
\end{aligned} \tag{4.2}$$

provided that f is a continuously differentiable function on an open neighborhood of $B(O,R)$, Schur convex on $B(O,R)$ and satisfying the condition (3.16).

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