Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 1 (2023), pp. 16-27

DOI: 10.4208/ata.OA-2019-0031
March 2023

Boundedness of Some Commutators of Marcinkiewicz Integrals on Hardy Spaces

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Received 1 October 2019; Accepted (in revised version) 9 February 2021

Abstract. Based on the results of the boundedness of μ_Ω^b on L^p spaces, by using the theory of atomic decomposition of Hardy spaces, we obtain the boundedness of μ_Ω^b on Hardy spaces.

Key Words: Marcinkiewicz integral, commutator, Lipschitz space, Hardy space.

AMS Subject Classifications: 42B25

1 Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbf{R^n}(n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in L^1(S^{n-1})$ be homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral is defined by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \Big)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Let $b \in \mathrm{L}^1_{loc}(\mathbf{R^n})$, the commutator generated by the Marcinkiewicz integral μ_Ω and b is defined by

$$\mu_{\Omega,b}(f)(x) = \Big(\int_0^\infty |F_{\Omega,b,t}(f)(x)|^2 \frac{dt}{t^3} \Big)^{1/2},$$

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where

$$F_{\Omega,b,t}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))f(y)dy.$$

Y. Ding [1] studied the continuity properties of higher order commutators generated by the homogeneous fractional integral and BMO functions on certain Hardy spaces, the special case of the main result in [1] is the following theorem:

Theorem 1.1 ([1])**.** *Let $b\in BMO(\mathbf{R^n})$, $0<\mu<n$ and $\Omega\in L^r(S^{n-1})(r>n/(n-\mu))$. If $\omega_r(\delta)$ satisfy*

$$\int_0^1\frac{\omega_r(\delta)}{\delta}\Big(\log\frac{1}{\delta}\Big)^m d\delta<\infty, \tag{1.2}$$

then $T^{b,m}_{\Omega,\mu}$ is bounded from $H^1_{b^m}(\mathbf{R^n})$ to $L^{n/(n-\mu)}(\mathbf{R^n})$, where

$$T^{b,m}_{\Omega,\mu}(f)(x)=\int_{\mathbf{R^n}}\frac{\Omega(x-y)}{|x-y|^{n-\mu}}(b(x)-b(y))^m f(y)dy,\quad m\in\mathbf{N}.$$

In 2007, H. Wang [2] gave the $(H^1,L^{n/(n-\beta)})$ type estimates for $\mu_{\Omega,b}$ with the kernel Ω satisfying the logarithmic type Lipschitz conditions.

Theorem 1.2 ([2])**.** *Let $b\in Lip_\beta(\mathbf{R^n})$, $0<\beta<1$. If Ω is a homogeneous function of degree zero and satisfies the following conditions:*

(1) $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$ and $\Omega\in L^r(S^{n-1})$ for some $r\ge n/(n-\beta)$;

(2) there exist constants $C>0$ and $\rho>1$ such that

$$|\Omega(y_1)-\Omega(y_2)|\le\frac{C}{(\ln\frac{2}{|y_1-y_2|})^\rho}$$

for any $y_1,y_2\in S^{n-1}$. Then $\mu_{\Omega,b}$ is bounded from $H^1(\mathbf{R^n})$ to $L^{n/n-\beta}(\mathbf{R^n})$.

In 2011, Y. He [3] obtained the $(L^p(\alpha),L^p(\beta))$ type estimates for $\mu_{\Omega,b}$ with the kernel satisfying the logarithmic type Lipschitz conditions. In 2012, by using Theorem 1.2, Jiang [4] proved that $\mu_{\Omega,b}$ is bounded from $H^1_b(\omega)$ to $L^1(\mathbf{R^n})$.

Theorem 1.3 ([3])**.** *Let $\Omega\in L^\infty(S^{n-1})$ satisfy the cancellation property* (1.1)*. In addition, suppose that there exist constants $C>0$ and $\rho>2$ such that*

$$|\Omega(y_1)-\Omega(y_2)|\le\frac{C}{(\ln\frac{1}{|y_1-y_2|})^\rho} \tag{1.3}$$

hold uniformly in $y_1,y_2\in S^{n-1}$, $1<p<\infty$, $\alpha,\beta\in A_p$, $b\in BMO(\omega)$, $\omega=(\alpha\beta^{-1})^{1/p}$. Then the following inequality hold:

$$||\mu_{\Omega,b}(f)||_{L^p(\beta)}\le C||b||_{BMO(\omega)}||f||_{L^p(\alpha)}.$$

Theorem 1.4 ([4]). *Let* $\Omega \in L^\infty(S^{n-1})$ $(n \geq 2)$ *satisfy the cancellation property* (1.1) *and* (1.3) *for some* $\rho > 2$, $\omega \in A_1$, $b \in BMO(\omega)$. *Then* $\mu_{\Omega,b}$ *is bounded from* $H_b^1(\omega)$ *to* $L^1(\mathbf{R^n})$.

Recently, Y. Zhao [5] studied the boundedness of $\mu_{\Omega,b}$ generated by μ_Ω and weighted Lipschitz function b on weighted L^p spaces.

Theorem 1.5 ([5]). *Let* $\Omega \in L^\infty(S^{n-1})$, $(n \geq 2)$ *satisfy the cancellation property* (1.1) *and* (1.3) *for some* $\rho > 2$, $\omega \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, $b \in Lip_\beta(\omega)$. *Then*

$$||\mu_{\Omega,b}(f)||_{L^q(\omega^{1-q})} \leq C||f||_{L^p(\omega)}.$$

Inspired by [1, 2], a natural problem is whether $\mu_{\Omega,b}$ has the similar conclusion in Theorem 1.2, when Ω satisfies some L^r-Dini condition and $b \in Lip_\beta(\mathbf{R^n})$. On the other hand, inspired by [3,4], applying Theorem 1.5, we establish the $(H_b^1(\omega), L^q(\omega^{1-q}))$ type boundedness of $\mu_{\Omega,b}$. To state our main results, we introduce the following definitions and auxiliary results.

First let us recall the definitions of A_p. A locally integrable nonnegative function ω is said to belong to A_p if

$$\begin{aligned}
&\sup_B \Big(\frac{1}{|B|}\int_B \omega(x)dx\Big)\Big(\frac{1}{|B|}\int_B \omega(x)^{-1/(p-1)}dx\Big)^{p-1} \leq C < \infty, && 1 < p < \infty,\\
&\frac{1}{|B|}\int_B \omega(x)dx \leq C \operatorname*{essinf}_B \omega(x), && p = 1,
\end{aligned}$$

for every ball $B \subset \mathbf{R^n}$. ω is said to satisfy the reverse Hölder condition and is written by $\omega \in RH_r$, if there exists $r > 1$ such that

$$\Big(\frac{1}{|B|}\int_B \omega(x)^r dx\Big)^{1/r} \leq C\Big(\frac{1}{|B|}\int_B \omega(x)dx\Big).$$

Remark 1.1. If $\omega \in A_p,\ p \geq 1$, then there exists $r > 1$ such that $\omega \in RH_r$.

Definition 1.1 ([1]). *Let* $0 < \beta \leq 1$, *the Lipschitz class* $Lip_\beta(\mathbf{R^n})$ *is defined by*

$$Lip_\beta(\mathbf{R^n}) = \Big\{f : ||f||_{Lip_\beta} = \sup_{x,y\in\mathbf{R^n},\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty\Big\}.$$

Definition 1.2 ([6]). *For* $0 < \beta \leq 1$, *the weighted Lipschitz space* $Lip_{\beta,p}(\omega)$ *is defined by*

$$\begin{aligned}
Lip_{\beta,p}(\omega) = \Big\{b : ||b||_{Lip_{\beta,p}(\omega)} =& \sup_B \frac{1}{\omega(B)^{\beta/n}}\Big(\frac{1}{\omega(B)}\int_B |b(x)-b_B|^p \omega(x)^{1-p}dx\Big)^{1/p}\\
&\leq C < \infty\Big\}.
\end{aligned}$$

Definition 1.3 ([7]). *A function* $a(x)$ *on* $\mathbf{R^n}$ *is said to be an* H^1 *atom, if there exists a ball* B, *such that*

(1) supp $a \subset B$;

(2) $||a||_{L^\infty} \leq |B|^{-1}$;

(3) $\int_{\mathbf{R^n}} a(x)dx = 0$.

It is said that $f \in H^1(\mathbf{R^n})$, if $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the case of distributions, where each a_j is an H^1 atom, $\lambda_j \in \mathbf{C}$ and $\sum_{j=-\infty}^{\infty} |\lambda_j| < \infty$. Furthermore, the $H^1(\mathbf{R^n})$ seminorm is defined as

$$||f||_{H^1} = \inf \sum_{j=-\infty}^{\infty} |\lambda_j|,$$

where the infimum is taken over all above decompositions of f.

Remark 1.2. Let $a(x)$ be an H^1 atom, then for any $p_0 \in [1,\infty]$, we have $||a||_{L^{p_0}} \leq |B|^{-1+1/p_0}$.

Definition 1.4 ([4]). *Let* $0 < p \leq 1$, $\omega \in A_\infty$, $b \in L_{loc}(\mathbf{R^n})$. *A function* $a(x)$ *on* $\mathbf{R^n}$ *is said to be* $\omega - (p,\infty,b)$ *atom, if*

(1) there exists $x_0 \in \mathbf{R^n}$ *and* $d > 0$ *such that supp*$a \subset B(x_0,d)$;

(2) $||a||_{L^\infty} \leq \omega(B(x_0,d))^{-1/p}$;

(3) $\int_{\mathbf{R^n}} a(x)dx = \int_{\mathbf{R^n}} a(x)b(x)dx = 0$.

It is said that $f \in H_b^p(\omega)$, if $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the case of distributions, where each a_j is an $\omega - (p,\infty,b)$ atom, $\lambda_j \in \mathbf{C}$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. Furthermore, the $H_b^p(\omega)$ seminorm is defined as

$$||f||_{H_b^p(\omega)} = \inf \Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \Big)^{1/p},$$

where the infimum is taken over all above decompositions of f.

Definition 1.5 ([1]). *For* $\Omega \in L^r(S^{n-1})$ $(r \geq 1)$, *the integral modulus* $\omega_r(\delta)$ *of continuity of order* r *of* Ω *is defined by*

$$\omega_r(\delta) = \sup_{|\rho| \leq \delta} \Big(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^r d\sigma(x') \Big)^{1/r},$$

where ρ *is a rotation in* S^{n-1}. *When* $\omega_r(\delta)$ *satisfies*

$$\int_0^1 \frac{\omega_r(\delta)}{\delta} d\delta < \infty, \tag{1.4}$$

we say that Ω *satisfies the* L^r*-Dini condition.*

Let us state our main results.

Theorem 1.6. *Let $b \in Lip_\beta(\mathbf{R^n})$, $0 < \beta < 1$ and $\Omega \in L^r(S^{n-1})$ $(r \geq n/(n-\beta))$. If $\omega_r(\delta)$ satisfies* (1.4)*, then $\mu_{\Omega,b}$ is bounded from $H^1(\mathbf{R^n})$ to $L^{n/n-\beta}(\mathbf{R^n})$.*

Theorem 1.7. *Let $\Omega \in L^\infty(S^{n-1})$ $(n \geq 2)$ satisfies* (1.1) *and* (1.3) *for some $\rho > 2$. $\omega \in A_1$, $b \in Lip_\beta(\omega)$, $0 < \beta < 1$, then $\mu_{\Omega,b}$ is bounded from $H^1_b(\omega)$ to $L^q(\omega^{1-q})$, where $q = n/(n-\beta)$.*

Remark 1.3. The kernel Ω in Theorem 1.2 satisfies the logarithmic type Lipschitz conditions, the kernel Ω in Theorem 1.6 satisfies the logarithmic type $L^r - Dini$ conditions, however Theorem 1.6 arrives at the same conclusion with Theorem 1.2.

2 Priliminaries

To prove our theorems, we need the following lemmas.

Lemma 2.1 ([8])**.** *Let $b \in Lip_\beta(\mathbf{R^n})$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$. If $\Omega \in L^r(S^{n-1})$ for some $r \geq n/(n-\beta)$ and* (1.1)*. Then $\mu_{\Omega,b}$ is bounded from $L^p(\mathbf{R^n})$ to $L^q(\mathbf{R^n})$.*

Lemma 2.2 ([1])**.** *Suppose that $0 < \alpha < n$, $r > 1$ and $\Omega \in L^r(S^{n-1})$ satisfies L^r−Dini condition* (1.4)*. If there exists a constant $a_0 > 0$ such that $|y| < a_0 R$, then*

$$\Big(\int_{R<|x|<2R}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(x)}{|x|^{n-\alpha}}\Big|^r dx\Big)^{1/r} \leq CR^{n/r-(n-\alpha)}\Big(\frac{|y|}{R}+\int_{|y|/2R}^{|y|/R}\frac{\omega_r(\delta)}{\delta}d\delta\Big).$$

Lemma 2.3 ([5])**.** *Let $b \in Lip_\beta(\omega)$, $0 < \beta < 1$, $\omega \in A_1$. Then*

$$\sup_{x\in B}|b(x)-b_B| \leq C||b||_{Lip_\beta(\omega)}\omega(B)^{1+\beta/n}|B|^{-1}.$$

For any ball B and any $\lambda > 0$, we denote by λB the ball with the same center as B but with λ times the radius. We have an estimate for $\omega(\lambda B)$ as follows.

Lemma 2.4 ([9])**.** *Let $\omega \in A_p$, $p \geq 1$. Then for any ball B and $\lambda > 1$*

$$\omega(\lambda B) \leq C\lambda^{np}\omega(B),$$

where C does not dependent on B nor on λ.

Lemma 2.5 ([9])**.** *Let $\omega \in A_p \bigcap RH_r$, $p \geq 1$, $r > 1$. Then there exist constants C_1, $C_2 > 0$ such that*

$$C_1\Big(\frac{|E|}{|B|}\Big)^p \leq \frac{\omega(E)}{\omega(B)} \leq C_2\Big(\frac{|E|}{|B|}\Big)^{(r-1)/r}$$

for any measurable subset E of a ball B.

3 Proof of theorems

Proof of Theorem 1.6. We need to prove that $||\mu_{\Omega,b}(a)||_{L^q} \leq C$ for any H^1 atom a, where $q = n/(n-\beta)$ and $\text{supp}a \subset B = B(x_0,d)$

$$
\begin{aligned}
||\mu_{\Omega,b}(a)||_{L^q} \leq & \Big(\int_{2B} |\mu_{\Omega,b}(a)(x)|^q dx \Big)^{1/q} + \Big(\int_{(2B)^c} |\mu_{\Omega,b}(a)(x)|^q dx \Big)^{1/q} \\
=: & I + J.
\end{aligned}
$$

Choose p_1 and q_1 such that $1 < p_1 < n/\beta$, $1/q_1 = 1/p_1 - \beta/n$. It is easy to see that $q < q_1$, by Hölder's inequality and Lemma 2.1, we get

$$
\begin{aligned}
I \leq & \Big(\int_{2B} |\mu_{\Omega,b}(a)(x)|^{qt} dx \Big)^{1/(qt)} \Big(\int_{2B} dx \Big)^{1/(qt')} \\
\leq & C \Big(\int_{2B} |\mu_{\Omega,b}(a)(x)|^{q_1} dx \Big)^{1/q_1} |2B|^{1/q - 1/q_1} \\
\leq & C ||\mu_{\Omega,b}(a)||_{L^{q_1}} |2B|^{1/q-1/q_1} \leq C ||a||_{L^{p_1}} |B|^{1/q-1/q_1} \\
\leq & C ||a||_\infty |B|^{1/p_1} |B|^{1/q-1/q_1} \leq C |B|^{-1+1/p_1} |B|^{1/q-1/q_1} \\
\leq & C,
\end{aligned}
$$

where $t = q_1/q$.

Let us turn to estimate J now,

$$
\begin{aligned}
J \leq & \Big\{ \int_{(2B)^c} \Big(\int_0^{|x-x_0|+2d} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) a(y) dy \Big|^2 \frac{dt}{t^3} \Big)^{q/2} dx \Big\}^{1/q} \\
& + \Big\{ \int_{(2B)^c} \Big(\int_{|x-x_0|+2d}^{\infty} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) a(y) dy \Big|^2 \frac{dt}{t^3} \Big)^{q/2} dx \Big\}^{1/q} \\
=: & J_1 + J_2.
\end{aligned}
$$

Using the fact that $|x-y| \sim |x-x_0| \sim |x-x_0| + 2d$ for any $y \in B$ and $x \in (2B)^c$, Minkowski's inequality and Hölder's inequality, we have

$$
\begin{aligned}
J_1 \leq & C \Big\{ \int_{(2B)^c} \Big[\int_{\mathbf{R^n}} \Big(\int_{|x-y|}^{|x-x_0|+2d} \frac{dt}{t^3} \Big)^{1/2} \frac{|\Omega(x-y)||a(y)|}{|x-y|^{n-1}} |b(x)-b(y)| dy \Big]^q dx \Big\}^{1/q} \\
\leq & C \int_B \Big\{ \int_{(2B)^c} \Big[\frac{d^{1/2} |\Omega(x-y)|}{|x-y|^{n+1/2}} |b(x)-b(y)| \Big]^q dx \Big\}^{1/q} |a(y)| dy \\
\leq & C \int_B \sum_{k=1}^{\infty} \Big\{ \int_{2^{k+1}B \setminus 2^k B} \Big[\frac{d^{1/2} |\Omega(x-y)|}{|x-y|^{n+1/2}} |b(x)-b(y)| \Big]^q dx \Big\}^{1/q} |a(y)| dy \\
\leq & C \int_B \sum_{k=1}^{\infty} 2^{-k/2} (2^k d)^{-n} (2^{k+1} d)^{\beta} ||b||_{Lip_\beta} \Big[\int_{2^{k+1}B} |\Omega(x-y)|^q dx \Big]^{1/q} |a(y)| dy
\end{aligned}
$$

$$
\begin{aligned}
&\le C||b||_{Lip_\beta}||\Omega||_{L^r(S^{n-1})}\int_B\sum_{k=1}^{\infty}2^{-k/2}(2^kd)^{-n}(2^{k+1}d)^{\beta}(2^{k+1}d)^{n/q}|a(y)|dy\\
&\le C\int_B\sum_{k=1}^{\infty}2^{-k[(1/2-\beta)+n(1-1/q)]}|a(y)|dy\\
&\le C||a||_{L^1}\le C||a||_\infty|B|\le C.
\end{aligned}
$$

In the above last inequality, we applied the fact that $-n+\beta+n/q=0$ and the series is convergent.

Note that $t\ge|x-x_0|+2d\ge|x-x_0|+|y-x_0|\ge|x-y|$ for any $y\in B$ and the cancellation property of a,

$$
\begin{aligned}
&\Big[\int_{|x-x_0|+2d}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big]^{1/2}\\
=&\Big[\int_{|x-x_0|+2d}^{\infty}\Big|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big]^{1/2}\\
=&\Big|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|\Big[\int_{|x-x_0|+2d}^{\infty}\frac{dt}{t^3}\Big]^{1/2}\\
=&C\Big|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|\frac{1}{|x-x_0|+2d}\\
\le&C\int_B|b(x)-b(x_0)|\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|a(y)|}{|x-x_0|+2d}dy\\
&+C\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|b(x)-b(x_0)||a(y)|}{|x-x_0|+2d}dy.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
J_2\le&C\Big\{\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|b(x)-b(x_0)|}{|x-x_0|+2d}|a(y)|dy\Big)^qdx\Big\}^{1/q}\\
&+C\Big\{\int_{(2B)^c}\Big(\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|b(y)-b(x_0)|}{|x-x_0|+2d}|a(y)|dy\Big)^qdx\Big\}^{1/q}\\
=:&J_{21}+J_{22}.
\end{aligned}
$$

Applying Minkowski's inequality, Hölder's inequality and Lemma 2.2, we have

$$
\begin{aligned}
J_{21}\le&C\int_B\Big\{\int_{(2B)^c}\Big(\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|b(x)-b(x_0)|}{|x-x_0|+2d}\Big)^qdx\Big\}^{1/q}|a(y)|dy\\
\le&C\int_B\Big\{\sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB}\Big(\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|b(x)-b(x_0)|}{|x-x_0|+2d}\Big)^qdx\Big\}^{1/q}|a(y)|dy\\
\le&C||b||_{Lip_\beta}\int_B\sum_{k=1}^{\infty}(2^kd)^{\beta-1}\Big(\int_{2^{k+1}B\backslash 2^kB}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^qdx\Big)^{1/q}|a(y)|dy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_B\sum_{k=1}^{\infty}(2^kd)^{\beta-1}\Big(\int_{2^{k+1}B\backslash 2^kB}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^r dx\Big)^{1/r}\Big(\int_{2^{k+1}B\backslash 2^kB}dx\Big)^{1/q-1/r}|a(y)|dy\\
&\leq C\int_B\sum_{k=1}^{\infty}(2^kd)^{\beta-1}(2^{k+1}d)^{n(1/q-1/r)}\Big(\int_{2^{k+1}B\backslash 2^kB}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^r dx\Big)^{1/r}|a(y)|dy\\
&\leq C\int_B\sum_{k=1}^{\infty}(2^kd)^{\beta-1}(2^{k+1}d)^{n(1/q-1/r)}(2^kd)^{n/r-n+1}\Big(\frac{1}{2^k}+\int_{|y-x_0|/2^{k+1}d}^{|y-x_0|/2^kd}\frac{\omega_r(\delta)}{\delta}d\delta\Big)|a(y)|dy\\
&\leq C\int_B\sum_{k=1}^{\infty}(2^kd)^{\beta+n/q-n}\Big(\frac{1}{2^k}+\int_{|y-x_0|/2^{k+1}d}^{|y-x_0|/2^kd}\frac{\omega_r(\delta)}{\delta}d\delta\Big)|a(y)|dy\\
&\leq C\int_B\sum_{k=1}^{\infty}\Big(\frac{1}{2^k}+\int_{|y-x_0|/2^{k+1}d}^{|y-x_0|/2^kd}\frac{\omega_r(\delta)}{\delta}d\delta\Big)|a(y)|dy\\
&\leq C\int_B\Big(\sum_{k=1}^{\infty}\frac{1}{2^k}+\sum_{k=1}^{\infty}\int_{|y-x_0|/2^{k+1}d}^{|y-x_0|/2^kd}\frac{\omega_r(\delta)}{\delta}d\delta\Big)|a(y)|dy\\
&\leq C\int_B|a(y)|dy\leq C||a||_\infty|B|\leq C,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{22}\leq& C\Big\{\int_{(2B)^c}\Big(\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|b(y)-b(x_0)|}{|x-x_0|+2d}|a(y)|dy\Big)^q dx\Big\}^{1/q}\\
\leq& C||b||_{Lip_\beta}\Big\{\int_{(2B)^c}\Big(\int_B\frac{|\Omega(x-y)|}{|x-y|^n}|y-x_0|^\beta|a(y)|dy\Big)^q dx\Big\}^{1/q}\\
\leq& C\int_B\Big\{\int_{(2B)^c}\frac{|\Omega(x-y)|^q}{|x-y|^{nq}}|y-x_0|^{\beta q}dx\Big\}^{1/q}|a(y)|dy\\
\leq& C\int_B\sum_{k=1}^{\infty}\Big(\int_{2^{k+1}B\backslash 2^kB}\frac{|\Omega(x-y)|^q}{|x-y|^{nq}}|y-x_0|^{\beta q}dx\Big)^{1/q}|a(y)|dy\\
\leq& C\int_B\sum_{k=1}^{\infty}d^\beta(2^kd)^{-n}\Big(\int_{2^{k+1}B\backslash 2^kB}|\Omega(x-y)|^q dx\Big)^{1/q}|a(y)|dy\\
\leq& C\int_B\sum_{k=1}^{\infty}d^\beta(2^kd)^{-n}|2^{k+1}B|^{1/q}||\Omega||_{L^r(S^{n-1})}|a(y)|dy\\
\leq& C\sum_{k=1}^{\infty}d^\beta(2^kd)^{-n}(2^{k+1}d)^{n/q}||a||_\infty|B|\\
\leq& C\sum_{k=1}^{\infty}2^{-kn(1-1/q)}d^{\beta-n+n/q}\leq C.
\end{aligned}
$$

We complete the proof of Theorem 1.6. □

Proof of Theorem 1.7. We need to prove that $||\mu_{\Omega,b}(a)||_{L^q(\omega^{1-q})}\leq C$ for any $\omega-(p,\infty,b)$

atom a, where $p=1$, $q=n/(n-\beta)$ and supp$a\subset B=B(x_0,d)$, $||a||_\infty\le\omega(B)^{-1}$.

$$\begin{aligned}||\mu_{\Omega,b}(a)||_{L^q(\omega^{1-q})}=&\Big(\int_{\mathbf{R^n}}|\mu_{\Omega,b}(a)(x)|^q\omega(x)^{1-q}dx\Big)^{1/q}\\ \le&\Big(\int_{2B}|\mu_{\Omega,b}(a)(x)|^q\omega(x)^{1-q}dx\Big)^{1/q}+\Big(\int_{(2B)^c}|\mu_{\Omega,b}(a)(x)|^q\omega(x)^{1-q}dx\Big)^{1/q}\\ =:&I_1+I_2.\end{aligned}$$

Choose p_1 and q_1 such that $1<q<q_1<\infty$, $1<p_1<\infty$, $1/q_1=1/p_1-\beta/n$. By Hölder's inequality, Theorem 1.5 and Lemma 2.4, we get

$$\begin{aligned}I_1=&\Big(\int_{2B}|\mu_{\Omega,b}(a)(x)|^q\omega(x)^{1-q}dx\Big)^{1/q}\\ =&\Big(\int_{2B}|\mu_{\Omega,b}(a)(x)|^q\omega(x)^{q/q_1-q}\omega(x)^{1-q/q_1}dx\Big)^{1/q}\\ \le&C\Big(\int_{2B}|\mu_{\Omega,b}(a)(x)|^{qt}\omega(x)^{(q/q_1-q)t}dx\Big)^{1/(qt)}\Big(\int_{2B}\omega(x)^{(1-q/q_1)t'}dx\Big)^{1/(qt')}\\ \le&C\Big(\int_{2B}|\mu_{\Omega,b}(a)(x)|^{q_1}\omega(x)^{1-q_1}dx\Big)^{1/q_1}\Big(\int_{2B}dx\Big)^{1/q-1/q_1}\\ \le&C||\mu_{\Omega,b}(a)||_{L^{q_1}(\omega^{1-q_1})}\omega(2B)^{1/q-1/q_1}\\ \le&C||a||_{L^{p_1}(\omega)}\omega(2B)^{1/q-1/q_1}\\ \le&C\Big(\int_B|a(x)|^{p_1}\omega(x)dx\Big)^{1/p_1}\omega(2B)^{1/q-1/q_1}\\ \le&C||a||_\infty\omega(B)^{1/p_1}\omega(2B)^{1/p-1/p_1}\\ \le&C\omega(B)^{-1/p}\omega(B)^{1/p_1}\omega(2B)^{1/p-1/p_1}\\ \le&C\Big(\frac{\omega(2B)}{\omega(B)}\Big)^{1/p-1/p_1}\le C,\end{aligned}$$

where $t=q_1/q$. For I_2, since $|x-x_0|>2d$, we have

$$\begin{aligned}I_2=&\Big(\int_{(2B)^c}|\mu_{\Omega,b}(a)(x)|^q\omega(x)^{1-q}dx\Big)^{1/q}\\ =&\Big\{\int_{(2B)^c}\Big(\int_0^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big)^{q/2}\omega(x)^{1-q}dx\Big\}^{1/q}\\ \le&\Big\{\int_{(2B)^c}\Big(\int_0^{|x-x_0|+2d}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big)^{q/2}\omega(x)^{1-q}dx\Big\}^{1/q}\\ &+\Big\{\int_{(2B)^c}\Big(\int_{|x-x_0|+2d}^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big)^{q/2}\omega(x)^{1-q}dx\Big\}^{1/q}\\ =:&I_{21}+I_{22}.\end{aligned}$$

The definitions of $Lip_{\beta,p}(\omega)$ and A_1 give

$$\begin{aligned}
&\Big(\int_{2^{k+1}B}|b(x)-b_{2^{k+1}B}|^q\omega(x)^{1-q}dx\Big)^{1/q}\\
&=\frac{\omega(2^{k+1}B)^{\beta/n}}{\omega(2^{k+1}B)^{\beta/n}}\Big(\frac{\omega(2^{k+1}B)}{\omega(2^{k+1}B)}\int_{2^{k+1}B}|b(x)-b_{2^{k+1}B}|^q\omega(x)^{1-q}dx\Big)^{1/q}\\
&\leq C||b||_{Lip_\beta(\omega)}\omega(2^{k+1}B)^{\beta/n}\omega(2^{k+1}B)^{1/q}\\
&=C||b||_{Lip_\beta(\omega)}\omega(2^{k+1}B)^{1/p}, \qquad (3.1a)\\
&\Big(\int_{2^{k+1}B}\omega(x)^{1-q}dx\Big)^{1/q}=\Big(\int_{2^{k+1}B}\Big(\frac{1}{\omega}\Big)^{q-1}dx\Big)^{1/q}\\
&\leq C\Big(\frac{|2^{k+1}B|}{\omega(2^{k+1}B)}\Big)^{(q-1)/q}|2^{k+1}B|^{1/q}\\
&=C\frac{|2^{k+1}B|}{\omega(2^{k+1}B)^{1-1/q}}. \qquad (3.1b)
\end{aligned}$$

From Minkowski's inequality, Lemmas 2.3, 2.5, (3.1a) and (3.1b), we obtain

$$\begin{aligned}
I_{21}\leq&C\Big\{\int_{(2B)^c}\Big[\int_{\mathbf{R^n}}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||a(y)|\Big(\int_{|x-y|}^{|x-x_0|+2d}\frac{dt}{t^3}\Big)^{1/2}dy\Big]^q\omega(x)^{1-q}dx\Big\}^{1/q}\\
\leq&Cd^{1/2}\Big\{\int_{(2B)^c}\Big[\int_B\frac{|b(x)-b(y)|}{|x-y|^{n+1/2}}|a(y)|dy\Big]^q\omega(x)^{1-q}dx\Big\}^{1/q}\\
\leq&Cd^{1/2}\int_B|a(y)|\Big\{\int_{(2B)^c}\frac{|b(x)-b(y)|^q}{|x-y|^{(n+1/2)q}}\omega(x)^{1-q}dx\Big\}^{1/q}dy\\
\leq&Cd^{1/2}\int_B|a(y)|\sum_{k=1}^{\infty}\Big(\int_{2^{k+1}B\backslash 2^kB}\frac{|b(x)-b_{2^{k+1}B}|^q}{|x-y|^{(n+1/2)q}}\omega(x)^{1-q}dx\Big)^{1/q}dy\\
&+Cd^{1/2}\int_B|a(y)||b(y)-b_{2^{k+1}B}|\sum_{k=1}^{\infty}\Big(\int_{2^{k+1}B\backslash 2^kB}\frac{1}{|x-y|^{(n+1/2)q}}\omega(x)^{1-q}dx\Big)^{1/q}dy\\
\leq&Cd^{1/2}\int_B|a(y)|\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}\Big(\int_{2^{k+1}B}|b(x)-b_{2^{k+1}B}|^q\omega(x)^{1-q}dx\Big)^{1/q}dy\\
&+Cd^{1/2}\int_B|a(y)||b(y)-b_{2^{k+1}B}|\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}\Big(\int_{2^{k+1}B}\omega(x)^{1-q}dx\Big)^{1/q}dy\\
\leq&Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}||b||_{Lip_\beta(\omega)}\omega(2^{k+1}B)^{1/p}\int_B|a(y)|dy\\
&+Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}\frac{|2^{k+1}B|}{\omega(2^{k+1}B)^{1-1/q}}\int_B|a(y)||b(y)-b_{2^{k+1}B}|dy\\
\leq&Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}\omega(2^{k+1}B)^{1/p}||a||_\infty|B|
\end{aligned}$$

$$
\begin{aligned}
&+Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}\frac{|2^{k+1}B|}{\omega(2^{k+1}B)^{1-1/q}}||a||_\infty\frac{\omega(2^{k+1}B)^{1+\beta/n}}{|2^{k+1}B|}|B|\\
\le&Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}|B|\Big(\frac{\omega(2^{k+1}B)}{\omega(B)}\Big)^{1/p}\\
&+Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}\omega(B)^{-1/p}\omega(2^{k+1}B)^{\beta/n+1/q}|B|\\
\le&Cd^{1/2}\sum_{k=1}^{\infty}(2^kd)^{-(n+1/2)}|B|\Big(\frac{|2^{k+1}B|}{|B|}\Big)^{1/p}\\
\le&C\sum_{k=1}^{\infty}2^{-k(n+1/2-n/p)}\le C,
\end{aligned}
$$

where $p=1$, $q=n/(n-\beta)$, $\beta/n+1/q=1/p$. Applying the estimate

$$
\begin{aligned}
&\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\\
\le&\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-y|^{n-1}}\Big|+\Big|\frac{\Omega(x-x_0)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\\
\le&\frac{C(1+|\Omega(x-x_0)|)}{|x-x_0|^{n-1}(\log\frac{|x-x_0|}{d})^{\rho}},
\end{aligned}
$$

Minkowski's inequality, (3.1a) and (3.1b), we have

$$
\begin{aligned}
I_{22}=&\Big\{\int_{(2B)^c}\Big(\int_{|x-x_0|+2d}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big)^{q/2}\omega(x)^{1-q}dx\Big\}^{1/q}\\
\le&C\Big\{\int_{(2B)^c}\Big(\int_{|x-x_0|+2d}^{\infty}\Big|\int_{|x-y|\le t}\Big(\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big)(b(x)-b(y))a(y)dy\Big|^2\frac{dt}{t^3}\Big)^{q/2}\cdot\omega(x)^{1-q}dx\Big\}^{1/q}\\
\le&C\Big\{\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big||b(x)-b(y)||a(y)|\Big(\int_{|x-x_0|+2d}^{\infty}\frac{dt}{t^3}\Big)^{1/2}dy\Big)^q\cdot\omega(x)^{1-q}dx\Big\}^{1/q}\\
\le&C\Big\{\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big||b(x)-b(y)||a(y)|\frac{1}{|x-x_0|+2d}dy\Big)^q\omega(x)^{1-q}dx\Big\}^{1/q}\\
\le&C\Big\{\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n}}\Big||b(x)-b(y)||a(y)|dy\Big)^q\omega(x)^{1-q}dx\Big\}^{1/q}\\
\le&C\Big\{\int_{(2B)^c}\Big(\int_B\frac{1}{|x-x_0|^n(\ln\frac{|x-x_0|}{d})^{\rho}}|b(x)-b(y)||a(y)|dy\Big)^q\omega(x)^{1-q}dx\Big\}^{1/q}\\
\le&C\int_B|a(y)|\Big(\int_{(2B)^c}\frac{1}{|x-x_0|^{nq}(\ln\frac{|x-x_0|}{d})^{\rho q}}|b(x)-b(y)|^q\omega(x)^{1-q}dx\Big)^{1/q}dy\\
\le&C\int_B|a(y)|\sum_{k=1}^{\infty}\Big(\int_{2^{k+1}B\backslash 2^kB}\frac{1}{|x-x_0|^{nq}(\ln\frac{|x-x_0|}{d})^{\rho q}}|b(x)-b(y)|^q\omega(x)^{1-q}dx\Big)^{1/q}dy\\
\le&C\int_B|a(y)|\sum_{k=1}^{\infty}(2^kd)^{-n}k^{-\rho}\Big(\int_{2^{k+1}B}|b(x)-b_{2^{k+1}B}|^q\omega(x)^{1-q}dx\Big)^{1/q}dy
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_B |a(y)||b(y)-b_{2^{k+1}B}|\sum_{k=1}^{\infty}(2^kd)^{-n}k^{-\rho}\Big(\int_{2^{k+1}B}\omega(x)^{1-q}dx\Big)^{1/q}dy\\
\leq& C\sum_{k=1}^{\infty}(2^kd)^{-n}k^{-\rho}\omega(2^{k+1}B)^{1/p}\int_B|a(y)|dy\\
&+C\sum_{k=1}^{\infty}(2^kd)^{-n}k^{-\rho}\frac{|2^{k+1}B|}{\omega(2^{k+1}B)^{1-1/q}}\int_B|a(y)||b(y)-b_{2^{k+1}B}|dy\\
\leq& C\sum_{k=1}^{\infty}(2^kd)^{-n}k^{-\rho}\omega(2^{k+1}B)^{1/p}||a||_\infty|B|\\
&+C\sum_{k=1}^{\infty}(2^kd)^{-n}k^{-\rho}\frac{|2^{k+1}B|}{\omega(2^{k+1}B)^{1-1/q}}\omega(B)^{-1/p}\frac{\omega(2^{k+1}B)^{1+\beta/n}}{|2^{k+1}B|}|B|\\
\leq& C\sum_{k=1}^{\infty}2^{-kn}d^{-n}k^{-\rho}\Big(\frac{\omega(2^{k+1}B)}{\omega(B)}\Big)^{1/p}|B|\\
\leq& C\sum_{k=1}^{\infty}2^{-kn(1-1/p)}k^{-\rho}=C\sum_{k=1}^{\infty}k^{-\rho}\leq C.
\end{aligned}
$$

This completes the proof of Theorem 1.7. □

Acknowledgements

The research supported by the National Natural Science Foundation of China (No. 11501262).

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