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Boundedness of the Multilinear Maximal Operator with the Hausdorff Content

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Abstract. In this paper, we establish the strong and weak boundedness of the multilinear maximal operator in the setting of the Choquet integral with respect to the α-dimensional Hausdorff content. Our results cover Orobitg and Verdera's results in [8].

Key Words: Multilinear maximal operator, Hausdorff content, Choquet integrals.

AMS Subject Classifications: 42B25, 42B35

1 Introduction

The purpose of this paper is to establish the strong and weak boundedness of the multilinear maximal operator on the Choquet space. For m-couple locally integrable functions $(f_1,\cdots,f_m)$ on $\mathbb{R}^n\times\cdots\times\mathbb{R}^n$, the multi(sub)linear maximal operator M is defined by

$$M(f_1,\cdots,f_m)(x):=\sup_{Q\ni x}\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q|f_i(y)|\,\mathrm{d}y, \tag{1.1}$$

where the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes. Very often it is much more convenient to work with dyadic multilinear maximal function $M_d(f_1,\cdots,f_m)$, which is defined by the right-hand side of (1.1), but the supremum is taken only on the family of dyadic cubes containing x. Clearly, when $m=1$, M is the classical Hardy-Littlewood maximal operator. These maximal operators are fundamental tools to study harmonic analysis, potential theory, and the theory of partial differential equations (see, e.g., [3,5]).

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For $E \subset \mathbb{R}^n$ and $0 < \alpha \le n$, the α-dimensional Hausdorff content of E is defined by

$$H^\alpha(E) := \inf \sum_{j=1}^{\infty} \ell(Q_j)^\alpha, \tag{1.2}$$

where the infimum is taken over all coverings of E by countable families of cubes Q_j with sides parallel to the coordinate axes and $\ell(Q)$ denotes the side length of the cube Q. If we take the infimum in (1.2) only on coverings of E by dyadic squares, we can obtain an equivalent quantity $H_d^\alpha(E)$ called the dyadic α-dimensional Hausdorff content. In [8], Orobitg and Verdera used the Choquet integral with respect to the α-dimensional Hausdorff content to extend some well-known estimates for Hardy-Littlewood maximal opertaor. They proved the strong type inequality

$$\int (Mf)^p \, \mathrm{d}H^\alpha \le C \int |f|^p \, \mathrm{d}H^\alpha \tag{1.3}$$

for $\alpha/n < p$, and the weak type inequality

$$H^\alpha\{x \colon Mf(x) > t\} \le Ct^{-\frac{\alpha}{n}} \int |f|^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha \tag{1.4}$$

for any $t > 0$ and $p = \alpha/n$. Here, the integrals are taken in the Choquet sense, that is, the Choquet integral of $\varphi \ge 0$ with respect to a set function Λ is defined by

$$\int \varphi \, \mathrm{d}\Lambda := \int_0^\infty \Lambda\{x \in \mathbb{R}^n \colon \varphi(x) > t\} \, \mathrm{d}t.$$

When $\alpha = n$, both (1.3) and (1.4) become the classical strong type inequality and weak type inequality, respectively. It is worth mentioning that the Orobitg-Verdera result came from their efforts to comprehend the special case $p = 1$ that is first proved by Adams in [1]–a result of the H^1-BMO duality theory applied to the characterization of the Riesz capacities. In fact, the Orobitg-Verdera's proof is a modification of arguments due to Carleson [4] and Hormander [6]. Moreover, Tang [10] generalized the preceding results and established the boundedness of maximal operators on the weighted Choquet space and the Choquet-Morrey space.

Motivated by these works, we investigate the strong and weak boundedness of the multilinear maximal operators in the frame of Choquet integrals with respect to the α-dimensional Hausdorff content.

Now, we formulate our main results as follows.

Theorem 1.1. *Let $0 < \alpha < n$, $0 < p \le p_i < \infty$ with $1 \le i \le m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{\alpha}{n} < \min\{p_1, \cdots, p_m\}$. Then, the following inequality*

$$\left(\int \big(M(f_1, \cdots, f_m)\big)^p \, \mathrm{d}H^\alpha \right)^{\frac{1}{p}} \le C \prod_{i=1}^{m} \left(\int |f_i|^{p_i} \, \mathrm{d}H^\alpha \right)^{\frac{1}{p_i}}$$

holds for some constant C depending on α, m, n and p_i.

Theorem 1.2. *Let $0<\alpha<\min\{n,np\}$ and $\alpha=\alpha_1+\cdots+\alpha_m$ with $\alpha_i>0$ for $1\le i\le m$. Then, the following inequality*

$$\int \big(M(f_1,\cdots,f_m)\big)^p \,\mathrm{d}H^\alpha \le C\prod_{i=1}^m \int |f_i|^p \,\mathrm{d}H^{\alpha_i}$$

holds for some constant C depending only on α, m, n and p.

Theorem 1.3. *Let $0<\alpha<n$. Then, for some constant C depending on α, m and n, we have that*

$$\big(H^\alpha\{x\colon M_d(f_1,\cdots,f_m)(x)>t\}\big)^m \le Ct^{-\frac{\alpha}{n}}\prod_{i=1}^m \int |f_i|^{\frac{\alpha}{n}} \,\mathrm{d}H^\alpha, \tag{1.5a}$$

$$H^\alpha\{x\colon M_d(f_1,\cdots,f_m)(x)>t\} \le C\sum_{i=1}^m t^{-\beta_i\frac{\alpha}{n}} \int |f_i|^{\frac{\alpha}{n}} \,\mathrm{d}H^\alpha, \tag{1.5b}$$

where $\beta_1+\cdots+\beta_m=1$ and $\beta_1,\cdots,\beta_m\ge 0$.

The rest of the present paper is organized as follows: In Section 2, we will give some facts and lemmas. Theorems 1.1–1.3 are proved in Section 3. A tacit understanding in the present paper is that all cubes in $\mathbb{R}^n$ are cubes with sides parallel to the coordinate axes. We denote by $|E|$ the Lebesgue measure of the set $E\subset\mathbb{R}^n$. The positive constant C varies from one occurrence to another.

Remark 1.1. In this paper, we merely give the proof with the case $m=2$ for the sake of clarity in writing, and the same is true for $m>2$.

2 Some facts and lemmas

We first give some properties of the Choquet integral with respect to the α-dimensional Hausdorff content.

Proposition 2.1. *Suppose that f and g are locally integrable nonnegative functions on $\mathbb{R}^n$ and C is a constant. Then we have*

$$\begin{aligned}
&\int Cf \,\mathrm{d}H^\alpha = C\int f\,\mathrm{d}H^\alpha,\\
&\int f\,\mathrm{d}H^\alpha = 0 \implies f=0, \quad H^\alpha\text{-a.e. on } \ \mathbb{R}^n,\\
&\int (f+g)\,\mathrm{d}H^\alpha \le 2\left(\int f\,\mathrm{d}H^\alpha + \int g\,\mathrm{d}H^\alpha\right).
\end{aligned}$$

In fact, a fundamental method in dealing with Choquet integrals with respect to Hausdorff content is that, for non-negative functions f_i, we have that

$$\int \sum_{i=1}^\infty f_i \,\mathrm{d}H^\alpha \le C\sum_{i=1}^\infty \int f_i\,\mathrm{d}H^\alpha$$

for some constant C depending on α and n. This follows from the non-trivial fact that the Choquet integral with respect to dyadic Hausdorff content is sublinear [1].

The following lemma shows that an inequality, similar to the Hölder inequality, holds for Choquet integrals with the Hausdorff content. In addition, this lemma will play an important role in the proofs of Theorems 1.1 and 1.2.

Lemma 2.1. *Let $0 < p, p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Assume that f and g are two measurable functions on $\mathbb{R}^n$. Then we have that*

$$\left(\int |fg|^p \,\mathrm{d}H^\alpha\right)^{\frac{1}{p}} \leq 2\left(\int |f|^{p_1} \,\mathrm{d}H^\alpha\right)^{\frac{1}{p_1}} \left(\int |g|^{p_2} \,\mathrm{d}H^\alpha\right)^{\frac{1}{p_2}}. \tag{2.1}$$

Proof. Indeed, it suffices to prove the case of $p = 1$ by observing that $1 = \frac{p}{p_1} + \frac{p}{p_2}$.

Let A and B be the two factors on the right side of (2.1). If $A = 0$, then $f = 0$, H^α-a.e. on $\mathbb{R}^n$; hence $fg = 0$, H^α-a.e. on $\mathbb{R}^n$, so (2.1) holds. If $A > 0$ and $B = \infty$, (2.1) is again trivial. So we need consider only the case $0 < A < \infty$ and $0 < B < \infty$. Put

$$F = \frac{f}{A}, \quad G = \frac{g}{B}.$$

This gives

$$\int F^{p_1} \,\mathrm{d}H^\alpha = \int G^{p_2} \,\mathrm{d}H^\alpha = 1.$$

Using the Young inequality, we see that

$$F(x)G(x) \leq \frac{1}{p_1} F(x)^{p_1} + \frac{1}{p_2} G(x)^{p_2} \tag{2.2}$$

for every $x \in \mathbb{R}^n$. Integrating the two sides of the inequality (2.2) yields that

$$\int FG \,\mathrm{d}H^\alpha \leq 2\left(\frac{1}{p_1} + \frac{1}{p_2}\right) = 2,$$

which is exactly the inequality (2.1). $\square$

Using a self-contained and direct argument, Orobitg-Verdera [8] proved the following strong type inequality.

Lemma 2.2. *Let $0 < \alpha < n$ such that $\frac{\alpha}{n} < p$. Then the following inequality*

$$\int (Mf)^p \,\mathrm{d}H^\alpha \leq C \int |f|^p \,\mathrm{d}H^\alpha$$

holds for some constant C depending only on α, n and p.

A well-known argument in [8] gives the following lemma which is also of great importance in proving Theorem 1.2. Here, we only give the statement of this lemma.

Lemma 2.3. *Let χ_Q be the characteristic function of the cube Q. Then for $\alpha/n < p$, we have*

$$\int M(\chi_Q)^p \,\mathrm{d}H^\alpha \le C\ell(Q)^\alpha,$$

where C depends only on α, n and p.

To prove Theorem 1.2, we need the following lemma which can be viewed as a corollary of Lemma 2.1 and Lemma 2.3 as well.

Lemma 2.4. *Let χ_{Q_1} and χ_{Q_2} be the two characteristic functions of the cubes Q_1 and Q_2, respectively. If $\frac{\alpha}{n} < p$, α_1, $\alpha_2 > 0$ and $\alpha = \alpha_1 + \alpha_2$, then we have*

$$\int \big(M(\chi_{Q_1},\chi_{Q_2})\big)^p \,\mathrm{d}H^\alpha \le C\ell(Q_1)^{\alpha_1}\ell(Q_2)^{\alpha_2},$$

where C depends only on α, n and p.

Proof. The proof is based on Lemmas 2.1 and 2.3. Actually, we conclude that

$$\begin{aligned}\int \big(M(\chi_{Q_1},\chi_{Q_2})\big)^p \,\mathrm{d}H^\alpha \le& \int \big(M(\chi_{Q_1})\big)^p \big(M(\chi_{Q_2})\big)^p \,\mathrm{d}H^\alpha \\ \le& 2\left(\int \big(M(\chi_{Q_1})\big)^{p\frac{\alpha}{\alpha_1}} \,\mathrm{d}H^\alpha\right)^{\frac{\alpha_1}{\alpha}} \left(\int \big(M(\chi_{Q_2})\big)^{p\frac{\alpha}{\alpha_2}} \,\mathrm{d}H^\alpha\right)^{\frac{\alpha_2}{\alpha}} \\ \le& C\ell(Q_1)^{\alpha\frac{\alpha_1}{\alpha}}\ell(Q_2)^{\alpha\frac{\alpha_2}{\alpha}} \\ =& C\ell(Q_1)^{\alpha_1}\ell(Q_2)^{\alpha_2},\end{aligned}$$

which is our desired result. □

We need the following auxiliary inequality to pass from integration with the Lebesgue measure to the Hausdorff content:

$$\int f(x)\,\mathrm{d}x \le \frac{n}{\alpha}\left(\int f^{\frac{\alpha}{n}} \,\mathrm{d}H^\alpha\right)^{\frac{n}{\alpha}} \tag{2.3}$$

for $f \ge 0$ and $0 < \alpha \le n$; see [8].

Besides, the key to the proof of Theorem 1.3 is a covering lemma. The version we need is also the one employed by Orobitg-Verdera [8], and is due to Melnikov [7].

Lemma 2.5. *Let $0 < \alpha < n$ and $\{Q_j\}_j$ be a family of non-overlapping dyadic cubes. Then there exists a subfamily $\{Q_{j_k}\}_k$ satisfying the following two properties:*

(i) $\sum_{Q_{j_k}\subset Q} \ell(Q_{j_k})^\alpha \le 2\ell(Q)^\alpha$, *for each dyadic cube Q,*

(ii) $H^\alpha\left(\bigcup_j Q_j\right) \leq 2\sum_k \ell(Q_{j_k})^\alpha$.

Based on part (i) of Lemma 2.5, one can show that the following inequality holds with some constant C independent of $f\geq 0$,

$$\sum_k \int_{Q_{j_k}} f\,\mathrm{d}H^\alpha \leq C\int_{\bigcup_k Q_{j_k}} f\,\mathrm{d}H^\alpha.$$

3 Proof of Theorems 1.1-1.3

Proof of Theorem 1.1. Without loss of generality, we assume $f_1\geq 0$ and $f_2\geq 0$. Since

$$\begin{aligned}
M(f_1,f_2)(x) =&\sup_{Q\ni x}\frac{1}{|Q|^2}\left(\int_Q f_1(y)\,\mathrm{d}y\right)\left(\int_Q f_2(y)\,\mathrm{d}y\right)\\
\leq&\left(\sup_{Q\ni x}\frac{1}{|Q|}\int_Q f_1(y)\,\mathrm{d}y\right)\left(\sup_{Q\ni x}\frac{1}{|Q|}\int_Q f_2(y)\,\mathrm{d}y\right)\\
=&(Mf_1)(x)(Mf_2)(x),
\end{aligned}$$

it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}
\left(\int \left(M(f_1,f_2)\right)^p\mathrm{d}H^\alpha\right)^{\frac{1}{p}} \leq&\left(\int (Mf_1)^p(Mf_2)^p\,\mathrm{d}H^\alpha\right)^{\frac{1}{p}}\\
\leq&2\left(\int (Mf_1)^{p_1}\,\mathrm{d}H^\alpha\right)^{\frac{1}{p_1}}\left(\int (Mf_2)^{p_2}\,\mathrm{d}H^\alpha\right)^{\frac{1}{p_2}}\\
\leq&C\left(\int f_1^{p_1}\,\mathrm{d}H^\alpha\right)^{\frac{1}{p_1}}\left(\int f_2^{p_2}\,\mathrm{d}H^\alpha\right)^{\frac{1}{p_2}}.
\end{aligned}$$

This finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Without loss of generality, we also assume $f_1\geq 0$ and $f_2\geq 0$.

For each integer k, let $\{Q^k_{j,i}\}_j$ be a family of non-overlapping dyadic cubes $Q^k_{j,i}$ such that

$$\begin{aligned}
&\left\{x\colon 2^k<f_i(x)\leq 2^{k+1}\right\}\subset\bigcup_j Q^k_{j,i},\\
&\sum_j \ell(Q^k_{j,i})^\alpha\leq 2H^\alpha_d\left\{x\colon 2^k<f_i(x)\leq 2^{k+1}\right\},
\end{aligned}$$

where $i=1,\,2$. Set

$$g_i=\sum_k 2^{(k+1)p}\chi_{A_{k,i}},$$

where $A_{k,i}=\bigcup_j Q^k_{j,i}$. Thus we have $f_i^p\le g_i$.

Assume first that $1\le p$. Then a direct calculation yields that

$$\begin{aligned}
\big(M(f_1,f_2)(x)\big)^p=&\sup_{Q\ni x}\prod_{i=1}^2\frac{1}{|Q|^p}\left(\int_Q f_i(y)\,\mathrm{d}y\right)^p\\
\le&\sup_{Q\ni x}\prod_{i=1}^2\frac{1}{|Q|^p}\left(\left(\int_Q (f_i(y))^p\ \mathrm{d}y\right)^{\frac{1}{p}}\left(\int_Q 1\,\mathrm{d}y\right)^{\frac{p-1}{p}}\right)^p\\
=&\sup_{Q\ni x}\prod_{i=1}^2\frac{1}{|Q|}\int_Q (f_i(y))^p\ \mathrm{d}y\\
=&M(f_1^p,f_2^p)(x)\\
\le&M(g_1,g_2)(x).
\end{aligned}$$

Moreover, it can be derived that

$$\begin{aligned}
&M(g_1,g_2)(x)\\
=&\sup_{Q\ni x}\prod_{i=1}^2\frac{1}{|Q|}\sum_k 2^{(k+1)p}\int_Q\chi_{A_{k,i}}(y)\,\mathrm{d}y\\
=&\sup_{Q\ni x}\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\left(\frac{1}{|Q|}\int_Q\chi_{A_{k,1}}(y)\,\mathrm{d}y\right)\left(\frac{1}{|Q|}\int_Q\chi_{A_{h,2}}(y)\,\mathrm{d}y\right)\\
=&\sup_{Q\ni x}\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\left(\frac{1}{|Q|}\int_Q\sum_v\chi_{Q^k_{v,1}}(y)\,\mathrm{d}y\right)\left(\frac{1}{|Q|}\int_Q\sum_j\chi_{Q^h_{j,2}}(y)\,\mathrm{d}y\right)\\
=&\sup_{Q\ni x}\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\sum_v\sum_j\left(\frac{1}{|Q|}\int_Q\chi_{Q^k_{v,1}}(y)\,\mathrm{d}y\right)\left(\frac{1}{|Q|}\int_Q\chi_{Q^h_{j,2}}(y)\,\mathrm{d}y\right)\\
\le&\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\sum_v\sum_j M(\chi_{Q^k_{v,1}}\chi_{Q^h_{j,2}})(x).
\end{aligned}$$

Combining the above two inequalities, one can immediately obtain that

$$\big(M(f_1,f_2)(x)\big)^p\le\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\sum_v\sum_j M(\chi_{Q^k_{v,1}},\chi_{Q^h_{j,2}})(x).$$

Hence an application of Lemma 2.4 to $\{M(\chi_{Q^k_{v,1}},\chi_{Q^h_{j,2}})\}_{v,j,k,h}$ gives that

$$\begin{aligned}
\int\big(M(f_1,f_2)\big)^p\,\mathrm{d}H^\alpha\le&C\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\sum_v\sum_j\int M(\chi_{Q^k_{v,1}},\chi_{Q^h_{j,2}})\,\mathrm{d}H^\alpha\\
\le&C\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\sum_v\sum_j\ell(Q^k_{v,1})^{\alpha_1}\ell(Q^h_{j,2})^{\alpha_2}\\
=&C\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\left(\sum_v\ell(Q^k_{v,1})^{\alpha_1}\right)\left(\sum_j\ell(Q^h_{j,2})^{\alpha_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{k}\sum_{h}2^{(k+1)p}2^{(h+1)p}H^{\alpha_1}\{x\colon 2^k<f_1\leq 2^{k+1}\}H^{\alpha_2}\{x\colon 2^h<f_2\leq 2^{h+1}\}\\
&=C\left(\sum_{k}2^{(k+1)p}H^{\alpha_1}\{x\colon 2^k<f_1\leq 2^{k+1}\}\right)\left(\sum_{h}2^{(h+1)p}H^{\alpha_2}\{x\colon 2^h<f_2\leq 2^{h+1}\}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{k}2^{(k+1)p}H^{\alpha_1}\{x\colon 2^k<f_1\leq 2^{k+1}\}\\
\leq&\sum_{k}2^{(k+1)p}H^{\alpha_1}\{x\colon f_1>2^k\}\\
=&\sum_{k}\frac{2^{2p}}{2^p-1}2^{(k-1)p}\left(2^p-1\right)H^{\alpha_1}\{x\colon f_1>2^k\}\\
=&\frac{2^{2p}}{2^p-1}\sum_{k}\int_{2^{(k-1)p}}^{2^{kp}}H^{\alpha_1}\{x\colon f_1^p>2^{kp}\}\,\mathrm{d}t\\
\leq&\frac{2^{2p}}{2^p-1}\sum_{k}\int_{2^{(k-1)p}}^{2^{kp}}H^{\alpha_1}\{x\colon f_1^p>t\}\,\mathrm{d}t\\
=&\frac{2^{2p}}{2^p-1}\int_{0}^{\infty}H^{\alpha_1}\{x\colon f_1^p>t\}\,\mathrm{d}t\\
=&\frac{2^{2p}}{2^p-1}\int f_1^p\,\mathrm{d}H^{\alpha_1}.
\end{aligned}
$$

By the same approach as above, we can deduce that

$$
\sum_{h}2^{(h+1)p}H^{\alpha_2}\{x\colon 2^h<f_2(x)\leq 2^{h+1}\}\leq\frac{2^{2p}}{2^p-1}\int f_2^p\,\mathrm{d}H^{\alpha_2}.
$$

Finally, we can therefore show that

$$
\int\left(M(f_1,f_2)\right)^p\mathrm{d}H^{\alpha}\leq C\left(\int f_1^p\,\mathrm{d}H^{\alpha_1}\right)\left(\int f_2^p\,\mathrm{d}H^{\alpha_2}\right),
$$

which leads to the proof of the case for $p\geq 1$.

Assume now that $\frac{\alpha}{n}<p<1$. Since $f_i\leq\sum_k 2^{k+1}\chi_{A_{k,i}}$ with $i=1,2$, it implies that

$$
M(f_1,f_2)\leq\sum_{k}\sum_{h}2^{k+1}2^{h+1}\sum_{v}\sum_{j}M(\chi_{Q^k_{v,1}},\chi_{Q^h_{j,2}}).
$$

Since $p<1$, we deduce that

$$
\left(M(f_1,f_2)\right)^p\leq\sum_{k}\sum_{h}2^{(k+1)p}2^{(h+1)p}\sum_{v}\sum_{j}\left(M(\chi_{Q^k_{v,1}},\chi_{Q^h_{j,2}})\right)^p,
$$

and hence

$$\begin{aligned}\int \big(M(f_1,f_2)\big)^p \, \mathrm{d}H^\alpha \leq & C\sum_k\sum_h 2^{(k+1)p}2^{(h+1)p}\sum_v\sum_j \ell(Q^k_{v,1})^{\alpha_1}\ell(Q^h_{j,2})^{\alpha_2} \\ \leq & C\left(\int f_1^p \, \mathrm{d}H^{\alpha_1}\right)\left(\int f_2^p \, \mathrm{d}H^{\alpha_2}\right).\end{aligned}$$

This completes the proof for the case $\frac{\alpha}{n} < p < 1$. □

Proof of Theorem 1.3. If one of the two integrals on the right-hand side of (1.5a) and (1.5b) is infinite, then there is nothing to prove. Hence, we assume that

$$\int |f_i|^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha < \infty$$

for $i = 1, 2$. Then by (2.3) we obatin that $f_i \in L^1(\mathbb{R}^n)$.

The proof of (1.5a) will be given first. We assume again, without loss of generality, that $f_1 \geq 0$ and $f_2 \geq 0$. Given $t > 0$, let $\{Q_j\}_j$ be the family of maximal dyadic cubes Q_j such that

$$\frac{1}{|Q_j|^2}\left(\int_{Q_j} f_1(y)\,\mathrm{d}y\right)\left(\int_{Q_j} f_2(y)\,\mathrm{d}y\right) > t. \tag{3.1}$$

Then we can obtain that

$$\{x\colon M_d(f_1,f_2)(x) > t\} = \bigcup_j Q_j.$$

For each $j \in \mathbb{N}$, we deduce from (2.3) and (3.1) that

$$\begin{aligned}\ell(Q_j)^{2\alpha} \leq & \left(\frac{1}{t}\left(\int_{Q_j} f_1(y)\,\mathrm{d}y\right)\left(\int_{Q_j} f_2(y)\,\mathrm{d}y\right)\right)^{\frac{\alpha}{n}} \\ \leq & Ct^{-\frac{\alpha}{n}}\left(\int_{Q_j} f_1^{\frac{\alpha}{n}}\,\mathrm{d}H^\alpha\right)\left(\int_{Q_j} f_2^{\frac{\alpha}{n}}\,\mathrm{d}H^\alpha\right).\end{aligned}$$

Applying Lemma 2.5 to the $\{Q_j\}_j$ we obatin some subfamily $\{Q_{j_k}\}_k$ for which we can write

$$\begin{aligned}&\big(H^\alpha\{x\colon M_d(f_1,f_2)(x) > t\}\big)^2 \\ =&\left(H^\alpha\left(\bigcup_j Q_j\right)\right)^2 \leq 4\left(\sum_k \ell(Q_{j_k})^\alpha\right)^2 \\ \leq & C\left(\sum_k t^{-\frac{\alpha}{2n}}\left(\int_{Q_{j_k}} f_1^{\frac{\alpha}{n}}\,\mathrm{d}H^\alpha\right)^{\frac{1}{2}}\left(\int_{Q_{j_k}} f_2^{\frac{\alpha}{n}}\,\mathrm{d}H^\alpha\right)^{\frac{1}{2}}\right)^2\end{aligned}$$

$$
\begin{aligned}
&\leq C t^{-\frac{\alpha}{n}} \left(\left(\sum_k \int_{Q_{j_k}} f_1^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha \right)^{\frac{1}{2}} \left(\sum_k \int_{Q_{j_k}} f_2^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha \right)^{\frac{1}{2}} \right)^2 \\
&\leq C t^{-\frac{\alpha}{n}} \left(\int f_1^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha \right) \left(\int f_2^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha \right).
\end{aligned}
$$

This finishes the proof of inequality (1.5a). Next, we give the proof of the second inequality by using an analogous process.

Given $t > 0$. Let $\{Q_{j,i}\}_j$ be the family of maximal dyadic cubes $Q_{j,i}$ such that

$$
\frac{1}{|Q_{j,i}|} \int_{Q_{j,i}} f_i(y) \, \mathrm{d}y > t^{\beta_i},
$$

where $i = 1, 2$. Then we can show that

$$
\{x \colon M_d(f_1, f_2)(x) > t\} \subset \bigcup_i \bigcup_j Q_{j,i}.
$$

In fact, if we choose x such that $M_d(f_1, f_2)(x) > t$, i.e.,

$$
\sup_{Q \ni x} \frac{1}{|Q|^2} \left(\int_Q f_1(y) \, \mathrm{d}y \right) \left(\int_Q f_2(y) \, \mathrm{d}y \right) > t,
$$

where the supremum is taken on the family of dyadic cubes containing x, then by the definition of the supremum there exists a dyadic cube $Q \ni x$ satisfying

$$
\left(\frac{1}{|Q|} \int_Q f_1(y) \, \mathrm{d}y \right) \left(\frac{1}{|Q|} \int_Q f_2(y) \, \mathrm{d}y \right) > t.
$$

It follows that

$$
\frac{1}{|Q|} \int_Q f_1(y) \, \mathrm{d}y > t^{\beta_1}
$$

or

$$
\frac{1}{|Q|} \int_Q f_2(y) \, \mathrm{d}y > t^{\beta_2}
$$

must hold. Then we can deduce from the maximum of $Q_{j,i}$ that

$$
x \in Q \subset \bigcup_i \bigcup_j Q_{j,i}.
$$

By inequality (2.3), we have that

$$
\ell(Q_{j,i})^\alpha \leq \left(\frac{1}{t^{\beta_i}} \int_{Q_{j,i}} f_i(y) \, \mathrm{d}y \right)^{\frac{\alpha}{n}} \leq C t^{-\beta_i \frac{\alpha}{n}} \int_{Q_{j,i}} f_i^{\frac{\alpha}{n}} \, \mathrm{d}H^\alpha.
$$

Applying Lemma 2.5 to the $\{Q_{j,i}\}$ we obatin some subfamily $\{Q_{j_k,i}\}$ for which one can write

$$\begin{aligned}&H^\alpha\{x\colon M_d(f_1,f_2)(x)>t\}\\ \leq&\sum_i H^\alpha\left(\bigcup_j Q_{j,i}\right)\leq 2\sum_i\sum_k \ell(Q_{j_k,i})^\alpha\\ \leq& C\sum_i t^{-\beta_i\frac{\alpha}{n}}\sum_k\int_{Q_{j_k,i}} f_i^{\frac{\alpha}{n}}\,\mathrm{d}H^\alpha\leq C\sum_i t^{-\beta_i\frac{\alpha}{n}}\int f_i^{\frac{\alpha}{n}}\,\mathrm{d}H^\alpha.\end{aligned}$$

This finishes our proof. □

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References

[1] D. R. Adams, A note on Choquet integrals with respect to Hausdorff capacity, Function Spaces and Applications, Lund 1986, Lecture Notes in Math., 1302, Springer-Verlag, 115–124, 1988.
[2] D. R. Adams, Choquet integrals in potential theory, Publ. Mat., 42(1) (1998), 3–66.
[3] D. R. Adams and L. I. Hedberg, Function Spaces and Potential Theory, Springer, Heidelberg, 1996.
[4] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math., 80(4) (1958), 921–930.
[5] L. Grafakos, Classical Fourier Analysis, volume 249 of Graduate Texts in Mathematics, Springer, New York, 3rd edition, 2014.
[6] L. Hormander, L^p estimate for (pluri-) subharmonic functions, Math. Scand., 20(1) (1967), 65–78.
[7] M. S. Melnikov, Metric properties of analytic α-capacity and approximation of analytic functions with a Hölder condition by rational functions, Math. USSR Sbornik, 8(1) (1969), 115–124.
[8] J. Orobitg and J. Verdera, Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator, Bull. London Math. Soc., 30(2) (1998), 145–150.
[9] W. Rudin, Real and Complex Analysis, New York: McGraw-Hill Publishing Co., 3rd edition, 1987.
[10] L. Tang, Choquet integrals, weighted Hausdorff content and maximal operators, Georgian Math. J., 19(3) (2011), 587–596.