Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 1 (2023), pp. 83-92

DOI: 10.4208/ata.OA-2021-0027
March 2023

On a Right Inverse of a Polynomial of the Laplace in the Weighted Hilbert Space $L^2(\mathbb{R}^n, e^{-|x|^2})$

Shaoyu Dai[1,*], Yang Liu[2] and Yifei Pan[3]

[1] *Department of Mathematics, Jinling Institute of Technology, Nanjing, Jiangsu 211169, China*
[2] *Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China*
[3] *Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, 46805-1499, USA*

Received 22 August 2021; Accepted (in revised version) 8 February 2022

Abstract. Let $P(\Delta)$ be a polynomial of the Laplace operator

$$\Delta=\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} \quad \text{on} \quad \mathbb{R}^n.$$

We prove the existence of a bounded right inverse of the differential operator $P(\Delta)$ in the weighted Hilbert space with the Gaussian measure, i.e., $L^2(\mathbb{R}^n, e^{-|x|^2})$.

Key Words: Laplace operator, polynomial, right inverse, weighted Hilbert space, Gaussian measure.

AMS Subject Classifications: 35A01, 35A25, 35D30, 35J05

1 Introduction

In this paper, we study the right inverse of the polynomial differential operator of the Laplace

$$P(\Delta)=\Delta^m+a_{m-1}\Delta^{m-1}+\cdots+a_1\Delta+a_0,$$

where $a_0, a_1, \cdots, a_{m-1}$ are complex numbers. We prove the existence of global weak solutions of the equation $P(\Delta)u=f$ in the weighted Hilbert space $L^2(\mathbb{R}^n, e^{-|x|^2})$ by the following result of L^2 estimates.

*Corresponding author. *Email addresses:* `dsy@jit.edu.cn` (S. Dai), `liuyang@zjnu.edu.cn` (Y. Liu), `pan@pfw.edu` (Y. Pan)

Theorem 1.1. *For each* $f \in L^2(\mathbb{R}^n, e^{-|x|^2})$*, there exists a weak solution* $u \in L^2(\mathbb{R}^n, e^{-|x|^2})$ *solving the equation*

$$P(\Delta)u = f$$

in $\mathbb{R}^n$ *with the norm estimate*

$$\int_{\mathbb{R}^n} |u|^2 e^{-|x|^2} dx \leq \frac{1}{(8n)^m} \int_{\mathbb{R}^n} |f|^2 e^{-|x|^2} dx.$$

The novelty of Theorem 1.1 is that the differential operator $P(\Delta)$ has a bounded right inverse

$$\begin{aligned} &Q : L^2(\mathbb{R}^n, e^{-|x|^2}) \longrightarrow L^2(\mathbb{R}^n, e^{-|x|^2}), \\ &P(\Delta)Q = I, \end{aligned}$$

with the norm estimate

$$\|Q\| \leq \frac{1}{(8n)^{\frac{m}{2}}}.$$

In particular, the Laplace operator Δ has a bounded right inverse

$$Q_0 : L^2(\mathbb{R}^n, e^{-|x|^2}) \longrightarrow L^2(\mathbb{R}^n, e^{-|x|^2}),$$

which, to the best of our knowledge, appears to be even new.

As a result, a natural question could be if Theorem 1.1 would be true for more general differential operators. For related results, see [1–4, 6]. The method employed in this paper was motivated from the Hörmander L^2 method [5] for Cauchy-Riemann equations in several complex variables.

The organization of this paper is as follows. In Section 2, we will prove several key lemmas based on functional analysis, while the proof of Theorem 1.1 will be given in Section 3. In Section 4, we will give some further remarks.

2 Several lemmas

In this section, we will prove the following lemma, which is key for the proof of Theorem 1.1.

Lemma 2.1. *Let* ξ *be a complex number. For each* $f \in L^2(\mathbb{R}^n, e^{-|x|^2})$*, there exists a weak solution* $u \in L^2(\mathbb{R}^n, e^{-|x|^2})$ *solving the equation*

$$\Delta u + \xi u = f \quad \text{in } \mathbb{R}^n,$$

with the norm estimate

$$\int_{\mathbb{R}^n} |u|^2 e^{-|x|^2} dx \leq \frac{1}{8n} \int_{\mathbb{R}^n} |f|^2 e^{-|x|^2} dx.$$

For the proof of Lemma 2.1, we need some preparation.

First we give some notations. Here, let $L^2_{loc}(\mathbb{R}^n)$ be the set of all locally square-integrable complex-valued functions on $\mathbb{R}^n$. We consider the weighted Hilbert space

$$L^2(\mathbb{R}^n, e^{-\varphi}) = \left\{ f \middle| f \in L^2_{loc}(\mathbb{R}^n);\ \int_{\mathbb{R}^n} |f|^2 e^{-\varphi} dx < +\infty \right\},$$

where φ is a nonnegative function on $\mathbb{R}^n$. We denote the weighted inner product for $f, g \in L^2(\mathbb{R}^n, e^{-\varphi})$ by

$$\langle f, g \rangle_\varphi = \int_{\mathbb{R}^n} \overline{f} g e^{-\varphi} dx$$

and the weighted norm of $f \in L^2(\mathbb{R}^n, e^{-\varphi})$ by

$$\|f\|_\varphi = \sqrt{\langle f, f \rangle_\varphi}.$$

Let $C_0^\infty(\mathbb{R}^n)$ denote the set of all smooth complex-valued functions with compact support. For $u, f \in L^2_{loc}(\mathbb{R}^n)$, we say that f is the Laplace of u in the weak sense, written $\Delta u = f$, provided

$$\int_{\mathbb{R}^n} u \Delta \phi dx = \int_{\mathbb{R}^n} f \phi dx$$

for all test functions $\phi \in C_0^\infty(\mathbb{R}^n)$.

In the following, let φ be a smooth and nonnegative function on $\mathbb{R}^n$ and ξ be a complex number throughout this section. For every $\phi \in C_0^\infty(\mathbb{R}^n)$, we first define the following formal adjoint of Δ with respect to the weighted inner product in $L^2(\mathbb{R}^n, e^{-\varphi})$. Let $u \in L^2_{loc}(\mathbb{R}^n)$. We calculate as follows:

$$\begin{aligned} \langle \phi, \Delta u \rangle_\varphi &= \int_{\mathbb{R}^n} \overline{\phi} \Delta u e^{-\varphi} dx = \int_{\mathbb{R}^n} u \Delta \left(\overline{\phi} e^{-\varphi} \right) dx \\ &= \int_{\mathbb{R}^n} e^{\varphi} u \Delta \left(\overline{\phi} e^{-\varphi} \right) e^{-\varphi} dx = \left\langle \overline{e^{\varphi} \Delta \left(\overline{\phi} e^{-\varphi} \right)}, u \right\rangle_\varphi \\ &=: \left\langle \Delta^*_\varphi \phi, u \right\rangle_\varphi, \end{aligned}$$

where $\Delta^*_\varphi \phi = \overline{e^{\varphi} \Delta \left(\overline{\phi} e^{-\varphi} \right)}$ is the formal adjoint of Δ with domain in $C_0^\infty(\mathbb{R}^n)$. Let $(\Delta + \xi)^*_\varphi$ be the formal adjoint of $\Delta + \xi$ with domain in $C_0^\infty(\mathbb{R}^n)$. Note that $I^*_\varphi = I$, where I is the identity operator. Then

$$(\Delta + \xi)^*_\varphi = \Delta^*_\varphi + \overline{\xi}.$$

Let ∇ be the gradient operator on $\mathbb{R}^n$. Now we give several lemmas based on functional analysis.

Lemma 2.2. *For each $f \in L^2(\mathbb{R}^n, e^{-\varphi})$, there exists a global weak solution $u \in L^2(\mathbb{R}^n, e^{-\varphi})$ solving the equation*

$$\Delta u + \xi u = f$$

in $\mathbb{R}^n$ *with the norm estimate*

$$\|u\|_\varphi^2 \le c$$

if and only if

$$|\langle f,\phi\rangle_\varphi|^2 \le c\left\|(\Delta+\xi)^*_\varphi\phi\right\|^2_\varphi, \quad \forall\phi\in C_0^\infty(\mathbb{R}^n),$$

where c is a constant.

Proof. Let $\Delta+\xi=H$. Then $(\Delta+\xi)^*_\varphi=H^*_\varphi$.

(Necessity). For every $\phi\in C_0^\infty(\mathbb{R}^n)$, from the definition of H^*_φ and Cauchy-Schwarz inequality, we have

$$|\langle f,\phi\rangle_\varphi|^2=|\langle Hu,\phi\rangle_\varphi|^2=\left|\left\langle u,H^*_\varphi\phi\right\rangle_\varphi\right|^2\le\|u\|^2_\varphi\left\|H^*_\varphi\phi\right\|^2_\varphi\le c\left\|H^*_\varphi\phi\right\|^2_\varphi.$$

(Sufficiency). Consider the subspace

$$E=\left\{H^*_\varphi\phi\mid\phi\in C_0^\infty(\mathbb{R}^n)\right\}\subset L^2(\mathbb{R}^n,e^{-\varphi}).$$

Define a linear functional $L_f: E\to\mathbb{C}$ by

$$L_f\left(H^*_\varphi\phi\right)=\langle f,\phi\rangle_\varphi=\int_{\mathbb{R}^n}\overline{f}\phi e^{-\varphi}dx.$$

Since

$$\left|L_f\left(H^*_\varphi\phi\right)\right|=|\langle f,\phi\rangle_\varphi|\le\sqrt{c}\left\|H^*_\varphi\phi\right\|_\varphi,$$

then L_f is a bounded functional on E. Let $\overline{E}$ be the closure of E with respect to the norm $\|\cdot\|_\varphi$ of $L^2(\mathbb{R}^n,e^{-\varphi})$. Note that $\overline{E}$ is a Hilbert subspace of $L^2(\mathbb{R}^n,e^{-\varphi})$. So by Hahn-Banach's extension theorem, L_f can be extended to a linear functional $\widehat{L}_f$ on $\overline{E}$ such that

$$\left|\widehat{L}_f(g)\right|\le\sqrt{c}\,\|g\|_\varphi,\quad\forall g\in\overline{E}. \tag{2.1}$$

Using the Riesz representation theorem for $\widehat{L}_f$, there exists a unique $u_0\in\overline{E}$ such that

$$\widehat{L}_f(g)=\langle u_0,g\rangle_\varphi,\quad\forall g\in\overline{E}. \tag{2.2}$$

Now we prove $Hu_0=f$. For every $\phi\in C_0^\infty(\mathbb{R}^n)$, apply $g=H^*_\varphi\phi$ in (2.2). Then

$$\widehat{L}_f\left(H^*_\varphi\phi\right)=\left\langle u_0,H^*_\varphi\phi\right\rangle_\varphi=\langle Hu_0,\phi\rangle_\varphi.$$

Note that

$$\widehat{L}_f\left(H^*_\varphi\phi\right)=L_f\left(H^*_\varphi\phi\right)=\langle f,\phi\rangle_\varphi.$$

Therefore,

$$\langle Hu_0, \phi\rangle_\varphi = \langle f, \phi\rangle_\varphi, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n),$$

i.e.,

$$\int_{\mathbb{R}^n} \overline{Hu_0}\phi e^{-\varphi} dx = \int_{\mathbb{R}^n} \overline{f}\phi e^{-\varphi} dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n).$$

Thus, $Hu_0 = f$. Next we give a bound for the norm of u_0. Let $g = u_0$ in (2.1) and (2.2). Then we have

$$\|u_0\|_\varphi^2 = |\langle u_0, u_0\rangle_\varphi| = \left|\widehat{L}_f(u_0)\right| \le \sqrt{c}\,\|u_0\|_\varphi.$$

Therefore, $\|u_0\|_\varphi^2 \le c$. Note that $u_0 \in \overline{E}$ and $\overline{E} \subset L^2(\mathbb{R}^n, e^{-\varphi})$. Then $u_0 \in L^2(\mathbb{R}^n, e^{-\varphi})$. Let $u = u_0$. So there exists $u \in L^2(\mathbb{R}^n, e^{-\varphi})$ such that $Hu = f$ with $\|u\|_\varphi^2 \le c$. The proof is completed. □

Lemma 2.3. *For every $\phi \in C_0^\infty(\mathbb{R}^n)$, we have*

$$\left\|(\Delta+\xi)_\varphi^*\phi\right\|_\varphi^2 = \|(\Delta+\xi)\phi\|_\varphi^2 + \left\langle \phi, \Delta\left(\Delta_\varphi^*\phi\right) - \Delta_\varphi^*(\Delta\phi)\right\rangle_\varphi.$$

Proof. Let $\Delta + \xi = H$. Then $(\Delta+\xi)_\varphi^* = H_\varphi^*$. For every $\phi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left\|H_\varphi^*\phi\right\|_\varphi^2 &= \left\langle H_\varphi^*\phi, H_\varphi^*\phi\right\rangle_\varphi = \left\langle \phi, HH_\varphi^*\phi\right\rangle_\varphi \\
&= \left\langle \phi, H_\varphi^*H\phi\right\rangle_\varphi + \left\langle \phi, HH_\varphi^*\phi - H_\varphi^*H\phi\right\rangle_\varphi \\
&= \langle H\phi, H\phi\rangle_\varphi + \left\langle \phi, HH_\varphi^*\phi - H_\varphi^*H\phi\right\rangle_\varphi \\
&= \|H\phi\|_\varphi^2 + \left\langle \phi, HH_\varphi^*\phi - H_\varphi^*H\phi\right\rangle_\varphi.
\end{aligned} \tag{2.3}$$

Note that

$$\begin{aligned}
HH_\varphi^*\phi &= (\Delta+\xi)(\Delta+\xi)_\varphi^*\phi \\
&= (\Delta+\xi)\left(\Delta_\varphi^*\phi + \overline{\xi}\phi\right) \\
&= \Delta\left(\Delta_\varphi^*\phi\right) + \overline{\xi}\Delta\phi + \xi\Delta_\varphi^*\phi + |\xi|^2\phi
\end{aligned}$$

and

$$\begin{aligned}
H_\varphi^*H\phi &= (\Delta+\xi)_\varphi^*(\Delta+\xi)\phi \\
&= \left(\Delta_\varphi^* + \overline{\xi}\right)(\Delta\phi + \xi\phi) \\
&= \Delta_\varphi^*(\Delta\phi) + \xi\Delta_\varphi^*\phi + \overline{\xi}\Delta\phi + |\xi|^2\phi.
\end{aligned}$$

Then

$$HH_{\varphi}^{*}\phi - H_{\varphi}^{*}H\phi = \Delta\left(\Delta_{\varphi}^{*}\phi\right) - \Delta_{\varphi}^{*}\left(\Delta\phi\right). \tag{2.4}$$

So by (2.3) and (2.4), we have

$$\left\|H_{\varphi}^{*}\phi\right\|_{\varphi}^{2} = \|H\phi\|_{\varphi}^{2} + \left\langle \phi, \Delta\left(\Delta_{\varphi}^{*}\phi\right) - \Delta_{\varphi}^{*}\left(\Delta\phi\right)\right\rangle_{\varphi}.$$

This lemma is proved. □

Lemma 2.4. *Let $\varphi = |x|^2$. Then for every $\phi \in C_0^\infty(\mathbb{R}^n)$, we have*

$$\left\langle \phi, \Delta\left(\Delta_{\varphi}^{*}\phi\right) - \Delta_{\varphi}^{*}\left(\Delta\phi\right)\right\rangle_{\varphi} = 8n\|\phi\|_{\varphi}^{2} + 8\|\nabla\phi\|_{\varphi}^{2}.$$

Proof. Note that for any smooth functions α and β on $\mathbb{R}^n$, the following formula holds

$$\Delta(\alpha\beta) = \beta\Delta\alpha + \alpha\Delta\beta + 2\nabla\alpha\cdot\nabla\beta.$$

Then for every $\phi \in C_0^\infty(\mathbb{R}^n)$, by the definition of Δ_{φ}^{*}, we have

$$\Delta_{\varphi}^{*}\phi = \overline{e^{\varphi}\Delta\left(\overline{\phi}e^{-\varphi}\right)} = \Delta\phi + \phi|\nabla\varphi|^2 - \phi\Delta\varphi - 2\nabla\phi\cdot\nabla\varphi. \tag{2.5}$$

From (2.5), we have

$$\begin{aligned}\Delta\left(\Delta_{\varphi}^{*}\phi\right) =& \Delta^2\phi + \Delta(\phi|\nabla\varphi|^2) - \Delta(\phi\Delta\varphi) - 2\Delta(\nabla\phi\cdot\nabla\varphi)\\ =& \Delta^2\phi + \Delta\phi|\nabla\varphi|^2 + \phi\Delta(|\nabla\varphi|^2) + 2\nabla\phi\cdot\nabla(|\nabla\varphi|^2)\\ &- \Delta\phi\Delta\varphi - \phi\Delta^2\varphi - 2\nabla\phi\cdot\nabla(\Delta\varphi) - 2\Delta(\nabla\phi\cdot\nabla\varphi)\end{aligned}$$

and

$$\Delta_{\varphi}^{*}(\Delta\phi) = \Delta^2\phi + \Delta\phi|\nabla\varphi|^2 - \Delta\phi\Delta\varphi - 2\nabla(\Delta\phi)\cdot\nabla\varphi.$$

Then

$$\begin{aligned}\Delta\left(\Delta_{\varphi}^{*}\phi\right) - \Delta_{\varphi}^{*}\left(\Delta\phi\right) =& \phi\Delta(|\nabla\varphi|^2) + 2\nabla\phi\cdot\nabla(|\nabla\varphi|^2) - \phi\Delta^2\varphi\\ &- 2\nabla\phi\cdot\nabla(\Delta\varphi) - 2\Delta(\nabla\phi\cdot\nabla\varphi) + 2\nabla(\Delta\phi)\cdot\nabla\varphi.\end{aligned} \tag{2.6}$$

Let $\varphi = |x|^2$. We have $\nabla\varphi = 2x$, $\Delta\varphi = 2n$, $|\nabla\varphi|^2 = 4|x|^2$, $\nabla(|\nabla\varphi|^2) = 8x$, $\Delta(|\nabla\varphi|^2) = 8n$. Then by (2.6) and the following formula

$$\Delta(\nabla\phi\cdot x) = \nabla(\Delta\phi)\cdot x + 2\Delta\phi, \quad \forall\phi \in C_0^\infty(\mathbb{R}^n),$$

we get

$$\Delta\left(\Delta_{\varphi}^{*}\phi\right) - \Delta_{\varphi}^{*}\left(\Delta\phi\right) = 8n\phi + 16(\nabla\phi\cdot x) - 8\Delta\phi.$$

Consequently,

$$\begin{aligned}
&\left\langle \phi, \Delta\left(\Delta_{\varphi}^{*}\phi\right)-\Delta_{\varphi}^{*}\left(\Delta\phi\right)\right\rangle_{\varphi}\\
=&\left\langle \phi, 8n\phi+16(\nabla\phi\cdot x)-8\Delta\phi\right\rangle_{\varphi}\\
=&8n\|\phi\|_{\varphi}^{2}+8\left\langle \phi, 2(\nabla\phi\cdot x)-\Delta\phi\right\rangle_{\varphi}.
\end{aligned}$$

Note, as the key step of the proof, that

$$\begin{aligned}
&\langle\phi,2(\nabla\phi\cdot x)-\Delta\phi\rangle_{\varphi}=\int_{\mathbb{R}^n}\overline{\phi}(2(\nabla\phi\cdot x)-\Delta\phi)e^{-\varphi}dx\\
=&\int_{\mathbb{R}^n}\overline{\phi}\sum_{j=1}^{n}\left(2x_j\frac{\partial\phi}{\partial x_j}-\frac{\partial^2\phi}{\partial x_j\partial x_j}\right)e^{-|x|^2}dx=-\int_{\mathbb{R}^n}\overline{\phi}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}e^{-|x|^2}\right)dx\\
=&-\sum_{j=1}^{n}\int_{\mathbb{R}^n}\overline{\phi}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}e^{-|x|^2}\right)dx=\sum_{j=1}^{n}\int_{\mathbb{R}^n}\frac{\partial\overline{\phi}}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}e^{-|x|^2}\right)dx\\
=&\sum_{j=1}^{n}\int_{\mathbb{R}^n}\left|\frac{\partial\phi}{\partial x_j}\right|^2e^{-|x|^2}dx=\int_{\mathbb{R}^n}|\nabla\phi|^2e^{-|x|^2}dx=\|\nabla\phi\|_{\varphi}^2.
\end{aligned}$$

Then

$$\left\langle \phi, \Delta\left(\Delta_{\varphi}^{*}\phi\right)-\Delta_{\varphi}^{*}\left(\Delta\phi\right)\right\rangle_{\varphi}=8n\|\phi\|_{\varphi}^{2}+8\|\nabla\phi\|_{\varphi}^{2}.$$

The lemma is proved. □

Now we give the proof of Lemma 2.1.

Proof. Let $\varphi=|x|^2$. By Lemmas 2.3 and 2.4, we have for every $\phi\in C_0^{\infty}(\mathbb{R}^n)$,

$$\begin{aligned}
\left\|(\Delta+\xi)_{\varphi}^{*}\phi\right\|_{\varphi}^{2}&=\|(\Delta+\xi)\,\phi\|_{\varphi}^{2}+\left\langle \phi, \Delta\left(\Delta_{\varphi}^{*}\phi\right)-\Delta_{\varphi}^{*}\left(\Delta\phi\right)\right\rangle_{\varphi}\\
&\geq\left\langle \phi, \Delta\left(\Delta_{\varphi}^{*}\phi\right)-\Delta_{\varphi}^{*}\left(\Delta\phi\right)\right\rangle_{\varphi}\\
&\geq 8n\|\phi\|_{\varphi}^{2}.
\end{aligned}\tag{2.7}$$

By Cauchy-Schwarz inequality and (2.7), we have for every $\phi\in C_0^{\infty}(\mathbb{R}^n)$,

$$\begin{aligned}
|\langle f,\phi\rangle_{\varphi}|^2&\leq\|f\|_{\varphi}^{2}\,\|\phi\|_{\varphi}^{2}\\
&=\left(\frac{1}{8n}\,\|f\|_{\varphi}^{2}\right)\left(8n\,\|\phi\|_{\varphi}^{2}\right)\\
&\leq\left(\frac{1}{8n}\,\|f\|_{\varphi}^{2}\right)\left\|(\Delta+\xi)_{\varphi}^{*}\phi\right\|_{\varphi}^{2}.
\end{aligned}$$

Let $c=\frac{1}{8n}\|f\|_\varphi^2$. Then

$$|\langle f,\phi\rangle_\varphi|^2\le c\left\|(\Delta+\xi)_\varphi^*\phi\right\|_\varphi^2,\quad \forall\phi\in C_0^\infty(\mathbb{R}^n).$$

By Lemma 2.2, there exists a global weak solution $u\in L^2(\mathbb{R}^n,e^{-\varphi})$ solving the equation

$$\Delta u+\xi u=f$$

in $\mathbb{R}^n$ with the norm estimate

$$\|u\|_\varphi^2\le c,$$

i.e.,

$$\Delta u+\xi u=f\quad\text{with}\ \int_{\mathbb{R}^n}|u|^2e^{-|x|^2}dx\le\frac{1}{8n}\int_{\mathbb{R}^n}|f|^2e^{-|x|^2}dx.$$

The proof is completed. □

3 Proof of Theorem 1.1

Now we give the proof of Theorem 1.1.

Proof. Let $\varphi=|x|^2$. By the fundamental theorem of algebra, the polynomial $P(\Delta)$ can be rewritten as

$$P(\Delta)=(\Delta+\xi_1)\cdots(\Delta+\xi_m),$$

where ξ_j is a complex number for $j=1,\cdots,m$.

Let $H_j=\Delta+\xi_j$ for $j=1,\cdots,m$. If $m=1$, then the theorem is proved by Lemma 2.1. Now assume that $m\ge 2$. For f and H_1, by Lemma 2.1, there exists $u_1\in L^2(\mathbb{R}^n,e^{-|x|^2})$ such that

$$H_1u_1=f\quad\text{with}\ \int_{\mathbb{R}^n}|u_1|^2e^{-|x|^2}dx\le\frac{1}{8n}\int_{\mathbb{R}^n}|f|^2e^{-|x|^2}dx.$$

For u_1 and H_2, by Lemma 2.1, there exists $u_2\in L^2(\mathbb{R}^n,e^{-|x|^2})$ such that

$$H_2u_2=u_1\quad\text{with}\ \int_{\mathbb{R}^n}|u_2|^2e^{-|x|^2}dx\le\frac{1}{8n}\int_{\mathbb{R}^n}|u_1|^2e^{-|x|^2}dx.$$

So by the same method, we have for $1\le j\le m-1$, u_j and H_{j+1}, there exists $u_{j+1}\in L^2(\mathbb{R}^n,e^{-|x|^2})$ such that

$$H_{j+1}u_{j+1}=u_j\quad\text{with}\ \int_{\mathbb{R}^n}|u_{j+1}|^2e^{-|x|^2}dx\le\frac{1}{8n}\int_{\mathbb{R}^n}|u_j|^2e^{-|x|^2}dx.$$

Thus, there exists $u_m \in L^2(\mathbb{R}^n, e^{-|x|^2})$ such that

$$H_1 \cdots H_m u_m = f \quad \text{with} \quad \int_{\mathbb{R}^n} |u_m|^2 e^{-|x|^2} dx \leq \frac{1}{(8n)^m} \int_{\mathbb{R}^n} |f|^2 e^{-|x|^2} dx,$$

i.e.,

$$P(\Delta) u_m = f \quad \text{with} \quad \int_{\mathbb{R}^n} |u_m|^2 e^{-|x|^2} dx \leq \frac{1}{(8n)^m} \int_{\mathbb{R}^n} |f|^2 e^{-|x|^2} dx.$$

Let $u_m = u$. Then by the above formula, the theorem is proved. □

4 Further remarks

Remark 4.1. Given $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, for the weight $\varphi = \lambda |x - x_0|^2$, we obtain the following corollary from Theorem 1.1.

Corollary 4.1. *For each $f \in L^2(\mathbb{R}^n, e^{-\lambda|x-x_0|^2})$, there exists a weak solution $u \in L^2(\mathbb{R}^n, e^{-\lambda|x-x_0|^2})$ solving the equation*

$$P(\Delta) u = f$$

with the norm estimate

$$\int_{\mathbb{R}^n} |u|^2 e^{-\lambda|x-x_0|^2} dx \leq \frac{1}{\lambda^{2m}(8n)^m} \int_{\mathbb{R}^n} |f|^2 e^{-\lambda|x-x_0|^2} dx.$$

Proof. From $f \in L^2(\mathbb{R}^n, e^{-\lambda|x-x_0|^2})$, we have

$$\int_{\mathbb{R}^n} |f|^2(x) e^{-\lambda|x-x_0|^2} dx < +\infty. \tag{4.1}$$

Let

$$x = \frac{y}{\sqrt{\lambda}} + x_0 \quad \text{and} \quad g(y) = f(x) = f\left(\frac{y}{\sqrt{\lambda}} + x_0\right).$$

Then by (4.1), we have

$$\frac{1}{\left(\sqrt{\lambda}\right)^n} \int_{\mathbb{R}^n} |g(y)|^2 e^{-|y|^2} dy < +\infty,$$

which implies that $g \in L^2(\mathbb{R}^n, e^{-|y|^2})$. For g, applying Theorem 1.1 with $P(\Delta)$ replaced by

$$\widetilde{P}(\Delta) = \Delta^m + \frac{a_{m-1}}{\lambda} \Delta^{m-1} + \frac{a_{m-2}}{\lambda^2} \Delta^{m-2} + \cdots + \frac{a_1}{\lambda^{m-1}} \Delta + \frac{a_0}{\lambda^m},$$

there exists a weak solution $v \in L^2(\mathbb{R}^n, e^{-|y|^2})$ solving the equation

$$\widetilde{P}(\Delta) v(y) = g(y) \tag{4.2}$$

in $\mathbb{R}^n$ with the norm estimate

$$\int_{\mathbb{R}^n} |v(y)|^2 e^{-|y|^2} dy \le \frac{1}{(8n)^m} \int_{\mathbb{R}^n} |g(y)|^2 e^{-|y|^2} dy. \tag{4.3}$$

Note that $y = \sqrt{\lambda}(x - x_0)$ and $g(y) = f(x)$. Let

$$u(x) = \frac{1}{\lambda^m} v(y) = \frac{1}{\lambda^m} v\left(\sqrt{\lambda}(x - x_0)\right).$$

Then (4.2) and (4.3) can be rewritten by

$$P(\Delta)u(x) = f(x), \tag{4.4a}$$

$$\int_{\mathbb{R}^n} |u(x)|^2 e^{-\lambda|x-x_0|^2} dx \le \frac{1}{\lambda^{2m}(8n)^m} \int_{\mathbb{R}^n} |f(x)|^2 e^{-\lambda|x-x_0|^2} dx. \tag{4.4b}$$

(4.4b) implies that $u \in L^2(\mathbb{R}^n, e^{-\lambda|x-x_0|^2})$. Then by (4.4a) and (4.4b), the proof is completed. □

Remark 4.2. When $f \in L^2(\mathbb{R}^n)$, the solutins of $P(\Delta)u = f$ are not necessary in $L^2(\mathbb{R}^n)$. For example: $n = 1$, $P(\Delta) = \Delta$,

$$f(x) = \begin{cases} \frac{1}{x}, & x \ge 1, \\ x, & 0 < x < 1, \\ 0, & x \le 0, \end{cases}$$

$$u(x) = \int_0^x (x - t) f(t) dt + c_1 x + c_2 = -\frac{x}{2} + x \ln x + \frac{2}{3} + c_1 x + c_2, \quad x \ge 1,$$

where c_1 and c_2 are arbitrary real constants. It is easy to see $u \notin L^2(\mathbb{R})$.

References

[1] J. P. Rosay, A very elementary proof of the Malgrange-Ehrenpreis theorem, The American Mathematical Monthly, 98(6) (1991), 518–523.

[2] S. Y. Dai, and Y. F. Pan, A right inverse of Cauchy-Riemann operator $\overline{\partial}^k + a$ in the weighted Hilbert space $L^2(\mathbb{C}, e^{-|z|^2})$, Annales de la Faculté des Sciences de Toulouse : Mathématiques, 30(3) (2021), 619–632.

[3] S. Y. Dai, and Y. F. Pan, A right inverse of differential operator $\frac{d^k}{dx^k} + a$ in weighted Hilbert space $L^2(\mathbb{R}, e^{-x^2})$, J. Math. Anal. Appl., 486 (2020), 123890.

[4] S. Y. Dai, and Y. F. Pan, On the Poincaré-Lelong equation in $\mathbb{C}^n$, Complex Variables and Elliptic Equations, 67(6) (2022), 1426–1446.

[5] L. Hörmander, L^2 estimates and existence theorems for the $\overline{\partial}$ operator, Acta Math., 113 (1965), 89–152.

[6] R. Meise, B. A. Taylor, and D. Vogt, Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse, Ann. Inst. Fourier (Grenoble), 40(3) (1990), 619–655.