Analysis in Theory and Applications
Anal. Theory Appl., Vol. **40**, No. 1 (2024), pp. 22-37

DOI: 10.4208/ata.OA-2022-0029
March 2024

A Remark about Time-Analyticity of the Linear Landau Equation with Soft Potential

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Received 8 October 2022; Accepted (in revised version) 14 September 2023

Abstract. In this note, we study the Cauchy problem of the linear spatially homogeneous Landau equation with soft potentials. We prove that the solution to the Cauchy problem enjoys the analytic regularizing effect of the time variable with an L^2 initial datum for positive time. So that the smoothing effect of Cauchy problem for the linear spatially homogeneous Landau equation with soft potentials is similar to the heat equation.

Key Words: Spatially homogeneous Landau equation, analytic smoothing effect, soft potentials.

AMS Subject Classifications: 35B65, 76P05, 82C40

1 Introduction

The Cauchy problem of spatially homogenous Landau equation reads

$$\begin{cases} \partial_t F = Q(F,F), \\ F|_{t=0} = F_0, \end{cases} \tag{1.1}$$

where $F = F(t,v) \geq 0$ is the density distribution function at time $t \geq 0$, with the velocity variable $v \in \mathbb{R}^3$. The Landau bilinear collision operator is defined by

$$Q(G,F)(v) = \sum_{j,k=1}^{3} \partial_j \left(\int_{\mathbb{R}^3} a_{jk}(v-v_*)[G(v_*)\partial_k F(v) - \partial_k G(v_*)F(v)]dv_* \right) \tag{1.2}$$

with

$$a_{jk}(v) = (\delta_{jk}|v|^2 - v_j v_k)|v|^\gamma, \quad \gamma \geq -3,$$

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is a symmetric non-negative matrix such that

$$\sum_{j,k=1}^{3} a_{jk}(v)v_jv_k = 0.$$

Here, γ is a parameter which leads to the classification of the hard potential if $\gamma > 0$, Maxwellian molecules if $\gamma = 0$, soft potential if $-3 < \gamma < 0$ and Coulombian potential if $\gamma = -3$.

The Landau equation was introduced as a limit of the Boltzmann equation when the collisions become grazing in [6, 18]. In the hard potential case, the existence, and the uniqueness of the solution to the Cauchy problem for the spatially homogeneous Landau equation have been addressed by Desvillettes and Villani in [7,19]. Meanwhile, they also proved the smoothness of the solution is $C^\infty(]0,\infty[;\mathcal{S}(\mathbb{R}^3))$. The analytic and the Gevrey regularity of the solution for any $t > 0$ have already been studied in [1,2].

We shall study the linearization of the Landau equation (1.1) near the Maxwellian distribution

$$\mu(v) = (2\pi)^{\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Considering the fluctuation of the density distribution function

$$F(t,v) = \mu(v) + \sqrt{\mu}(v) f(t,v),$$

since $Q(\mu,\mu) = 0$, the Cauchy problem (1.1) takes the form

$$\begin{cases} \partial_t f + \mathcal{L} f = \Gamma(f,f), \\ f|_{t=0} = f_0, \end{cases}$$

with $F_0 = \mu + \sqrt{\mu} f_0$, where

$$\begin{aligned} &\Gamma(f,f) = \mu^{-\frac{1}{2}} Q(\sqrt{\mu} f, \sqrt{\mu} f), \\ &\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \quad \text{with } \mathcal{L}_1 f = -\Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 f = -\Gamma(f, \sqrt{\mu}). \end{aligned}$$

The spatially homogeneous Landau equation and non-cutoff Boltzmann equation in a close-to-equilibrium framework have been studied in [10] and the Gelfand-Shilov smoothing effect has been proved in [11, 15]. Guo [8] constructed the classical solution for the spatially inhomogeneous Landau equation near a global Maxwellian in a periodic box. The smoothness of the solutions has been studied in [3, 9, 12]. In addition, the analytic smoothing effect of the velocity variable for the nonlinear Landau equation has been treated in [13,16]. The variant regularity results under a close-to-equilibrium setting have been considered in [4, 5, 17].

In this work, we consider the Cauchy problem of the linear Landau equation, such as

$$\begin{cases} \partial_t f + \mathcal{L} f = g, \\ f|_{t=0} = f_0, \end{cases} \tag{1.3}$$

where g is a analytic function with respect to the variable t and v. The diffusion part $\mathcal{L}_1$ is written as follows

$$\mathcal{L}_1 f = -\nabla_v \cdot [A(v)\nabla_v f] + \left(A(v)\frac{v}{2}\cdot\frac{v}{2}\right) f - \nabla_v \cdot \left[A(v)\frac{v}{2}\right] f \tag{1.4}$$

with $A(v) = (\bar{a}_{jk})_{1\le j,k\le 3}$ is a symmetric matrix, and

$$\bar{a}_{jk} = a_{jk} * \mu = \int_{\mathbb{R}^3} \left(\delta_{jk}|v-v'|^2 - (v_j - v'_j)(v_k - v'_k)\right) |v-v'|^\gamma \mu(v')dv'.$$

We say that $u \in \mathcal{A}(\Omega)$ is an analytic function, where $\Omega \subset \mathbb{R}^n$ is an open domain, if $u \in C^\infty(\Omega)$ and there exists a constant C such that for all multi-indices $\alpha \in \mathbb{N}^n$,

$$\|\partial^\alpha u\|_{L^\infty(\Omega)} \le C^{|\alpha|+1}\alpha!.$$

Remark that, by using the Sobolev embedding, we can replace the L^∞ norm by the L^2 norm, or norm in any Sobolev space in the above definition.

We study the linear Landau equation (1.3), with $-3 < \gamma < 0$, and show that the solution to the Cauchy problem (1.3) with the $L^2(\mathbb{R}^3)$ initial datum enjoys the analytic regularizing effect of the time variable. The main result reads as follows.

Theorem 1.1. *For the soft potential $-3 < \gamma < 0$, for any $T > 0$ and the initial datum $f_0 \in L^2(\mathbb{R}^3)$. Let f be the solution of the Cauchy problem* (1.3)*, then there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, we have*

$$\|\partial_t^k f(t)\|_{L^2(\mathbb{R}^3)} \le \frac{C^{k+1}}{t^k} k!, \quad \forall t \in [0,T]. \tag{1.5}$$

For the linear operator with only the diffusion part of $\mathcal{L}_1$, the paper [14] prove that the Cauchy problem (1.3) admits a unique weak solution, and the solution satisfies for any $\alpha \in \mathbb{N}^3$, $\tilde{t} = \min(t,1)$,

$$\|\tilde{t}^{\frac{|\alpha|}{2}} \langle\cdot\rangle^{\frac{\gamma|\alpha|}{2}} \partial^\alpha f(t)\|_{L^2(\mathbb{R}^3)} \le C^{|\alpha|+1}\alpha!, \quad \forall t > 0.$$

With the similar computation, one can obtain the same analytical results as above, then using again the equation of (1.3), on have

$$f \in C^\infty([0,+\infty]\mathcal{A}(\mathbb{R}^3)).$$

So that we just need to prove (1.5) for the smooth solution of Cauchy problem (1.3).

2 Analysis of the Landau linear operator

In the following, the notation $A \lesssim B$ means there exists a constant $C > 0$ such that $A \le CB$. For simplicity, with $\gamma \in \mathbb{R}$, we denote the weighted Lebesgue spaces

$$\|\langle\cdot\rangle^\gamma f\|_{L^p(\mathbb{R}^3)} = \|f\|_{p,\gamma}, \quad 1 \le p \le \infty,$$

where we use the notation $\langle v\rangle = (1+|v|^2)^{\frac{1}{2}}$. And for the matrix A defined in (1.4), we denote

$$\|f\|_A^2 = \sum_{j,k=1}^{3}\int\left(\bar{a}_{jk}\partial_j f\partial_k f + \frac{1}{4}\bar{a}_{jk}v_jv_kf^2\right)dv.$$

From corollary 1 in [8], for $\gamma > -3$, there exists a constant $C_1 > 0$ such that

$$\|f\|_A^2 \geq C_1\left(\|\mathbf{P}_v\nabla f\|^2_{2,\frac{\gamma}{2}} + \|(\mathbf{I}-\mathbf{P}_v)\nabla f\|^2_{2,1+\frac{\gamma}{2}} + \|f\|^2_{2,1+\frac{\gamma}{2}}\right), \tag{2.1}$$

where for any vector-valued function $G(v) = (G_1, G_2, G_3)$ define the projection to the vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ as

$$(\mathbf{P}_vG)_j = \sum_{k=1}^{3} G_kv_k\frac{v_j}{|v|^2}, \quad 1 \leq j \leq 3.$$

Since

$$\nabla f = \mathbf{P}_v\nabla f + (\mathbf{I}-\mathbf{P}_v)\nabla f,$$

combining the inequality (2.1), we have

$$\|f\|_A \geq C_1\left(\|\nabla f\|_{2,\frac{\gamma}{2}} + \|f\|_{2,1+\frac{\gamma}{2}}\right). \tag{2.2}$$

For later use, we need the following results for the coefficients to the linear Landau operator, which have been proved in [14].

Lemma 2.1 ([14])**.** *For any $\beta \in \mathbb{R}^3$ with $|\beta| \geq 1$ and $\bar{a}_{jk}$ was defined in* (1.4) *with $-3 < \gamma < 0$, then we have*

$$|\partial^\beta \bar{a}_{jk}(v)| \lesssim \langle v\rangle^{\gamma+1}\sqrt{\beta!}. \tag{2.3}$$

Moreover, for any $\beta \in \mathbb{R}^3$,

$$\left|\partial^\beta\left(\sum_{j,k=1}^{3}\partial_j a_{jk} * (v_k\mu)\right)\right| \lesssim \langle v\rangle^{\gamma+1}(|\beta|+1)\sqrt{\beta!}, \tag{2.4a}$$

$$\left|\partial^\beta\left(\sum_{j,k=1}^{3}\bar{a}_{jk}v_jv_k\right)\right| \lesssim \langle v\rangle^{\gamma+1}(|\beta|+1)\sqrt{\beta!}. \tag{2.4b}$$

Lemma 2.2 ([14])**.** *Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^3)$, $\bar{a}_{jk}$ was defined in* (1.4) *with $-3 < \gamma < 0$. For any $\beta \in \mathbb{R}^3$, we have*

$$\left|\sum_{j,k=1}^{3}(\partial^\beta\bar{a}_{jk}\partial_k f_1, \partial_j f_2)_{L^2(\mathbb{R}^3)}\right| \lesssim \sqrt{\beta!}\|f_1\|_A\|f_2\|_A. \tag{2.5}$$

By using the results of the coefficients to the linear Landau operator in [14], we can obtain the following estimates. Firstly, for any $\gamma > -3$ and $\delta > 0$, we have

$$\int_{\mathbb{R}^3}|v-w|^\gamma e^{-\delta|w|^2}dw \lesssim \langle v\rangle^\gamma. \tag{2.6}$$

Lemma 2.3. *Let $f \in \mathcal{S}(\mathbb{R}^3)$, and $-3<\gamma<0$, then for any $0<\epsilon_1<1$, there exists a constant $C_{\epsilon_1}>0$ such that*

$$(1-\epsilon_1)\|f\|_A^2 \le (\mathcal{L}_1 f, f)_{L^2} + C_{\epsilon_1}\|f\|^2_{2,\frac{\gamma}{2}}.$$

Proof. By using the representation (1.4), and integrating by parts, we have

$$\begin{aligned}-(\mathcal{L}_1 f, f)_{L^2} &= -\int_{\mathbb{R}^3}\left(\bar{a}_{jk}\partial_j f\partial_k f + \frac{1}{4}\bar{a}_{jk}v_j v_k f^2\right) - \frac{1}{2}\int_{\mathbb{R}^3}\partial_j(\bar{a}_{jk}v_k)f^2 \\ &= -\|f\|_A^2 + R_0.\end{aligned}$$

Since

$$\sum_j a_{jk}v_j = \sum_k a_{jk}v_k = 0, \tag{2.7}$$

we have

$$\partial_j(\bar{a}_{jk}v_k) = \partial_j(a_{jk}*(v_k\mu)) = \partial_j a_{jk}*(v_k\mu).$$

Therefore from (2.4) and the Cauchy-Schwarz inequality, it follows that

$$|R_0| \lesssim \int_{\mathbb{R}^3}\langle v\rangle^{\gamma+1}f^2(v)dv \lesssim \|f\|_{2,\frac{\gamma}{2}}\|f\|_{2,1+\frac{\gamma}{2}},$$

then by using (2.2) and the Cauchy-Schwarz inequality, for any $0<\epsilon_1<1$, we have

$$|R_0| \le C_2\|f\|_{2,\frac{\gamma}{2}}\|f\|_A \le \epsilon_1\|f\|_A^2 + \frac{4C_2^2}{\epsilon_1}\|f\|^2_{2,\frac{\gamma}{2}}.$$

Let $C_{\epsilon_1} = \frac{4C_2^2}{\epsilon_1}$, then we can conclude

$$(1-\epsilon_1)\|f\|_A^2 \le (\mathcal{L}_1 f, f)_{L^2} + C_{\epsilon_1}\|f\|^2_{2,\frac{\gamma}{2}}.$$

This completes the proof. □

Proposition 2.1. *Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^3)$ and $-3<\gamma<0$, then there exists a constant $C_2>0$ such that*

$$|(\mathcal{L}_1 f_1, f_2)_{L^2}| \le C_2\|f_1\|_A\|f_2\|_A.$$

Proof. By using the representation (1.4), and integrating by parts, we have

$$\begin{aligned}(\mathcal{L}_1 f_1, f_2)_{L^2} = &\int_{\mathbb{R}^3}\bar{a}_{jk}\partial_j f_1\partial_k f_2 + \frac{1}{4}\int_{\mathbb{R}^3}\bar{a}_{jk}v_j v_k f_1 f_2 \\ &+ \frac{1}{2}\int_{\mathbb{R}^3}\partial_j(\bar{a}_{jk}v_k)f_1 f_2 \\ =&R_1+R_2+R_3.\end{aligned}$$

Since $-3<\gamma<0$, by the inequality (2.5), we obtain

$$|R_1| \lesssim \|f_1\|_A \|f_2\|_A.$$

For the term R_2 and R_3, using (2.7), then from (2.3) and (2.4), it follows that

$$|R_2|+|R_3| \lesssim \int_{\mathbb{R}^3} \langle v\rangle^{\gamma+1} |f_1(v) f_2(v)| dv,$$

then using the Cauchy-Schwarz inequality and (2.2), we have

$$|R_2|+|R_3| \lesssim \|f_1\|_{2,1+\frac{\gamma}{2}} \|f_2\|_{2,1+\frac{\gamma}{2}} \lesssim \|f_1\|_A \|f_2\|_A.$$

Finally, combining R_1-R_3 to get

$$|(\mathcal{L}_1 f_1, f_2)_{L^2}| \leq C_2 \|f_1\|_A \|f_2\|_A.$$

This completes the proof. □

Now, we shall estimate $(\mathcal{L}_2 f_1, f_2)_{L^2}$. Firstly, we give the representation of the operator $\mathcal{L}_2$. For $f \in \mathcal{S}(\mathbb{R}^3)$, from (1.2), it follows that

$$\begin{aligned}
\mathcal{L}_2 f &= -\mu^{-\frac{1}{2}} Q(\sqrt{\mu} f, \mu) \\
&= \mu^{-\frac{1}{2}} \partial_j \left(\int_{\mathbb{R}^3} a_{jk}(v-v') \left[\mu^{\frac{1}{2}}(v') f(v') v_k + \partial_k \left(\mu^{\frac{1}{2}} f \right)(v') \right] dv' \mu(v) \right) \\
&= \mu^{-\frac{1}{2}} \partial_j \left[\mu \left(a_{jk} * (v_k \mu^{\frac{1}{2}} f) + \partial_k a_{jk} * (\mu^{\frac{1}{2}} f) \right) \right].
\end{aligned} \tag{2.8}$$

Proposition 2.2. *Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^3)$ and $-3<\gamma<0$, then there exists a constant $C_3>0$ such that*

$$|(\mathcal{L}_2 f_1, f_2)_{L^2}| \leq C_3 \left(\|f_1\|_{2,\frac{\gamma}{2}} \|f_2\|_A + \|f_1\|_A \|f_2\|_{2,\frac{\gamma}{2}} \right). \tag{2.9}$$

Proof. Using integration by parts with (2.8), we have

$$\begin{aligned}
(\mathcal{L}_2 f_1, f_2)_{L^2} = &- \left(a_{jk} * (v_k \mu^{\frac{1}{2}} f_1), \mu^{\frac{1}{2}} \left(\frac{v_j}{2} f_2 + \partial_j f_2 \right) \right)_{L^2} \\
&+ \left(\partial_{jk} a_{jk} * (\mu^{\frac{1}{2}} f_1), \mu^{\frac{1}{2}} f_2 \right)_{L^2} - \left(\partial_k a_{jk} * (\mu^{\frac{1}{2}} f_1), \mu^{\frac{1}{2}} v_j f_2 \right)_{L^2} \\
= &I_1 + I_2 + I_3.
\end{aligned}$$

Since

$$|\partial^\alpha a_{jk}(v)| \leq |v|^{\gamma+2-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^3, \tag{2.10}$$

and

$$\langle v\rangle^\beta \mu^\rho(v) \in L^\infty(\mathbb{R}^3), \quad \forall \beta \in \mathbb{R}, \quad \rho>0, \tag{2.11}$$

by Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|a_{jk}*(v_k\mu^{\frac{1}{2}}f_1)\right| &\leq \int_{\mathbb{R}^3}|v-v'|^{\gamma+2}\langle v'\rangle^{1-\frac{\gamma}{2}}\mu^{\frac{1}{2}}(v')\langle v'\rangle^{\frac{\gamma}{2}}|f_1(v')|dv' \\ &\lesssim \left(\int_{\mathbb{R}^3}|v-v'|^{2(\gamma+2)}\mu^{\frac{1}{2}}(v')dv'\right)^{\frac{1}{2}}\|f_1\|_{2,\frac{\gamma}{2}}.\end{aligned}$$

For any $-3<\gamma<0$, we have $2(\gamma+2)>-3$, then by using the inequality (2.6), it follows that

$$\left|a_{jk}*(v_k\mu^{\frac{1}{2}}f_1)\right| \lesssim \langle v\rangle^{\gamma+2}\|f_1\|_{2,\frac{\gamma}{2}}.$$

Using the Cauchy-Schwarz inequality and the inequality (2.2), we can conclude

$$\begin{aligned}|I_1| &\lesssim \|f_1\|_{2,\frac{\gamma}{2}}\int_{\mathbb{R}^3}\langle v\rangle^{\gamma+3}\mu^{\frac{1}{2}}(v)\left(|f_2(v)|+|\nabla f_2(v)|\right)dv \\ &\lesssim \|f_1\|_{2,\frac{\gamma}{2}}\left(\|f_2\|_{2,1+\frac{\gamma}{2}}+\|\nabla f_2\|_{2,\frac{\gamma}{2}}\right)\lesssim \|f_1\|_{2,\frac{\gamma}{2}}\|f_2\|_A.\end{aligned}$$

For the term I_2, from (2.10), one has

$$\left|\partial_{jk}a_{jk}*(\mu^{\frac{1}{2}}f_1)\right| \lesssim \int_{\mathbb{R}^3}|v-v'|^{\gamma}\mu^{\frac{1}{2}}(v')\left|f_1(v')\right|dv'.$$

Consider two sets $\{|v-v'|\leq 1\}$ and $\{|v-v'|\geq 1\}$, that is

$$\begin{aligned}&\int_{\mathbb{R}^3}|v-v'|^{\gamma}\mu^{\frac{1}{2}}(v')\left|f_1(v')\right|dv' \\ &=\int_{|v-v'|\leq 1}+\int_{|v-v'|\geq 1}=A_1+A_2.\end{aligned}$$

For the term A_1, since $-3<\gamma<0$, we have

$$\begin{aligned}A_1 &=\sum_{j\leq 0}\int_{2^{j-1}\leq|v-v'|\leq 2^j}|v-v'|^{\gamma}\mu^{\frac{1}{2}}(v')\left|f_1(v')\right|dv' \\ &\leq \sum_{j\leq 0}\left(2^{j-1}\right)^{\gamma}\int_{|v-v'|\leq 2^j}\mu^{\frac{1}{2}}(v')\left|f_1(v')\right|dv' \\ &=8\sum_{j\leq 0}\left(2^{j-1}\right)^{\gamma+3}\frac{1}{2^{3j}}\int_{|v-v'|\leq 2^j}\mu^{\frac{1}{2}}(v')\left|f_1(v')\right|dv' \\ &\leq 8\sum_{j\leq 0}\left(2^{j-1}\right)^{\gamma+3}M(\mu^{\frac{1}{2}}f_1)\lesssim M(\mu^{\frac{1}{2}}f_1),\end{aligned}$$

where M is the Hardy-Littlewood maximal function. For term A_2, from (2.6),

$$\begin{aligned}A_2 =& \int_{|v-v'|\ge 1} |v-v'|^\gamma \mu^{\frac{1}{2}}(v') \left|f_1(v')\right| dv' \\ \le& \int_{\mathbb{R}^3} |v-v'|^{\gamma+2} \mu^{\frac{1}{2}}(v') \left|f_1(v')\right| dv' \\ \lesssim& \left(\int_{\mathbb{R}^3} |v-v'|^{2(\gamma+2)} \mu^{\frac{1}{2}}(v') dv'\right)^{\frac{1}{2}} \|f_1\|_{2,\frac{\gamma}{2}} \\ \lesssim& \langle v\rangle^{\gamma+2} \|f_1\|_{2,\frac{\gamma}{2}}.\end{aligned}$$

Combining A_1 and A_2, using Cauchy-Schwarz inequality to get

$$\begin{aligned}|I_2| \lesssim& \int_{\mathbb{R}^3} |M(\mu^{\frac{1}{2}} f_1)(v) \mu^{\frac{1}{2}}(v) f_2(v)| dv + \|f_1\|_{2,\frac{\gamma}{2}} \int_{\mathbb{R}^3} \langle v\rangle^{\gamma+2} \mu^{\frac{1}{2}}(v) |f_2(v)| dv \\ \lesssim& \|M(\mu^{\frac{1}{2}} f_1)\|_{L^2} \|\mu^{\frac{1}{2}} f_2\|_{L^2} + \|f_1\|_{2,\frac{\gamma}{2}} \|f_2\|_{2,\frac{\gamma}{2}} \\ \lesssim& \|\mu^{\frac{1}{2}} f_1\|_{L^2} \|\mu^{\frac{1}{2}} f_2\|_{L^2} + \|f_1\|_{2,\frac{\gamma}{2}} \|f_2\|_{2,\frac{\gamma}{2}} \\ \lesssim& \|f_1\|_{2,\frac{\gamma}{2}} \|f_2\|_{2,\frac{\gamma}{2}} \lesssim \|f_1\|_{2,\frac{\gamma}{2}} \|f_2\|_A.\end{aligned}$$

For I_3, from (2.10), we have

$$\left|\partial_k a_{jk} * (\mu^{\frac{1}{2}} f_1)\right| \lesssim \int_{\mathbb{R}^3} |v-v'|^{\gamma+1} \mu^{\frac{1}{2}}(v') |f_1(v')| dv'.$$

Note that

$$\frac{3}{2}(\gamma+1) > -3$$

with $-3 < \gamma < 0$. Using Hölder's inequality, (2.11) and (2.6), we have

$$\begin{aligned}\left|\partial_k a_{jk} * (\mu^{\frac{1}{2}} f_1)\right| \lesssim& \left(\int_{\mathbb{R}^3} |v-v'|^{\frac{3}{2}(\gamma+1)} \mu^{\frac{1}{2}}(v') dv'\right)^{\frac{2}{3}} \|f_1\|_{3,\frac{\gamma}{2}} \\ \lesssim& \langle v\rangle^{\gamma+1} \|f_1\|_{3,\frac{\gamma}{2}}.\end{aligned}$$

Now, we want to show $\|f_1\|_{3,\frac{\gamma}{2}}$ can be bounded by $\|f_1\|_A$. By applying Hölder's inequality, $\langle v\rangle^{\gamma/2} f_1(v)$ in $L^3(\mathbb{R}^3)$ can be bounded by

$$\left(\|\langle\cdot\rangle^{\gamma/2} f_1\|_{L^2} \|\langle\cdot\rangle^{\gamma/2} f_1\|_{L^6}\right)^{\frac{1}{2}},$$

and Sobolev embedding implies

$$\|\langle\cdot\rangle^{\gamma/2} f_1\|_{L^6} \lesssim \|\nabla[\langle\cdot\rangle^{\gamma/2} f_1]\|_{L^2},$$

thus we get

$$\|f_1\|_{3,\frac{\gamma}{2}} \lesssim \left(\|\langle\cdot\rangle^{\gamma/2} f_1\|_{L^2} \|\nabla[\langle\cdot\rangle^{\gamma/2} f_1]\|_{L^2}\right)^{\frac{1}{2}}.$$

Notice that

$$\nabla[\langle v\rangle^{\gamma/2}f_1(v)] = \frac{\gamma}{2}\langle v\rangle^{\gamma/2-2}f_1(v)v + \langle v\rangle^{\gamma/2}\nabla f_1(v),$$

from (2.2), we have

$$\|\nabla[\langle\cdot\rangle^{\gamma/2}f_1]\|_{L^2} \le \left\|\frac{\gamma}{2}\langle v\rangle^{\gamma/2-2}f_1 v\right\|_{L^2} + \|\langle\cdot\rangle^{\gamma/2}\nabla f_1\|_{L^2} \lesssim \|f_1\|_A,$$

which implies

$$\|f_1\|_{3,\frac{\gamma}{2}} \lesssim \|f_1\|_A. \tag{2.12}$$

Finally, using the Cauchy-Schwarz inequality and (2.12) to get

$$|I_3| \lesssim \|f_1\|_A \int_{\mathbb{R}^3}\langle v\rangle^{\gamma+2}\mu^{\frac{1}{2}}(v)|f_2(v)|dv \lesssim \|f_1\|_A\|f_2\|_{2,\frac{\gamma}{2}}.$$

Combining $I_1 - I_3$, we obtain

$$|(\mathcal{L}_2 f_1, f_2)_{L^2}| \le C_3\left(\|f_1\|_{2,\frac{\gamma}{2}}\|f_2\|_A + \|f_1\|_A\|f_2\|_{2,\frac{\gamma}{2}}\right).$$

This completes the proof. □

Remark 2.1.

(1). For $f_1, f_2 \in \mathcal{S}(\mathbb{R}^3)$ and $\gamma > -5/2$, we have

$$|(\mathcal{L}_2 f_1, f_2)_{L^2}| \lesssim \|f_1\|_{2,\frac{\gamma}{2}}\|f_2\|_A.$$

(2). For $-3 < \gamma < 0$, if $f_1 = f_2$, then for any $\epsilon_2 > 0$, there exists a constant $C_{\epsilon_2} > 0$ such that

$$|(\mathcal{L}_2 f_1, f_1)_{L^2}| \le \epsilon_2\|f_1\|_A^2 + C_{\epsilon_2}\|f_1\|^2_{2,\frac{\gamma}{2}}. \tag{2.13}$$

(3). From (2.2),

$$|(\mathcal{L}_2 f_1, f_2)_{L^2}| \le C_4\|f_1\|_A\|f_2\|_A. \tag{2.14}$$

3 Energy estimates

In this section, we study the energy estimates of the solution to the Cauchy problem (1.3).

Lemma 3.1. *For $-3 < \gamma < 0$. Lef f be the solution of Cauchy problem* (1.3). *Assume $f_0 \in L^2(\mathbb{R}^3)$. Then there exists a constant $C_5 > 0$ such that for any $T > 0$ and $t \in [0,T]$,*

$$\|f(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|f(s)\|_A^2 ds \le (C_5)^2.$$

Proof. Since f is the solution of Cauchy problem (1.3),

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|^2_{L^2(\mathbb{R}^3)}+(\mathcal{L}_1 f,f)_{L^2(\mathbb{R}^3)}=(g,f)_{L^2(\mathbb{R}^3)}-(\mathcal{L}_2 f,f)_{L^2(\mathbb{R}^3)}.$$

Since $\gamma<0$, by using Lemma 2.3 and (2.13), we have

$$\begin{aligned}&\frac{d}{dt}\|f(t)\|^2_{L^2(\mathbb{R}^3)}+2(1-\epsilon_1)\|f(t)\|^2_A\\ \le&2\|f(t)\|_{L^2}\|g(t)\|_{L^2}+2\epsilon_2\|f(t)\|^2_A+2(C_{\epsilon_1}+C_{\epsilon_2})\|f(t)\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Applying Cauchy-Schwarz inequality, and choosing $\epsilon_1=\epsilon_2=\frac{1}{4}$, we can get

$$\frac{d}{dt}\|f(t)\|^2_{L^2(\mathbb{R}^3)}+\|f(t)\|^2_A\le c_1\|f(t)\|^2_{L^2}+\|g(t)\|^2_{L^2}.$$

Since g is analytic with respect to t and v, for any $T>0$ and $t\in[0,T]$, there exists a constant $A>0$ such that

$$\|g(t)\|_{L^2}\le A. \tag{3.1}$$

Therefore by applying Gronwall inequality, for any $T>0$ and $t\in[0,T]$, taking

$$C_5\ge e^{\frac{1}{2}c_1T}\sqrt{\|f_0\|^2_{L^2}+TA^2},$$

one can obtain

$$\|f(t)\|^2_{L^2(\mathbb{R}^3)}+\int_0^t\|f(s)\|^2_A ds\le e^{c_1T}\left(\|f_0\|^2_{L^2}+TA^2\right)\le(C_5)^2.$$

This completes the proof. □

Lemma 3.2. *For $-3<\gamma<0$. Lef f be the solution of Cauchy problem* (1.3)*. Assume $f_0\in L^2(\mathbb{R}^3)$. Then there exists a constant $C_6>0$ such that for any $T>0$ and $t\in[0,T]$,*

$$\|t\partial_t f\|^2_{L^\infty(]0,T];L^2(\mathbb{R}^3))}+\int_0^T\|t\partial_t f\|^2_A dt\le(C_6)^2. \tag{3.2}$$

Proof. Since the solution of Cauchy problem (1.3) belongs to $C^\infty(]0,T[;\mathcal{S}(\mathbb{R}^3))$, we have that

$$\partial_t(t\partial_t f)+\mathcal{L}_1(t\partial_t f)=\partial_t f-\mathcal{L}_2(t\partial_t f)+t\partial_t g,$$

and for $0<t\le T$,

$$\begin{aligned}&\frac{1}{2}\|t\partial_t f\|^2_{L^2(\mathbb{R}^3)}+\int_0^t(\mathcal{L}_1(s\partial_s f),s\partial_s f)_{L^2(\mathbb{R}^3)}ds\\ =&\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)}ds-\int_0^t(\mathcal{L}_2(s\partial_s f),s\partial_s f)_{L^2(\mathbb{R}^3)}+\int_0^t(s\partial_s f,s\partial_s g)_{L^2(\mathbb{R}^3)}ds\\ =&S_1+S_2+S_3.\end{aligned}$$

Firstly, since $\gamma < 0$, by Lemma 2.3 for all $0 < t \leq T$, we can conclude

$$\begin{aligned}&\int_0^t (\mathcal{L}_1(s\partial_s f), s\partial_s f)_{L^2(\mathbb{R}^3)} ds \\ \geq&(1-\epsilon_1)\int_0^t \|s\partial_s f\|_A^2 ds - C_{\epsilon_1}\int_0^t \|s\partial_s f\|^2_{2,\frac{\gamma}{2}} ds \\ \geq&(1-\epsilon_1)\int_0^t \|s\partial_s f\|_A^2 ds - TC_{\epsilon_1}\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds.\end{aligned}$$

For the term S_1, since f is the solution of (1.3), using Proposition 2.1 and (2.14), for all $0 < t \leq T$, we have

$$\begin{aligned}&\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds \\ =&\int_0^t (g, s\partial_s f)_{L^2(\mathbb{R}^3)} ds - \int_0^t (\mathcal{L}_1 f, s\partial_s f)_{L^2(\mathbb{R}^3)} ds - \int_0^t (\mathcal{L}_2 f, s\partial_s f)_{L^2(\mathbb{R}^3)} ds \\ \leq&\int_0^t \|g(s)\|_{L^2(\mathbb{R}^3)}\|s\partial_s f\|_{L^2(\mathbb{R}^3)} ds + C_2\int_0^t \|f(s)\|_A\|s\partial_s f\|_A ds \\ &+ C_4\int_0^t \|f(s)\|_A\|s\partial_s f\|_A ds.\end{aligned}$$

For all $0 < t \leq T$, by Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}&\int_0^t \|g(s)\|_{L^2(\mathbb{R}^3)}\|s\partial_s f\|_{L^2(\mathbb{R}^3)} ds \\ \leq&T^{\frac{1}{2}}\int_0^t \|g(s)\|_{L^2(\mathbb{R}^3)} s^{\frac{1}{2}}\|\partial_s f\|_{L^2(\mathbb{R}^3)} ds \\ \leq&\frac{1}{2}\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds + \frac{T}{2}\int_0^t \|g(s)\|^2_{L^2(\mathbb{R}^3)} ds.\end{aligned}$$

Using Cauchy-Schwarz inequality, since $\gamma < 0$, for any $0 < \delta < 1$, we have

$$\begin{aligned}&\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds \\ \leq&\delta\int_0^t \|s\partial_s f\|_A^2 ds + T\int_0^t \|g(s)\|^2_{L^2(\mathbb{R}^3)} ds + C_\delta\int_0^t \|f(s)\|_A^2 ds,\end{aligned}$$

with C_δ depends on C_2, C_4. Combining (3.1) and Lemma 3.1, for $0 < \delta < 1$,

$$S_1 = \int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds \leq \delta\int_0^t \|s\partial_s f\|_A^2 ds + T^2A^2 + C_\delta C_5^2. \tag{3.3}$$

For the term S_2, let $f_1 = s\partial_s f$ in (2.13), then for all $0 < t \leq T$,

$$|S_2| \leq \epsilon_2\int_0^t \|s\partial_s f\|_A^2 ds + C_{\epsilon_2}T\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds,$$

by using (3.3) with $c_3T\delta \le \epsilon_2$,

$$|S_2| \le 2\epsilon_2 \int_0^t \|s\partial_s f\|_A^2 ds + \tilde{C}_{\epsilon_2},$$

with $\tilde{C}_{\epsilon_2}$ depends on C_2, C_4, C_5, A and T.

Finally, for the term S_3, by Cauchy-Schwarz inequality, it follows that

$$|S_3| \le \int_0^t \|s\partial_s g\|^2_{L^2(\mathbb{R}^3)} + T\int_0^t s\|\partial_s f\|^2_{L^2(\mathbb{R}^3)} ds.$$

Since g is analytic with respect to t and v, for all $0 < t \le T$, we have

$$\|t\partial_t g\|_{L^2(\mathbb{R}^3)} \le A^2,$$

applying (3.3) with $T\delta \le \epsilon_2$ to get S_3 can be bounded by

$$\epsilon_2 \int_0^t \|s\partial_s f\|_A^2 ds + TA^4 + T\left(T^2A^2 + C_{\epsilon_2}C_5^2\right).$$

Therefore, combining the results above and using (3.3) with $TC_{\epsilon_1}\delta < \epsilon_1$, let $\epsilon_1 = \epsilon_2 = \frac{1}{16}$, $0 < \delta \le \frac{1}{4}$, and taking $C_6 \ge \sqrt{\tilde{C}_5}$, we get

$$\|t\partial_t f\|^2_{L^\infty(]0,T];L^2(\mathbb{R}^3))} + \int_0^T \|t\partial_t f\|_A^2 dt \le \tilde{C}_5 \le (C_6)^2.$$

with C_6 depend on C_2, C_4, C_5, A and T. □

4 Analytic smoothing effect for time variable

In this section, we will show the analytic regularity of the time variable for $t > 0$. We construct the following estimate, which implies Theorem 1.1 immediately.

Proposition 4.1. *For $-3 < \gamma < 0$. Let f be the solution of Cauchy problem* (1.3)*, and $f_0 \in L^2(\mathbb{R}^3)$. Then there exists a constant $B > 0$ such that for any $T > 0$, $t \in [0,T]$ and $k \in \mathbb{N}$,*

$$\|t^k\partial_t^k f\|^2_{L^\infty(]0,T];L^2(\mathbb{R}^3))} + \int_0^T \|t^k\partial_t^k f\|_A^2 dt \le \left(B^{k+1}k!\right)^2. \tag{4.1}$$

Proof. We prove this proposition by induction on the index k. For $k = 1$, it is enough to take in (3.2). Assume (4.1) holds true, for any $1 \le m \le k-1$ with $k \ge 2$,

$$\|t^m\partial_t^m f\|^2_{L^\infty(]0,T];L^2(\mathbb{R}^3))} + \int_0^T \|t^m\partial_t^m f\|_A^2 dt \le \left(B^{m+1}m!\right)^2. \tag{4.2}$$

We shall prove (4.1) holds true for $m = k$.

Since μ is the function with respect to v, which implies

$$t^k\partial_t^k\mathcal{L}_1 f=\mathcal{L}_1(t^k\partial_t^k f),\quad t^k\partial_t^k\mathcal{L}_2 f=\mathcal{L}_2(t^k\partial_t^k f).$$

Then by (1.3), we have

$$\partial_t(t^k\partial_t^k f)+\mathcal{L}_1(t^k\partial_t^k f)=kt^{k-1}\partial_t^k f-\mathcal{L}_2(t^k\partial_t^k f)+t^k\partial_t^k g.$$

Taking the $L^2(\mathbb{R}^3)$ inner product of both sides with respect to $t^k\partial_t^k f$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|t^k\partial_t^k f\|^2_{L^2(\mathbb{R}^3)}+(\mathcal{L}_1(t^k\partial_t^k f),t^k\partial_t^k f)_{L^2(\mathbb{R}^3)}\\ =&k(t^{k-1}\partial_t^k f,t^k\partial_t^k f)_{L^2(\mathbb{R}^3)}-(\mathcal{L}_2(t^k\partial_t^k f),t^k\partial_t^k f)_{L^2(\mathbb{R}^3)}+(t^k\partial_t^k g,t^k\partial_t^k f)_{L^2(\mathbb{R}^3)}.\end{aligned}$$

For all $0<t\le T$, integrating from 0 to t, since $\gamma<0$, by using Lemma 2.3, it follows that

$$\begin{aligned}&\int_0^t(\mathcal{L}_1(s^k\partial_s^k f),s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}ds\\ \ge&(1-\epsilon_1)\int_0^t\|s^k\partial_s^k f\|_A^2ds-C_{\epsilon_1}\int_0^t\|s^k\partial_s^k f\|^2_{2,\frac{\gamma}{2}}ds\\ \ge&(1-\epsilon_1)\int_0^t\|s^k\partial_s^k f\|_A^2ds-TC_{\epsilon_1}\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds,\end{aligned}$$

and let $f_1=s^k\partial_s^k f$ in (2.13) to get

$$\int_0^t|(\mathcal{L}_2(s^k\partial_s^k f),s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}|ds\le\epsilon_2\int_0^t\|s^k\partial_s^k f\|_A^2ds+TC_{\epsilon_2}\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds,$$

then using Cauchy-Schwarz inequality to get

$$\begin{aligned}&\int_0^t|(s^k\partial_s^k g,s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}|ds\\ \le&\frac{1}{2}\int_0^t\|s^k\partial_s^k g\|^2_{L^2(\mathbb{R}^3)}ds+\frac{1}{2}\int_0^t\|s^k\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds\\ \le&\frac{1}{2}\int_0^t\|s^k\partial_s^k g\|^2_{L^2(\mathbb{R}^3)}ds+\frac{T}{2}\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds.\end{aligned}$$

Combining the results above, and taking $\epsilon_1=\epsilon_2=\frac{1}{8}$, we have for all $0<t\le T$,

$$\begin{aligned}&\|t^k\partial_t^k f\|^2_{L^2(\mathbb{R}^3)}+\frac{3}{2}\int_0^t\|s^k\partial_s^k f\|_A^2ds\\ \le&\int_0^t\|s^k\partial_s^k g\|^2_{L^2(\mathbb{R}^3)}ds+C_7\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds,\end{aligned}\tag{4.3}$$

with C_7 depends on T.

Since f is the solution of (1.3) and $k \geq 2$, we have

$$\partial_t^k f = \partial_t^{k-1} g - \mathcal{L}_1(\partial_t^{k-1} f) - \mathcal{L}_2(\partial_t^{k-1} f),$$

which implies

$$\begin{aligned}&\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds\\ =&\int_0^t (s^{k-1}\partial_s^{k-1} g, s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}ds - \int_0^t (\mathcal{L}_1(s^{k-1}\partial_s^{k-1} f), s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}ds\\ &-\int_0^t (\mathcal{L}_2(s^{k-1}\partial_s^{k-1} f), s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}ds.\end{aligned}$$

For all $0 < t \leq T$, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_0^t (s^{k-1}\partial_s^{k-1} g, s^k\partial_s^k f)_{L^2(\mathbb{R}^3)}ds\\ \leq& T^{\frac{1}{2}}\int_0^t \|s^{k-1}\partial_s^{k-1} g\|_{L^2(\mathbb{R}^3)} s^{\frac{2k-1}{2}}\|\partial_s^k f\|_{L^2(\mathbb{R}^3)}\\ \leq& \frac{T}{2}\int_0^t \|s^{k-1}\partial_s^{k-1} g\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds.\end{aligned}$$

By using Proposition 2.1, (2.14) and Cauchy-Schwarz inequality, for any $0 < \delta < 1$, there exists a constant $C_\delta > 0$ such that for all $0 < t \leq T$,

$$\begin{aligned}&\int_0^t s^{2k-1}\|\partial_s^k f\|^2_{L^2(\mathbb{R}^3)}ds\\ \leq& \delta\int_0^t \|s^k\partial_s^k f\|^2_A ds + T\int_0^t \|s^{k-1}\partial_s^{k-1} g\|^2_{L^2(\mathbb{R}^3)} + C_\delta\int_0^t \|s^{k-1}\partial_s^{k-1} f\|^2_A ds,\end{aligned} \tag{4.4}$$

with C_δ depends on C_2, C_4. Let $C_7\delta \leq \frac{1}{2}$, substituting (4.4) into (4.3), we get

$$\begin{aligned}&\|t^k\partial_t^k f\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|s^k\partial_s^k f\|^2_A ds\\ \leq& \tilde{C}_7\left(\int_0^t \|s^{k-1}\partial_s^{k-1} g\|^2_{L^2(\mathbb{R}^3)}ds + \int_0^t \|s^{k-1}\partial_s^{k-1} f\|^2_A ds\right) + \int_0^t \|s^k\partial_s^k g\|^2_{L^2(\mathbb{R}^3)}ds,\end{aligned}$$

with $\tilde{C}_7$ depends on C_2, C_4, C_7 and T.

Finally, since g is analytic with respect to t and v, for any $k \in \mathbb{N}$, there exists a constant $A > 0$ such that for any $0 < t \leq T$,

$$\|t^k\partial_t^k g\|_{L^2(\mathbb{R}^3)} \leq A^{k+1}k!,$$

taking $B \geq \max\{A, \sqrt{2\tilde{C}_7}\}$, using the induction hypothesis (4.2), we obtain

$$\begin{aligned}&\left\|t^k\partial_t^k f\right\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|s^k\partial_s^k f\|^2_A ds\\ \leq&\tilde{C}_7\left((A^k(k-1)!)^2 + (B^k(k-1)!)^2\right) + (A^{k+1}k!)^2\\ \leq&(B^{k+1}k!)^2,\end{aligned}$$

with B depends on C_1, C_2, C_4, A and T. We finish the proof of Proposition 4.1. □

Acknowledgements

This work was supported by the NSFC (No. 12031006) and the Fundamental Research Funds for the Central Universities of China.

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