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Dynamics Analysis for a Hierarchical System with Nonlinear Cut-Off Interaction and Free Will

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Abstract. Noting that the particles communicate with each other within a finite distance, in the paper, we investigate the hierarchical model with free will and cut-off function (linear or nonlinear). As our observation, the cut-off size r is a sensitive coefficient to achieve the flocking behavior. Besides we will use energy function to achieve some sufficient conditions to achieve a flock. The free-will represents the tendency of the particle itself to move. It will either facilitate or prevent cluster generation. Some numerical simulations will validate our conclusions.

Key Words: Motsch-Tadmor model, hierarchical structure, free will, flocking behavior.

AMS Subject Classifications: 34M03, 70E55

1 Introduction

In recent years, cluster phenomenon has attracted more and more researchers' attention. For example, migrating birds, schools of fish regroup after encountering predators, honeybee colonies that gather in nature, and so on. Scientists use scientific means to study its internal dynamics law, and have achieved fruitful results. The terminology flocking represents the phenomenon that self-propelled agents achieve a consentaneous velocity asymptotically.

Since 1995 when Vicsek [1] proposed an interactive dynamics model of N particles, this work has been favored by many researchers. Among them, Cucker and Smale [2] extended models have attracted a lot of attention from scholars, which is given by

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \dfrac{d\mathbf{v}_i}{dt} = \dfrac{\lambda}{N} \displaystyle\sum_{j=1}^{N} \varphi(|\mathbf{x}_j - \mathbf{x}_i|)(\mathbf{v}_j - \mathbf{v}_i), \quad i = 1, 2, \cdots, N. \end{cases} \tag{1.1}$$

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Here λ is a positive constant denoting the coupling strength and $\varphi(\cdot)$ is the interaction function. This model resembles a Newton-type N-body system for an interacting particle system. A classical conclusion about $1/2$ is presented. It should be noted that the interchange function $\varphi(\mathbf{x})$ exists in the whole process in the C-S model, i.e.,

$$\varphi(\mathbf{x}) = \frac{1}{(1+||\mathbf{x}_i(t)-\mathbf{x}_j(t)||^2)^\beta}.$$

However, in the above model, the interaction function of particles still exists with the increase of distance. If a small group moves away from a much larger group, it is found that the dynamics of the small group is almost masked by $\frac{1}{N}$. So to remedy this deficiency, Motsch and Tadmor [3] proposed the following model:

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \dfrac{d\mathbf{v}_i}{dt} = \dfrac{\lambda}{\sum_{j=1}^N \psi(|\mathbf{x}_j-\mathbf{x}_i|)} \sum_{j=1}^N \psi(|\mathbf{x}_j-\mathbf{x}_i|)(\mathbf{v}_j-\mathbf{v}_i), \quad i=1,2,\cdots,N. \end{cases} \tag{1.2}$$

In the Motsch-Tadmor model (1.2), ψ stands for communication function. Hence, it is of certain significance to study this paper based on M-T model due to the relative distance. Furthermore, the researchers considered multi-particle models with realistic delay factors and new analysis methods [5–13]).

In fact, the function ψ in the above model is global in space. But the reality is the case that the impact function should be falling sharply at some distance. The Vicsek model captures this property, i.e., local interaction ($\chi_r(s)$). In this paper, $\chi_r(s)$ is defined as:

$$\chi_r(s) = \begin{cases} \varphi(s), & |s| < r, \\ 0, & |s| \geq r. \end{cases} \tag{1.3}$$

In $\chi_r(s)$ (1.3), $\varphi(s) = 1$ or $(1+s^2)^{-\beta}$, s is the displacement difference between the particles. r stands for maximum communication distance, and communication is interrupted when the distance between particles exceeds r. When $r=\infty$,

$$\varphi(\mathbf{x}) = (1+||\mathbf{x}_i(t)-\mathbf{x}_j(t)||^2)^{-\beta},$$

it is the classical CS model (1.1). When $r=0$, it is a trivial solution and the clustering depends strictly on the initial value.

In 2003, Scaruffi revealed the existence of free will in the insect world, listing and particles in his monograph. Then biologists disclosed that from single-celled tissue to human beings has free will. In the existing models, little attention has been paid to the intrinsic dynamics of particles, which means the whole flock eventually moves along a straight-line direction. In motion, however, a shoal of fish or a flock of birds takes on a more complex trajectory. This is because each particle is affected not only by its neighbours, but also due to an intrinsic dynamical process. So it makes sense to explore the mechanisms that produce complex motion. In [4], Cucker added free will ($C_i(t)\mathbf{d}_i(t)$) to the C-S model, where the definition of free will depends on relative velocity with other individuals. Where $C_i(t) \geq 0$ is some local measure of the alignment consensus at time t (with absolute consensus for $C_i(t) = 0$) and $\mathbf{d}_i(t) \in \mathbb{E}$ (Euclidean space) is the preferred direction of agent i. In 2007, Shen [15] considered free will in a continuous hierarchical leadership (HL) model. In 2013, Dong [16] studied the discrete HL model with free will. Moreover, for uav, I prefer its free will as a kind of controller or a tendentious movement. At the beginning, free will makes its movement consistent. With the increase of consistency, its free will should gradually decline.

Hence in this paper we mainly consider interaction functions with cut-off and free will $f_i(t)$. In [17], the discrete form of truncation function model is considered. In this paper, we consider the continuous form of free will and cut-off. Previous researchers mainly focused on the free will model under the global communication function, while this paper focuses on the free will model under the local communication function and extends the local function to nonlinearity. In this paper, we want to get the minimum interaction distance needed to form a cluster by controlling the speed difference. Clustering can occur only if r meets certain conditions, and indicates that the choice of r leads to multi-cluster. Besides we give the conditions for choosing r. In the proof, we introduce the energy function for consideration, discuss by constructing the appropriate energy function including the velocity difference and displacement difference, and draw the conclusion through the property between the velocity and displacement. In follow-up studies, we found that free will function can promote the happening of the cluster, also can prevent the generation of the cluster, when free will function make particles close to the center node role, can shorten the time of the cluster formation, but when free will function has a tendency to deviate from the collective, can prevent the cluster formation, thus it can be seen, when the cluster itself is not cluster behavior occurs, we can choose the proper function of free will to make it form a cluster.

We consider an adaptive complex system with N particles and denote $\mathcal{N} := \{1, \cdots, N\}$. Let $\mathbf{v}_i(t) \in \mathbb{R}^n$ and $\mathbf{x}_i(t) \in \mathbb{R}^n$ represent the position and the velocity of particle i at time t, the integer $n \geq 1$, the function $\mathbf{f}_i(t) \in \mathbb{R}^n$ denote the free will of individual i, then $(x_i(t), v_i(t))$ satisfies:

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i(t), \quad i \in \mathcal{N}, \\ \dot{\mathbf{v}}_i(t) = \alpha \displaystyle\sum_{j=1,\, j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)(\mathbf{v}_j(t) - \mathbf{v}_i(t)) + \mathbf{f}_i(t), \end{cases} \tag{1.4}$$

where interaction coefficient is given as

$$b_{ij}(t) = \frac{\chi_r(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)}{N_i(t)}, \tag{1.5a}$$

$$N_i(t) = card\left\{j : |\mathbf{x}_j(t) - \mathbf{x}_i(t)| < r\right\}. \tag{1.5b}$$

Note that $\chi_r(s) \leq 1$, thus

$$\sum_{j=1,j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t)-\mathbf{x}_i(t)||) < 1.$$

Further, let

$$b_{ii}(t) = 1 - \sum_{j\neq i} b_{ij}(t),$$

which implies that

$$\sum_{j=1}^{N} b_{ij}(||\mathbf{x}_j(t)-\mathbf{x}_i(t)||) = 1.$$

From the particularity of the communication function $\chi_r(s)$ our can know even if the initial time meet the requirements of connectivity, but over time may also be disconnected because of this influence function, more attention should be paid to the change of its connectivity. Roughly, if you look at the free will besides satisfy certain conditions, for on the edge of the particles have the speed to the role of the center, to keep connectivity has certain positive meaning. Due to the features of local function, so during the process of proof we can use the properties of $\chi_r(s)$ to push some conclusions.

Structure of this paper is as follows, in the second section we discuss the model without free will, showing that the formation of the cluster is connected with the r,α, and to explore its cluster formation conditions, its theoretical proofs are given. In the third section, we study the model with free will, giving its cluster formation conditions, and free will meet the conditions required for the cluster to form. In the fourth section, the experimental results and situation of two particles are discussed, and pointed out that the effect of free will function, in the subsequent experiments we present r plays an important role in the formation of the cluster, the formation of free will can promote or prevent cluster. So in the initial stage we can choose the appropriate free will function to promote the formation of the cluster.

For the convenience of the following proof we first give the definition of flocking:

Definition 1.1 ([20]). *The Motsch-Tadmor system* (1.2) *has a time-asymptotic flocking if and only if the solutions* $\{\mathbf{x}_i,\mathbf{v}_i\}$, $i=1,\cdots,N$ *to* (1.2) *satisfy the following two-conditions:*

1. *The velocity fluctuations go to zero time-asymptotically (velocity alignment):*

$$\text{exists a constant } \mathbf{v}_c, \ \ s.t. \ \lim_{t\to\infty}\sum_{i=1}^{N}||\mathbf{v}_i(t)-\mathbf{v}_c||^2 = 0.$$

2. *The position fluctuations are uniformly bounded in time t (forming a group):*

$$\sup_{\substack{0\leq t<\infty\\ i,j\in\mathcal{N}}} ||\mathbf{x}_i(t)-\mathbf{x}_j(t)||^2 < \infty.$$

2 Dynamics analysis for two-particle system

Due to the particularity of $\chi_r(s)$, we see the network topology may be broken in the middle, so we only take two particles as examples to prove the advantage of $f(t)$. As two particles, let $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$, $f_i(t) \in \mathbb{R}$, we have the follow function

$$\begin{cases} \dot{x}_1(t) = v_1, \\ \dot{v}_1(t) = \alpha b_{12}(v_2 - v_1), \end{cases} \qquad \begin{cases} \dot{x}_2(t) = v_2, \\ \dot{v}_2(t) = \alpha b_{21}(v_1 - v_2). \end{cases} \tag{2.1}$$

Without loss of generality, we might let $v_1(0) \le v_2(0)$, then we can see $\frac{dv_1}{dt} \ge 0$, $\frac{dv_2}{dt} \le 0$, while

$$b_{12} = \chi_r(||x_2(t) - x_1(t)||) = b_{21},$$

so $v_1(t)$ will increase and $v_2(t)$ will decrease. If there exists t_0 s.t. $||x_1(t_0) - x_2(t_0)|| > r$, then at time t_0 we add free will $f_i(t)$ to the particles, so the above system of equations will become

$$\begin{cases} \dot{x}_1(t) = v_1, \\ \dot{v}_1(t) = f_1(t), \end{cases} \qquad \begin{cases} \dot{x}_2(t) = v_2, \\ \dot{v}_2(t) = f_2(t). \end{cases} \tag{2.2}$$

So at time t we can get

$$v_1(t) = v_1(t_0) + \int_{t_0}^{t} f_1(s)ds, \quad v_2(t) = v_2(t_0) + \int_{t_0}^{t} f_2(s)ds, \tag{2.3}$$

then

$$x_1(t) = x_1(t_0) + \int_{t_0}^{t} v_1(t)dt = x_1(t_0) + v_1(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{\theta} f_1(s)dsd\theta, \tag{2.4a}$$

$$x_2(t) = x_2(t_0) + \int_{t_0}^{t} v_2(t)dt = x_2(t_0) + v_2(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{\theta} f_2(s)dsdd\theta, \tag{2.4b}$$

so

$$x_2(t) - x_1(t) = x_2(t_0) - x_1(t_0) + (v_2(t_0) - v_1(t_0))(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{\theta} (f_2(s) - f_1(s))dsd\theta,$$

at time t_0, we identify that $|x_1(t_0) - x_2(t_0)| > r$, we consider two cases about $x_1(0)$ and $x_2(0)$, firstly, if $x_1(0) < x_2(0)$, then it can find time t_0 makes $|x_1(t_0) - x_2(t_0)| > r, x_1(t_0) < x_2(t_0)$, secondly, if $x_1(0) > x_2(0)$, we see that the distance between $x_1(t)$ and $x_2(t)$ becoming decrease before $x_2(t_*) = x_1(t_*)$ at time t_* from $v_1(0) \le v_2(0)$ (we don't care about the flocking case that exists time t_2 such that $v_2(t_2) = v_1(t_2), x_2(t_2) \le x_1(t_2)$). So it's exists time t_3 makes $v_2(t_3) > v_1(t_3)$, $x_2(t_3) > x_1(t_3)$ true, then we can think of t_3 as the initial time and it can be classified under the first case. so if $x_2(t_0) - x_1(t_0) > r$, at time t we hope the two particles begin to connect, then we need $x_2(t) - x_1(t) < r$, so we get

$$\int_{t_0}^{t}\int_{t_0}^{\theta}(f_2(s) - f_1(s))dsd\theta < r - (x_2(t_0) - x_1(t_0)) - (v_2(t_0) - v_1(t_0))(t - t_0), \tag{2.5}$$

in the inequality (2.5) we can see $x_2(t_0) - x_1(t_0) > r$, $v_2(t_0) - v_1(t_0) > 0$, if $v_2(t_0) - v_1(t_0) \leq 0$, from the initial condition $v_1(0) < v_2(0)$, we get when $v_1(t^*) = v_2(t^*)$, $\frac{dv_1}{dt} = \frac{dv_2}{dt} = 0$ before time t_0, then they will be flocking and we won't add free will. Hence we only need to consider $v_2(t_0) - v_1(t_0) > 0$ and the sign on the right-hand side of (2.5) is negative.

$$r - (x_1(t_0) - x_2(t_0)) - (v_1(t_0) - v_2(t_0))(t - t_0) < 0,$$

then we easily find t_1 and $f_2(s) - f_1(s) = \mathcal{O}(\frac{1}{t+1})$, $f_2(t) - f_1(t) < 0$ satisfy

$$\int_{t_0}^{t_1}\int_{t_0}^{\theta}(f_1(s) - f_2(s))dsd\theta < r - (x_1(t_0) - x_2(t_0)) - (v_1(t_0) - v_2(t_0))(t - t_0).$$

To sum up, $f_i(t)$ is conducive to the formation of clusters.

Remark 2.1. Through the analysis of the above two particles, we can see that free will has a positive effect on the formation of clusters. When the particles cannot be connected, free will can help to establish connections. So on the other hand, we can strengthen the free will at the beginning, which can help the topological connectivity in the cluster process, and the experiments Fig. 1 also verified this part. We finally remark that it's meaningful to study the multi-particle systems.

3 Main results for multi-particle systems

This section mainly discusses the emergence conditions of flocking in a multi-particle system with and without free will. Before that, we introduce some preliminaries.

Based on graph theory, let $G(t) = (\mathcal{N}, \uparrow(t))$ an undirected graph associated with system (3.2), where $\uparrow(t) = \{(i,j) : |\mathbf{x}_i(t) - \mathbf{x}_j(t)| < r,\ i,j \in \mathcal{N}\}$. A graph is called connected at time t if there exists a path between any two vertexes of the graph at t, i.e., for $\forall i,j \in \mathcal{N}$, there exist $k_1, k_2, \cdots, k_q \in \mathcal{N}$ such that $(i,k_1), (k_1,k_2), \cdots, (k_q,j) \in \uparrow(t)$. For $\forall i,j$ let $a_{ij} = |e_{ij}|$, $e_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle \in \uparrow(t)$. We define $A = (a_{ij})_{N\times N}$ as adjacent matrix. The

$$A^k = (a_{ij}^{(k)})_{N\times N}, \quad \left(a_{ij}^{(k)} = \sum_{h=1}^{m} a_{ih}^{(k-1)} a_{hj}\right),$$

while $a_{ij}^{(k)}$ expresses the arbitrary i, j through k nodes can be connected. In other words $a_{ij}^{(k)}$ indicates form i to j can through k paths. $a_{ij}^{(k)} = 0$ shows there no paths.

Lemma 3.1. *If*

$$\sum_{k=1}^{N-1} a_{ij}^{(k)} = 0,$$

then from indicates i to j has no paths.

In order to visualize the flocking phenomenon of multi-particle swarms, we denote the diameter of position space and velocity space,

$$d_{\mathbf{X}_1}(t) = \max_{i,j\in\mathbb{N}}\{||\mathbf{x}_i(t) - \mathbf{x}_j(t)\}||, \quad d_{\mathbf{V}_1}(t) = \max_{i,j\in\mathbb{N}}\{||\mathbf{v}_i(t) - \mathbf{v}_j(t)||\}. \tag{3.1}$$

Reviewing Definition 1.1, we just need to prove that

$$\sup_{t>0} d_{\mathbf{X}_1}(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} d_{\mathbf{V}_1}(t) = 0$$

to explore the flocking dynamics of system (3.2).

3.1 Flocking without free will

We discuss the situation without free will in detail, that is $\mathbf{f}_i(t) = 0$ for all $i \in \mathcal{N}$, which means that (1.4) can be reduced to

$$\begin{cases} \dot{\mathbf{x}}_i(t) = \mathbf{v}_i(t), \quad i = 1, 2, \cdots, N, \\ \dot{\mathbf{v}}_i(t) = \alpha \displaystyle\sum_{j=1,\, j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)(\mathbf{v}_j(t) - \mathbf{v}_i(t)). \end{cases} \tag{3.2}$$

Lemma 3.2. *Consider system* (3.2) *and let* $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ *be the solution to the system, then for* $i, j \in \Omega(t)$

$$\frac{d}{dt} d_{\mathbf{X}_{ij}}(t) \le d_{\mathbf{V}_{ij}}(t), \tag{3.3a}$$

$$\frac{d}{dt} d_{\mathbf{V}_{ij}}(t) \le \alpha \left(1 - \frac{1}{N}\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right)\right) d_{\mathbf{V}_{1}(t)} - \alpha d_{\mathbf{V}_{ij}(t)}, \tag{3.3b}$$

where d_X *and* d_V *are as shown in* (3.1).

Proof. We proceed to prove this lemma with two steps, we first show that

$$\frac{d}{dt} d_{\mathbf{X}_{ij}}(t) \le d_{\mathbf{V}_{ij}}(t),$$

then show that

$$\frac{d}{dt} d_{\mathbf{V}_{ij}}(t) \le \alpha \left(1 - \frac{1}{N}\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right)\right) d_{\mathbf{V}_{1}(t)} - \alpha d_{\mathbf{V}_{ij}(t)}.$$

Step 1. We can prove $\frac{d}{dt}d_{\mathbf{X}_{ij}}(t) \leq d_{\mathbf{V}_{ij}}(t)$. For a given time t, we choose the particle $i,j \in \Omega(t)$, $\Omega(t) = \{(i,j) : \exists k \text{ s.t. } a_{ij}^{(k)} \neq 0\}$ (there's a path between i and j),

$$d_{\mathbf{X}_{ij}} = ||\mathbf{x}_i(t) - \mathbf{x}_j(t)||, \qquad d_{\mathbf{V}_{ij}}(t) = ||\mathbf{v}_i(t) - \mathbf{v}_j(t)||,$$
$$d_{\mathbf{X}_1} = \max_{i,j\in\mathcal{N}} ||\mathbf{x}_i(t) - \mathbf{x}_j(t)||, \qquad d_{\mathbf{V}_1}(t) = \max_{i,j\in\mathcal{N}} ||\mathbf{v}_i(t) - \mathbf{v}_j(t)||,$$

then

$$\begin{aligned}\frac{d}{dt}d_{\mathbf{X}_{ij}}(t)^2 =&2\langle \dot{\mathbf{x}}_i(t) - \dot{\mathbf{x}}_j(t), \mathbf{x}_i(t) - \mathbf{x}_j(t)\rangle \\ \leq& 2||\mathbf{v}_i(t) - \mathbf{v}_j(t)|| d_{\mathbf{X}_{ij}}(t) \\ \leq& 2 d_{\mathbf{V}_{ij}}(t) d_{\mathbf{X}_{ij}}(t) \end{aligned} \tag{3.4}$$

and then there are

$$\frac{d}{dt}d_{\mathbf{X}_{ij}}(t) \leq d_{\mathbf{V}_{ij}}(t).$$

Step 2. Now, because of $\sum_{j=1}^N b_{ij}(t) = 1$, we consider $d_{\mathbf{V}_{ij}}(t)$, the particle i's velocity satisfies:

$$\begin{aligned}&\left\langle \dot{\mathbf{v}}_i(t) - \dot{\mathbf{v}}_j(t), \mathbf{v}_i(t) - \mathbf{v}_j(t)\right\rangle \\ =&\Bigg\langle \alpha \sum_{k=1}^N b_{ik}(||\mathbf{x}_k(t) - \mathbf{x}_i(t)||)(\mathbf{v}_k(t) - \mathbf{v}_i(t)) \\ &-\alpha \sum_{p=1}^N b_{jp}(||\mathbf{x}_p(t) - \mathbf{x}_j(t)||)(\mathbf{v}_p(t) - \mathbf{v}_j(t)), \mathbf{v}_i(t) - \mathbf{v}_j(t) \Bigg\rangle \\ =&\alpha \sum_{k=1}^N b_{ik} \sum_{p=1,p\neq k}^N b_{jp} \left\langle \mathbf{v}_k(t) - \mathbf{v}_p(t), \mathbf{v}_i(t) - \mathbf{v}_j(t)\right\rangle \\ &- \alpha \left\langle \mathbf{v}_i(t) - \mathbf{v}_j(t), \mathbf{v}_i(t) - \mathbf{v}_j(t)\right\rangle \\ =&\alpha \sum_{k=1}^N b_{ik} \sum_{p=1,p\neq k}^N b_{jp} \left\langle \mathbf{v}_k(t) - \mathbf{v}_p(t), \mathbf{v}_i(t) - \mathbf{v}_j(t)\right\rangle \\ &+ \alpha \sum_{k=1}^N b_{ik} \sum_{p=1,p=k}^N b_{jp} d_{\mathbf{V}_1(t)} d_{\mathbf{V}_{ij}(t)} \\ &- \alpha \sum_{k=1}^N b_{ik} \sum_{p=1,p=k}^N b_{jp} d_{\mathbf{V}_1(t)} d_{\mathbf{V}_{ij}(t)} - \alpha d^2_{\mathbf{V}_{ij}(t)} \\ \leq& \alpha d_{\mathbf{V}_1(t)} d_{\mathbf{V}_{ij}(t)} - \alpha \sum_{k=1}^N b_{ik} \sum_{p=1,p=k}^N b_{jp} d_{\mathbf{V}_1(t)} d_{\mathbf{V}_{ij}(t)} - \alpha d^2_{\mathbf{V}_{ij}(t)}, \end{aligned} \tag{3.5}$$

then

$$
\begin{aligned}
&\langle \dot{\mathbf{v}}_i(t)-\dot{\mathbf{v}}_j(t), \mathbf{v}_i(t)-\mathbf{v}_j(t)\rangle \\
\leq&\alpha\left(1-\sum_{k=1}^{N} b_{ik}\sum_{p=1,p=k}^{N} b_{jp}\right) d_{\mathbf{V}_1(t)} d_{\mathbf{V}_{ij}(t)} - \alpha d^2_{\mathbf{V}_{ij}(t)} \\
\leq&2\alpha\left(1-\sum_{k=1}^{N} b_{ik}\sum_{p=1,p=k}^{N} b_{jp}\right) d_{\mathbf{V}_1(t)} d_{\mathbf{V}_{ij}(t)} - 2\alpha d^2_{\mathbf{V}_{ij}(t)},
\end{aligned}
\tag{3.6}
$$

for a given time t, we have

$$
\begin{aligned}
\frac{d}{dt} d_{\mathbf{V}_{ij}}(t) \leq& \alpha\left(1-\frac{1}{N_i(t)N_j(t)} N\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right)\right) d_{\mathbf{V}_1(t)} - \alpha d_{\mathbf{V}_{ij}(t)} \\
\leq& \alpha\left(1-\frac{1}{N}\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right)\right) d_{\mathbf{V}_1(t)} - \alpha d_{\mathbf{V}_{ij}(t)},
\end{aligned}
\tag{3.7}
$$

when $d_{\mathbf{V}_{ij}(t)} = d_{\mathbf{V}_1(t)}$, we have

$$
\frac{d}{dt} d_{\mathbf{V}_1}(t) \leq -\alpha\frac{1}{N}\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right) d_{\mathbf{V}_1(t)}.
$$

The proof is completed. □

Theorem 3.1. *Let* $(\mathbf{x}_j(t), \mathbf{v}_j(t))$ *be a solution to system* (3.2). $\chi_r(s)$ *is defined in* (1.3). *Suppose that initial configurations and r satisfies the following conditions,*

(1). there exists k such that $a^{(k)}_{ij}(0) \neq 0$ *for all* $i,j\in\mathcal{N}$, *and* $\alpha \min_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0) \geq d_{\mathbf{X}_{ij}}(0)$ *for all* $i,j\in\Omega(0)$;

(2). $\frac{\alpha^2}{N}\int^{r}_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}} \chi_r^2(s)ds > \max\limits_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}(0)} + \frac{N}{\chi^2(\kappa)} d_{\mathbf{V}_1}(0).$

Then the system will achieve flocking.

Remark 3.1. Condition (1). ensures initial connectivity and gives the lower bound on α. Condition (2). defines the lower bound of r by α, $\min_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}$, $\max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}(0)}$, $d_{\mathbf{V}_1}(0)$.

Proof. Roughly speaking, we prove this theorem by three steps. Firstly we introduce a function $E_{ij}(t)$ and then we prove the not increasing property of $E_{ij}(t)$. Secondly we show that bounded property of $d_{\mathbf{X}_{ij}}(t)$ through the $E_{ij}(t)$. Lastly we use the Gronwall inequality to prove $\lim_{t\to\infty} d_{\mathbf{V}_1}(t) = 0$.

For a given time t, we choose the particle $i,j\in\Omega(t)$, $d_{\mathbf{X}_{ij}} = ||\mathbf{x}_i(t)-\mathbf{x}_j(t)||$, $d_{\mathbf{V}_{ij}}(t) = ||\mathbf{v}_i(t)-\mathbf{v}_j(t)||$, we introduce a function

$$
E_{ij}(t) = \max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \alpha\frac{1}{N}\int_0^{d_{\mathbf{X}_{ij}}(t)} \chi_r^2(s)ds + \frac{N}{\chi^2(\kappa)} d_{\mathbf{V}_1}(t).
$$

Taking the derivative of the energy function $E_{ij}(t)$, then we apply the Lemma 3.2, let $\kappa = \max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)$, besides due to $\chi^2(\kappa) \le 1$, then

$$\begin{aligned}
E_{ij}^{'}(t) =& \frac{d}{dt}\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \alpha\frac{1}{N}\chi_r^2\left(d_{\mathbf{X}_{ij}}(t)\right)\frac{d}{dt}d_{\mathbf{X}_{ij}}(t) + \frac{N}{\chi^2(\kappa)}\frac{d}{dt}d_{\mathbf{V}_{1(t)}} \\
\le& \alpha\left(1-\frac{\chi^2(\kappa)}{N}\right)d_{\mathbf{V}_{1(t)}} - \alpha\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \alpha\frac{\chi^2\left(d_{\mathbf{X}_{ij}}(t)\right)}{N}d_{\mathbf{V}_{ij}}(t) - \alpha d_{\mathbf{V}_1}(t) \\
=& -\alpha\frac{\chi^2(\kappa)}{N}d_{\mathbf{V}_{1(t)}} - \alpha\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \alpha\frac{\chi^2\left(d_{\mathbf{X}_{ij}}(t)\right)}{N}d_{\mathbf{V}_{ij}}(t) \\
\le& -\alpha\frac{\chi^2(\kappa)}{N}d_{\mathbf{V}_{1(t)}} - \alpha d_{\mathbf{V}_{ij}}(t) + \alpha\frac{\chi^2\left(d_{\mathbf{X}_{ij}}(t)\right)}{N}d_{\mathbf{V}_{ij}}(t) \\
\le& -\alpha\frac{\chi^2(\kappa)}{N}d_{\mathbf{V}_{1(t)}} - \alpha d_{\mathbf{V}_{ij}}(t) + \alpha d_{\mathbf{V}_{ij}}(t) \le 0.
\end{aligned} \tag{3.8}$$

So the energy function is not increasing. Due to

$$\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}}^{r} \chi_r^2(s)ds > \max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}(0)} + \frac{N}{\chi^2(\kappa)}d_{\mathbf{V}_1}(0),$$

we can find a constant C_{ij}, $C_{ij} > \min_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)$, let

$$\max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}(0)} + \frac{N}{\chi^2(\kappa)}d_{\mathbf{V}_1}(0) = \frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)}^{C_{ij}} \chi_r^2(s)ds,$$

we fix t and from the not increasing function we can know that

$$\begin{aligned}
&\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \alpha\frac{1}{N}\int_0^{d_{\mathbf{X}_{ij}}(t)} \chi_r^2(s)ds + \frac{N}{\chi^2(\kappa)}d_{\mathbf{V}_1}(t) \\
\le& \max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}}(0) + \alpha\frac{1}{N}\int_0^{d_{\mathbf{X}_{ij}}(0)} \chi_r^2(s)ds + \frac{N}{\chi^2(\kappa)}d_{\mathbf{V}_1}(0),
\end{aligned}$$

then,

$$\begin{aligned}
&\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \frac{N}{\chi^2(\kappa)}d_{\mathbf{V}_1}(t) \\
\le& \max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}}(0) + \frac{N}{\chi^2(\kappa)}d_{\mathbf{V}_1}(0) + \alpha\frac{1}{N}\int_{d_{\mathbf{X}_{ij}}(t)}^{d_{\mathbf{X}_{ij}}(0)} \chi_r^2(s)ds,
\end{aligned}$$

from the above condition,

$$\begin{aligned}
&\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \frac{N}{\chi^2(\kappa)} d_{\mathbf{V}_1}(t)\\
\le&\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)}^{C_{ij}} \chi_r^2(s)ds + \frac{\alpha}{N}\int_{d_{\mathbf{X}_{ij}}(t)}^{d_{\mathbf{X}_{ij}}(0)} \chi_r^2(s)ds\\
=&\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)}^{C_{ij}} \chi_r^2(s)ds + \frac{\alpha^2}{N}\int_{\frac{d_{\mathbf{X}_{ij}}(t)}{\alpha}}^{\frac{d_{\mathbf{X}_{ij}}(0)}{\alpha}} \chi_r^2(\alpha t)dt\\
\le&\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)}^{C_{ij}} 1ds + \frac{\alpha^2}{N}\int_{\frac{d_{\mathbf{X}_{ij}}(t)}{\alpha}}^{\frac{d_{\mathbf{X}_{ij}}(0)}{\alpha}} 1dt,
\end{aligned}\tag{3.9}$$

because

$$\alpha \ge \frac{d_{\mathbf{X}_{ij}}(0)}{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)},$$

we get

$$\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t) + \frac{N}{\chi^2(\kappa)} d_{\mathbf{V}_1}(t) \le \frac{\alpha^2}{N}\int_{\frac{d_{\mathbf{X}_{ij}}(t)}{\alpha}}^{C_{ij}} 1ds.$$

Due to $\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t)$, $\frac{N}{\chi^2(\kappa)} d_{\mathbf{V}_1}(t)$ both positive, then we can derive: $d_{\mathbf{X}_{ij}}(t) < \alpha C_{ij}$, while $t \in (0,\infty)$, we can find $i,j \in \uparrow(t)$ as $d_{\mathbf{X}_1}(t) = \max_{i,j\in\mathbb{N}} ||\mathbf{x}_i(t) - \mathbf{x}_j(t)||$, then we can get

$$d_{\mathbf{X}_1}(t) < \alpha(C_{ik_1} + C_{k_1k_2} + \cdots + C_{k_qj}) < M.$$

From above, there are

$$\frac{d}{dt} d_{\mathbf{V}_1}(t) \le -\frac{\alpha}{N}\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right) d_{\mathbf{V}_1(t)},$$

then $d_{\mathbf{V}_1}(t) \le d_{\mathbf{V}_1}(0)e^{-C_1^* t}$ can be derived from the Gronwall inequality, while

$$C_1^* = \frac{\alpha}{N}\chi^2\left(\max_{i,j\in\uparrow(t)} d_{\mathbf{X}_{ij}}(t)\right),$$

and we know

$$\lim_{t\to\infty} d_{\mathbf{V}_1}(t) = 0.$$

This completes the proof of Theorem 3.1. □

3.2 Flocking with free will

The following are our main theorems about M-T model with free will.

Theorem 3.2. *Let* $(\mathbf{x}_i(t),\mathbf{v}_i(t))$ *be the solution to the system* (1.4). $\chi_r(s)$ *is defined in* (1.3). *Suppose that the initial configurations and the free will functions satisfy,*

(1). there exists k such that $a_{ij}^{(k)}(0)\neq 0$ *for all* $i,\ j\in\mathcal{N}$, *and* $d_{\mathbf{X}_{ij}}(0)\leq \alpha \min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}}(0)$;

(2). $\max\limits_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}}(0)+(1+\frac{1}{N\chi^2(\kappa)})\int_0^\infty ||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds+\frac{d_{\mathbf{V}_{ij}}(0)}{N\chi^2(\kappa)}<\frac{\alpha^2}{N}\int^r_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}}\chi_r^2(s)ds$, *for all* $i,\ j\in\mathcal{N}$;

Then the system (1.4) *will achieve flocking.*

Remark 3.2. Condition (2) ensures the boundedness of the free will function.

Proof. The proof is the same as Theorem 3.1. For any pair of individuals i and j, $i,j\in\Omega(t)$, their velocities $\mathbf{v}_i(t)$ and $\mathbf{v}_j(t)$ satisfies

$$
\begin{aligned}
&2\left\langle \dot{\mathbf{v}}_i(t)-\dot{\mathbf{v}}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
=&2\Bigg\langle \alpha\sum_{k=1,k\neq i}^N b_{ik}(||\mathbf{x}_k(t)-\mathbf{x}_i(t)||)(\mathbf{v}_k(t)-\mathbf{v}_i(t))\\
&-\alpha\sum_{l=1,l\neq j}^N b_{jl}(||\mathbf{x}_l(t)-\mathbf{x}_j(t)||)(\mathbf{v}_l(t)-\mathbf{v}_j(t)),\mathbf{v}_i(t)-\mathbf{v}_j(t)\Bigg\rangle\\
&+2\left\langle \mathbf{f}_i(t)-\mathbf{f}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
=&2\Bigg\langle \alpha\sum_{k=1}^N b_{ik}(||\mathbf{x}_k(t)-\mathbf{x}_i(t)||)(\mathbf{v}_k(t)-\mathbf{v}_i(t))\\
&-\alpha\sum_{l=1}^N b_{jl}(||\mathbf{x}_l(t)-\mathbf{x}_j(t)||)(\mathbf{v}_l(t)-\mathbf{v}_j(t)),\mathbf{v}_i(t)-\mathbf{v}_j(t)\Bigg\rangle\\
&+2\left\langle \mathbf{f}_i(t)-\mathbf{f}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
=&2\alpha\sum_{k=1}^N\sum_{l=1,l\neq k}^N b_{ik}b_{jl}\left\langle \mathbf{v}_k(t)-\mathbf{v}_l(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
&-2\alpha\left\langle \mathbf{v}_i(t)-\mathbf{v}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle+2\left\langle \mathbf{f}_i(t)-\mathbf{f}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
=&2\alpha\sum_{k=1}^N\sum_{l=1,l\neq k}^N b_{ik}b_{jl}\left\langle \mathbf{v}_k(t)-\mathbf{v}_l(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
&+2\alpha\sum_{k=1}^N\sum_{l=1,l=k}^N b_{ik}b_{jl}d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)-2\alpha\sum_{k=1}^N\sum_{l=1,l=k}^N b_{ik}b_{jl}d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)
\end{aligned}
$$

$$
\begin{aligned}
&-2\alpha\left\langle \mathbf{v}_i(t)-\mathbf{v}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle+2\left\langle \mathbf{f}_i(t)-\mathbf{f}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle\\
\leq& 2\alpha d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)-2\alpha\sum_{k=1}^{N}\sum_{l=1,l=k}^{N}b_{ik}b_{jl}d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)\\
&-2\alpha\left\langle \mathbf{v}_i(t)-\mathbf{v}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle+2\left\langle \mathbf{f}_i(t)-\mathbf{f}_j(t),\mathbf{v}_i(t)-\mathbf{v}_j(t)\right\rangle.
\end{aligned}
\tag{3.10}
$$

For any given time t, we can choose two individuals i and j, $i,j\in\Omega(t)$, then

$$
\begin{aligned}
\frac{d}{dt}d^2_{\mathbf{V}_{ij}}(t)\leq& 2\alpha d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)-2\alpha\sum_{k=1}^{N}\sum_{l=1,l=k}^{N}b_{ik}b_{jl}d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)\\
&-2\alpha d^2_{\mathbf{V}_{ij}}(t)+2||\mathbf{f}_i(t)-\mathbf{f}_j(t)||d_{\mathbf{V}_{ij}}(t)\\
\leq& 2\alpha d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)-2\alpha N\chi^2\left(\max_{i,j\in\uparrow(t)}d_{\mathbf{X}_{ij}}(t)\right)d_{\mathbf{V}}(t)d_{\mathbf{V}_{ij}}(t)\\
&-2\alpha d^2_{\mathbf{V}_{ij}}(t)+2||\mathbf{f}_i(t)-\mathbf{f}_j(t)||d_{\mathbf{V}_{ij}}(t),
\end{aligned}
\tag{3.11}
$$

which means that

$$
\begin{aligned}
\frac{d}{dt}d_{\mathbf{V}_{i,j}}(t)\leq&\alpha d_{\mathbf{V}}(t)-\alpha N\chi^2\left(\max_{i,j\in\uparrow(t)}d_{\mathbf{X}_{ij}}(t)\right)d_{\mathbf{V}}(t)\\
&-\alpha d_{\mathbf{V}_{ij}}(t)+||\mathbf{f}_i(t)-\mathbf{f}_j(t)||,
\end{aligned}
\tag{3.12}
$$

when $d_{\mathbf{V}_{ij}}(t)=d_{\mathbf{V}}(t)=\max_{i,j\in\mathcal{N}}||\mathbf{v}_i(t)-\mathbf{v}_j(t)||$, there is

$$
\frac{d}{dt}d_{\mathbf{V}}(t)\leq-\alpha N\chi^2\left(\max_{i,j\in\uparrow(t)}d_{\mathbf{X}_{ij}}(t)\right)d_{\mathbf{V}}(t)+||\mathbf{f}_i(t)-\mathbf{f}_j(t)||.
$$

Similar to the discussion in Theorem 3.1, let $\kappa=\max_{i,j\in\uparrow(t)}d_{\mathbf{X}_{ij}}(t)$, $\forall i,j\in\Omega(t)$, we construct the energy function

$$
\begin{aligned}
E_1(t)=&\max_{i,j\in\Omega(t)}d_{\mathbf{V}_{ij}}(t)-\left(1+\frac{1}{N\chi^2(\kappa)}\right)\int_0^t||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds\\
&+\frac{\alpha}{N}\int_0^{d_{\mathbf{X}_{ij}}(t)}\chi^2(s)ds+\frac{d_{\mathbf{V}_{ij}}(t)}{N\chi^2(\kappa)},
\end{aligned}
$$

we easily know that $E_1'\leq 0$. it's similar to Theorem 3.1, from the condition (1) and (2) we can find a positive constant d^*_{11}:

$$
\begin{aligned}
&\max_{i,j\in\Omega(0)}d_{\mathbf{V}_{ij}(0)}+\left(1+\frac{1}{N\chi^2(\kappa)}\right)\int_0^t||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds\\
&+\frac{d_{\mathbf{V}_{ij}}(0)}{N\chi^2(\kappa)}=\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)}d_{\mathbf{X}_{ij}(0)}}^{d^*_{11}}\chi_r^2(s)ds,
\end{aligned}
$$

due to the non-increasing function we get that:

$$\begin{aligned}
&\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t)-\Big(1+\frac{1}{N\chi^2(\kappa)}\Big)\int_0^t ||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds+\frac{\alpha}{N}\int_0^{d_{\mathbf{X}_{ij}}(t)}\chi^2(s)ds+\frac{d_{\mathbf{V}_{ij}}(t)}{N\chi^2(\kappa)}\\
\le&\max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}}(0)+\frac{\alpha}{N}\int_0^{d_{\mathbf{X}_{ij}}(0)}\chi^2(s)ds+\frac{d_{\mathbf{V}_{ij}}(0)}{N\chi^2(\kappa)},
\end{aligned}\tag{3.13}$$

then we know,

$$\begin{aligned}
&\max_{i,j\in\Omega(t)} d_{\mathbf{V}_{ij}}(t)+\frac{d_{\mathbf{V}_{ij}}(t)}{N\chi^2(\kappa)}\\
\le&\max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}}(0)+\Big(1+\frac{1}{N\chi^2(\kappa)}\Big)\int_0^t ||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds+\frac{\alpha}{N}\int_{d_{\mathbf{X}_{ij}}(t)}^{d_{\mathbf{X}_{ij}}(0)}\chi^2(s)ds+\frac{d_{\mathbf{V}_{ij}}(0)}{N\chi^2(\kappa)}\\
\le&\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}}^{d_{11}^*}\chi_r^2(s)ds+\frac{\alpha}{N}\int_{d_{\mathbf{X}_{ij}}(t)}^{d_{\mathbf{X}_{ij}}(0)}\chi^2(s)ds\\
\le&\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}}^{d_{11}^*}\chi_r^2(s)ds+\frac{\alpha^2}{N}\int_{\frac{d_{\mathbf{X}_{ij}}(t)}{\alpha}}^{\frac{d_{\mathbf{X}_{ij}}(0)}{\alpha}}\chi^2(\alpha t)dt\\
\le&\frac{\alpha^2}{N}\int_{\min\limits_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)}}^{d_{11}^*}1ds+\frac{\alpha^2}{N}\int_{\frac{d_{\mathbf{X}_{ij}}(t)}{\alpha}}^{\frac{d_{\mathbf{X}_{ij}}(0)}{\alpha}}1dt\overset{condition(1)}{\le}\frac{\alpha^2}{N}\int_{\frac{d_{\mathbf{X}_{ij}}(t)}{\alpha}}^{d_{11}^*}1ds,
\end{aligned}\tag{3.14}$$

then $d_{\mathbf{X}}(t)\le d_{11}^*$, due to the $\chi(s)$ is positive and non-increasing, so $\chi^2\left(d_{\mathbf{X}}(t)\right)\ge\chi^2(d_{11}^*)$, let $C_{11}^*=\alpha N\chi^2(d_{11}^*)$, then

$$\frac{d}{dt}d_{\mathbf{V}}(t)\le -C_{11}^*d_{\mathbf{V}}(t)+\sum_{i,j\in\mathbb{N}}||\mathbf{f}_i(t)-\mathbf{f}_j(t)||,$$

from the above equation we can obtain that

$$d_{\mathbf{V}}(t)\le e^{-C_{11}^*t}d_{\mathbf{V}}(0)+\sum_{i,j\in\mathbb{N}}e^{-C_{11}^*t}\int_0^t e^{C_{11}^*s}||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds,\tag{3.15}$$

because

$$\lim_{t\to\infty}e^{-C_{11}^*t}d_{\mathbf{V}}(0)=0,$$

and

$$\lim_{t\to\infty}e^{-C_{11}^*t}\int_0^t e^{C_{11}^*s}||\mathbf{f}_i(s)-\mathbf{f}_j(s)||=0,$$

so

$$\lim_{t\to\infty}d_{\mathbf{V}}(t)=0.$$

From the above description, it can be known that the system is clustered. □

Remark 3.3. In the above theorem, we let simply $\varphi(\mathbf{x}) = 1$, then we can get

$$\frac{N}{\alpha^2}\max_{i,j\in\Omega(0)} d_{\mathbf{V}_{ij}}(0) + \left(\frac{N}{\alpha^2}+\frac{1}{\alpha^2}\right)\int_0^\infty ||\mathbf{f}_i(s)-\mathbf{f}_j(s)||ds + \frac{d_{\mathbf{V}_{ij}}(0)}{\alpha^2} + \min_{i,j\in\Omega(0)} d_{\mathbf{X}_{ij}(0)} < r$$

for all $i,\ j \in \mathcal{N}$ from Theorem 3.2. It is means that we get the lower bound of r.

4 Numerical simulations

In this section, a simple one-dimensional case is selected for numerical experiments to illustrate the problem.

• **Scenario 1 for two-particle systems:** In Fig. 1, there is no free will at the beginning. So it will be disconnected in (a) and form flocking in (b). This means that free will can facilitate communication between particles and facilitate flocking.

• **Scenario 2 for multi-particle systems:** Fig. 2 is to verify Theorem 3.1. It can be seen that they all meet the connectivity at the initial time, but with the reduction of r, they do not meet the conditions of the Theorem 3.1, so it can be seen that no cluster is formed in (b). Thus r has an important influence on the formation of flocking.

• **Scenario 3 for multi-particle systems:** Fig. 3 is to verify Theorem 3.2. They all satisfy the initial connectivity condition. In (b), it don't form flocking. On the contrary, in (a) it can achieve flocking. On the other hand, this experiment also shows that r has a greater influence on the model than free will.

• **Scenario 4 for multi-particle systems:** Fig. 4 is to verify the free will functions can speed up the generation of clusters. This experiment shows that under the same con-

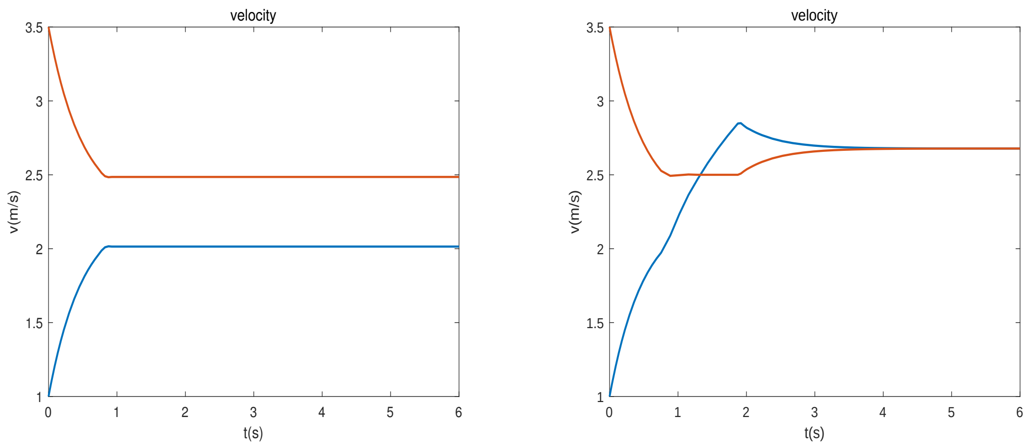

Figure 1: Let $\varphi(\mathbf{x}) = 1$ and set $x_1(0) = -4$, $v_1(0) = 1$; $x_2(0) = 0$, $v_2(0) = 3.5$. In (a), two particles are disconnected around 0.8s and cannot form a cluster, but in (b), at the same time point, we add the free will function $f_1(t) = \frac{1}{t}$, $f_2(t) = 0$ to find that it will form a flocking.

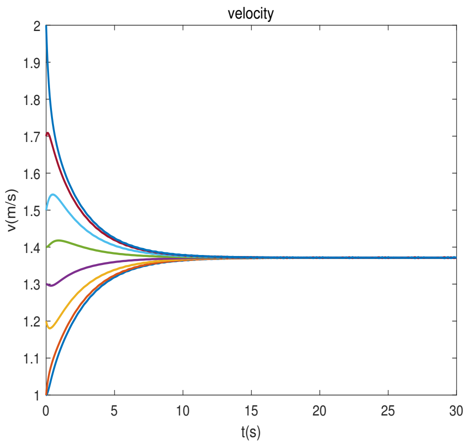

Figure 2: Let's $\varphi(\mathbf{x}) = 1$. Set $\alpha = 4$, $X(0) = (-4, 0, 4, 8, 12, 16, 20, 24)$, $V(0) = (1, 1, 1.2, 1.3, 1.4, 1.5, 1.7, 2)$, in (a) we take $r = 8$, and in (b) we take $r = 4.3$. In (a), the flocking forms in about 17s ($d_V(t) < 0.001$).

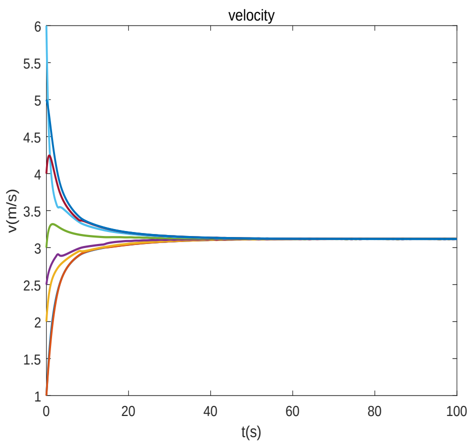

Figure 3: We take $\varphi(\mathbf{x}) = 1$. Set $\alpha = 2$, $X(0) = (-4, 0, 4, 8, 12, 16, 20, 24)$, $V(0) = (1, 1, 2, 2.5, 3, 6, 4, 5)$. In both (a) and (b), we take $f_1(t) = \frac{1}{t^2+2}$, $f_3(t) = \exp(-t)$, $f_6(t) = -\frac{1}{t^2+1}$, other $f_i(t)$ are zero. We take $r = 12$ and $r = 8$ in (a) and (b) separately. It can achieve flocking in (a) about 70s.

dition of clustering, appropriate selection of free will can accelerate the occurrence of clustering behavior.

• **Scenario 5 for noise perturbations:** Fig. 5 uses two kinds of noise for simulation. Theorem 3.2 tells us that $\int_0^\infty ||\mathbf{f}_i(s) - \mathbf{f}_j(s)||ds$ has to be bounded. In the noise term

$$\sigma\alpha \sum_{j=1, j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)(\mathbf{v}_j(t) - \mathbf{v}_i(t))dw_i,$$

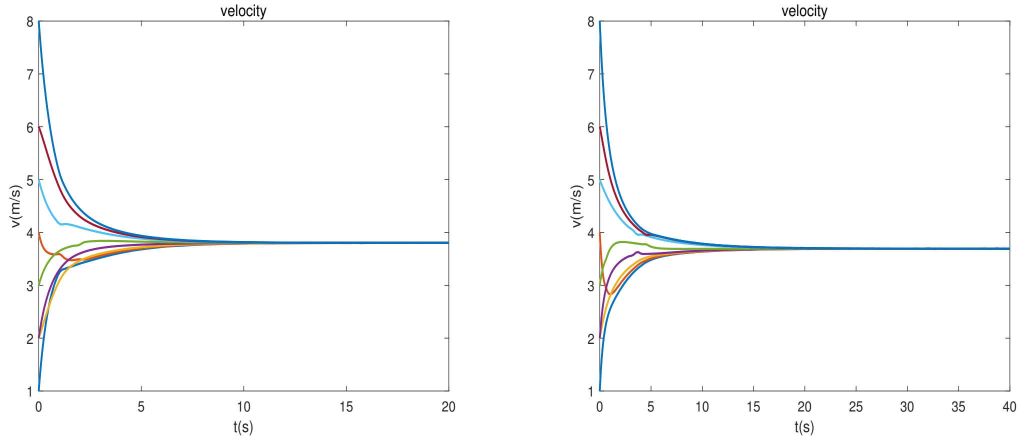

Figure 4: Set $\alpha = 4$, $r = 12$, $X(0) = (-4,0,4,8,12,16,20,24)$, $V(0) = (1,4,2,2,3,5,6,8)$. We let $\varphi(\mathbf{x}) = \frac{1}{(1+||\mathbf{x}_i(t)-\mathbf{x}_j(t)||^2)^\beta}$. In (a), we add free will in a fix time($d_{\mathbf{V}}(t) < 2$), taking $f_1(t) = \frac{1}{t^3+1}$, $f_2(t) = \frac{1}{t^2+0.5}$, $f_6(t) = -\frac{1}{t^2+1}$, $f_8(t) = -\frac{1}{t^3+2}$ other $f_i(t)$ are zero, but in (b), $\forall i$, $f_i(t) = 0$. In (b) the flocking will be formed in about 34s $(d_V(t) < 0.001)$ but in (a) it about 18s.

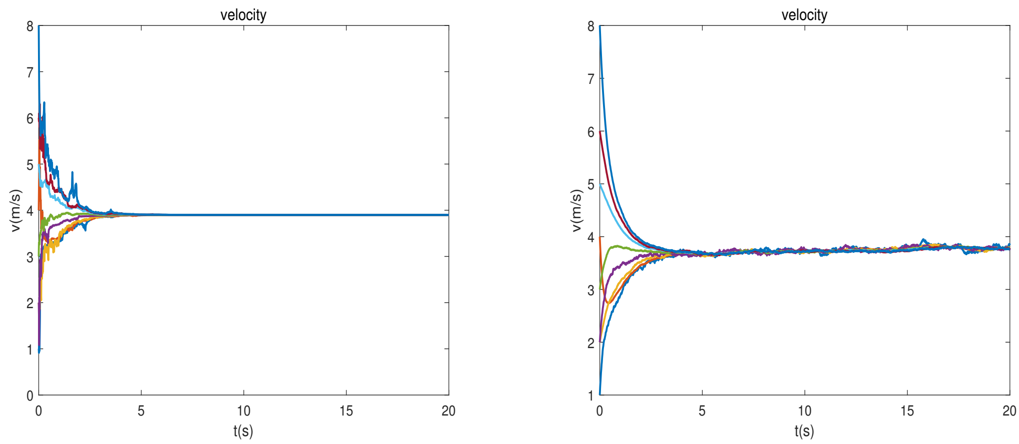

Figure 5: Set $\alpha = 4$, $X(0) = (-4,0,4,8,12,16,20,24)$, $V(0) = (1,4,2,2,3,5,6,8)$. In (a), the expression of the second equation of (1.4) becomes $\dot{\mathbf{v}}_i(t) = \alpha \sum\limits_{j=1,j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)(\mathbf{v}_j(t) - \mathbf{v}_i(t)) + \sigma\alpha \sum\limits_{j=1,j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)(\mathbf{v}_j(t) - \mathbf{v}_i(t))dw_i$, where $W(t) = (w_1(t), \cdots, w_8(t))$ is the 8-dimensional Brownian motions, besides, we let noise intensity σ is 50. In (b), the expression of the second equation of (1.4) becomes $\dot{\mathbf{v}}_i(t) = \alpha \sum_{j=1,j\neq i}^{N} b_{ij}(||\mathbf{x}_j(t) - \mathbf{x}_i(t)||)(\mathbf{v}_j(t) - \mathbf{v}_i(t)) + \sigma_i dw_i$ and the σ is $[0.1, 0.02, 0.05, 0.09, 0.03, 0.015, 0.025, 0.033]$.

if $v_i(t) = v_j(t)$ is true, then $||\mathbf{f}_i(s) - \mathbf{f}_j(s)|| = 0$ and the integral is bounded. But in $\sigma_i dw_i$, $||\mathbf{f}_i(s) - \mathbf{f}_j(s)|| > 0$ and the integral is unbounded. So the flocking will be occur in (a).

5 Conclusions

Through the above discussion, we get the condition of r forming cluster through α, $\min_{i,j\in\uparrow(0)} d_{\mathbf{X}_{ij}(0)}$, $\max_{i,j\in\uparrow(0)} d_{\mathbf{V}_{ij}(0)}$, $d_{\mathbf{V}_1}(0)$, $\mathbf{f}_i(t)$. What can be further discussed in this paper is the change of connection topology in the process of particle motion. It is certain that the connection topology will change with the distance between particles due to the particularity of communication function. In addition to the above conclusions, it can be known from Lemma 3.2 and (3.15) that

$$\frac{d}{dt}d_{\mathbf{X}_1}(t) \leq e^{-C_{11}^* t} d_{\mathbf{V}}(0) + \sum_{i,j\in\mathbb{N}} e^{-C_{11}^* t}\int_0^t e^{C_{11}^* s}||\mathbf{f}_i(s) - \mathbf{f}_j(s)||ds$$

is true, so when the reduction rate of r is smaller than that, no cluster may be generated. Therefore, there should be a critical value of the reduction rate of r theoretically. Finally, in future work, I would like to study what happens when the hierarchy has the communication function in this article.

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