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Bilinear Pseudo-Differential Operator and Its Commutator on Generalized Fractional Weighted Morrey Spaces

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Abstract. The aim of this paper is to establish the boundedness of bilinear pseudo-differential operator T_σ and its commutator $[b_1, b_2, T_\sigma]$ generated by T_σ and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ on generalized fractional weighted Morrey spaces $L^{p,\eta,\varphi}(\omega)$. Under assumption that a weight satisfies a certain condition, the authors prove that T_σ is bounded from products of spaces $L^{p_1,\eta_1,\varphi}(\omega_1) \times L^{p_2,\eta_2,\varphi}(\omega_2)$ into spaces $L^{p,\eta,\varphi}(\vec{\omega})$, where $\vec{\omega} = (\omega_1, \omega_2) \in A_{\vec{P}}$, $\vec{P} = (p_1, p_2)$, $\eta = \eta_1 + \eta_2$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p_1, p_2 \in (1, \infty)$. Furthermore, the authors show that the $[b_1, b_2, T_\sigma]$ is bounded from products of generalized fractional Morrey spaces $L^{p_1,\eta_1,\varphi}(\mathbb{R}^n) \times L^{p_2,\eta_2,\varphi}(\mathbb{R}^n)$ into $L^{p,\eta,\varphi}(\mathbb{R}^n)$. As corollaries, the boundedness of the T_σ and $[b_1, b_2, T_\sigma]$ on generalized weighted Morrey spaces $L^{p,\varphi}(\omega)$ and on generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ is also obtained.

Key Words: Generalized fractional weighted Morrey space, bilinear pseudo-differential operator, commutator, space $\mathrm{BMO}(\mathbb{R}^n)$.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction

In 1967, Hörmander first introduced the definition of a pseudo-differential operator (see [13]), that is, let $\sigma(x,\xi)$ be a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$, then the pseudo-differential operator $\widetilde{T}_\sigma$ is defined by

$$\widetilde{T}_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi \quad \text{for } f \in \mathcal{S}, \tag{1.1}$$

where $\widehat{f}$ represents the Fourier transform of f, and the smooth function σ belongs to the symbol classes $S^m_{\rho,\delta}$, which consist of all σ with satisfying the differential inequality

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}$$

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for multi-indices $\alpha,\beta \in \mathbb{N}^n$, where $m \in \mathbb{R}$ and $0 \le \rho,\delta \le 1$. Such operators not only generalize the definition of differential operators with variable coefficients, but also have a key application in PDE. Therefore, the study of the pseudo-differential operator $\widetilde{T}_\sigma$ is widely focused. For example, Calderón and Vaillancourt in [5] proved that $\widetilde{T}_\sigma$ is bounded on space $L^2(\mathbb{R}^n)$. In 1988, Cardery and Seeger obtained the boundedness of pseudo-differential operator $\widetilde{T}_\sigma$ on spaces L^p (see [4]). The more researches about the pseudo-differential operators $\widetilde{T}_\sigma$ on various of function spaces can be seen [1,2,10,11,14] and the references therein.

However, in 1975, Coifman and Meyer obtained the definition of bilinear pseudo-differential operators and their some properties (see [8]). Namely, let $m \in \mathbb{R}$ and $\rho,\delta \in [0,1]$. A symbol in $BS^m_{\rho,\delta}$ is a smooth function $\sigma(x,\xi,\eta)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices $\alpha,\beta,\gamma \in \mathbb{N}^n$, the following inequality

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{m-\rho(|\beta|+|\gamma|)+\delta|\alpha|}$$

holds. Respectively, the bilinear pseudo-differential operators T_σ associated with the above function $\sigma(x,\xi,\eta) \in BS^m_{\rho,\delta}$ is defined by

$$T_\sigma(f_1,f_2)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi,\eta)\widehat{f_1}(\xi)\widehat{f_2}(\eta)e^{ix\cdot(\xi+\eta)}d\xi d\eta \quad \text{for } f_1,f_2 \in \mathcal{S}. \tag{1.2}$$

In this paper, we will mainly consider the symbol $\sigma(x,\xi,\eta) \in BS^0_{1,0}$, that is,

$$\begin{aligned}&|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)|\\ \le& C_{\alpha,\beta,\gamma}(1+|\xi|+|\eta|)^{-(|\beta|+|\gamma|)} \quad \text{for all multi-indices } \alpha,\beta,\gamma \in \mathbb{N}^n.\end{aligned} \tag{1.3}$$

If we denote $\kappa(x,y,z)$ by the inverse Fourier transform (in the ξ-variable and η-variable) of the function $\sigma(x,\xi,\eta)$ (i.e., $\kappa(x,y,z) = \mathcal{F}_\xi^{-1}\mathcal{F}_\eta^{-1}\sigma(x,\xi,\eta)$), then

$$T_\sigma(f_1,f_2)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \kappa(x,y,z)f_1(x-y)f_2(x-z)dydz. \tag{1.4}$$

Further, if we set $K(x,y,z) = \kappa(x,x-y,x-z)$, then the bilinear pseudo-differential operators T_σ defined as in (1.4) is changed into the following standard form

$$T_\sigma(f_1,f_2)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y,z)f_1(y)f_2(z)dydz. \tag{1.5}$$

Since then, the research about T_σ defined as in (1.5) on various function spaces is widely focused. For example, Bényi and Torres proved that T_σ is bounded from the products of spaces $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ for all $1 < p,q < \infty$ (see [3]). In 2012, Xiao et al. [26] showed that T_σ is bounded on the products of local Hardy spaces. More researches on the bilinear pseudo-differential operators can be seen [18–20, 25].

Before stating the organization of this paper, we first recall the definition of bound mean oscillation space $=$ BMO$(\mathbb{R}^n)$) in [15].

Definition 1.1. *A function* $f \in L^1_{\rm loc}(\mathbb{R}^n)$ *is said to be in the space* $\mathrm{BMO}(\mathbb{R}^n)$ *if*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|}\int_B |f(y) - f_B|dy, \tag{1.6}$$

where f_B *represents the mean value of function* f *over ball* B*, that is*

$$f_B = \frac{1}{|B|}\int_B f(y)dy.$$

Regard as an important type of non-convolution Calderón-Zygmund operator, Coifman-Rochberg-Weiss in [9] obtained the definition of commutator defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x) \quad \text{for any } \ x \in \mathbb{R}^n.$$

Moreover, such operator has key applications in PDE (see [6, 7, 12]). Given $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, the commutator $[b_1, b_2, T_\sigma]$ generated by b_1, b_2 and T_σ is defined by

$$\begin{aligned}[b_1,b_2,T_\sigma](f_1,f_2)(x) =& b_1(x)b_2(x)T_\sigma(f_1,f_2)(x) - b_1(x)T_\sigma(f_1,b_2f_2)(x)\\ &- b_2(x)T_\sigma(b_1f_1,f_2)(x) + T_\sigma(b_1f_1,b_2f_2)(x).\end{aligned} \tag{1.7}$$

Also, the commutators $[b_1, T_\sigma]$ and $[b_2, T_\sigma]$ are respectively defined as follows:

$$[b_1,T_\sigma](f_1,f_2)(x) = b_1(x)T_\sigma(f_1,f_2)(x) - T_\sigma(b_1f_1,f_2)(x), \tag{1.8}$$

and

$$[b_2,T_\sigma](f_1,f_2)(x) = b_2(x)T_\sigma(f_1,f_2)(x) - T_\sigma(f_1,b_2f_2)(x). \tag{1.9}$$

The following definitions of Muckenhoupt's weight and multiple-weight are introduced in [15] and [17], respectively.

Definition 1.2. *Let* $p \in (1,\infty)$*. A non-negative* μ*-measurable function* ω *is called an* $A_p(\mathbb{R}^n)$ *weight if there exists a positive constant* C *such that, for all balls* $B \subset \mathcal{X}$*,*

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left\{\frac{1}{|B|}\int_B [\omega(x)]^{1-p'}dx\right\}^{p-1} \le C. \tag{1.10}$$

And a weight ω *is called an* $A_1(\mathbb{R}^n)$ *weight if there exists a positive constant* C *such that, for all balls* $B \subset \mathcal{X}$*,*

$$\frac{1}{|B|}\int_B \omega(x)dx \le C \inf_{y\in B}\omega(y). \tag{1.11}$$

As in the classical setting, let

$$A_\infty(\mathbb{R}^n) = \bigcup_{p=1}^{\infty} A_p(\mathbb{R}^n).$$

Definition 1.3. *Let $0 < p < \infty$ and $1 \leq p_1, p_2 < \infty$ with satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Given $\vec{\omega} = (\omega_1, \omega_2)$ and $\vec{P} = (p_1, p_2)$, for all $x \in \mathbb{R}^n$, set*

$$\nu_{\vec{\omega}} := \prod_{i=1}^{2} [\omega_i(x)]^{\frac{p}{p_i}}.$$

Multiple-weight $\vec{\omega}$ is said to satisfy the $A_{\vec{P}}$ condition if there exists a positive constant C such that,

$$\left\{ \frac{1}{|B|} \int_B \prod_{i=1}^{2} [\omega_i(x)]^{\frac{p}{p_i}} \right\}^{\frac{1}{p}} \times \prod_{i=1}^{2} \left\{ \frac{1}{|B|} \int_B [\omega_i(x)]^{1-p_i'} dx \right\}^{\frac{1}{p_i'}} \leq C. \tag{1.12}$$

When $p_i = 1$, the term $\left\{ \frac{1}{|B|} \int_B [\omega_i(x)]^{1-p_i'} dx \right\}^{\frac{1}{p_i'}}$ is understood as $(\inf_B \omega_i)^{-1}$.

We recall the notion of a generalized fractional weighted Morrey space $L^{p,\eta,\varphi}(\omega)$ in [24].

Definition 1.4. *Let φ be a positive constant, increasing function on $(0, \infty)$ and there exists a constant $\widetilde{C} > 0$ such that*

$$\varphi(2t) \leq \widetilde{C} \varphi(t) \quad \text{for} \quad t \geq 0.$$

The above best possible constant $\widetilde{C}$ is called doubling constant for φ.

Let ω be a non-negative weight function on $\mathbb{R}^n$, $\eta \in [0, n)$, $p \in [1, \frac{n}{\eta})$ and $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then the generalized fractional weighted Morrey space $L^{p,\eta,\varphi}(\omega)$ is defined by

$$L^{p,\eta,\varphi}(\omega) := \left\{ f \in L^p_{\rm loc}(\mathbb{R}^n) : \|f\|_{L^{p,\eta,\varphi}(\omega)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\eta,\varphi}(\omega)} = \sup_{x \in \mathbb{R}^n,\, r>0} [\varphi(r)]^{\frac{\eta}{n} - \frac{1}{p}} \left(\int_{B(x,r)} |f(y)|^p \omega(y) dy \right)^{\frac{1}{p}}, \tag{1.13}$$

and $B(x, r)$ is a ball with center at x and radius $r > 0$.

Remark 1.1. (i) If we take $\omega \equiv 1$ in (1.13), then the generalized fractional weighted Morrey space $L^{p,\eta,\varphi}(\omega)$ is just generalized fractional Morrey space $L^{p,\eta,\varphi}(\mathbb{R}^n)$, that is

$$\|f\|_{L^{p,\eta,\varphi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} [\varphi(r)]^{\frac{\eta}{n} - \frac{1}{p}} \left(\int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}}. \tag{1.14}$$

(ii) If we take $\eta = 0$ in (1.13), then the generalized fractional weighted Morrey space $L^{p,\eta,\varphi}(\omega)$ is just the generalized weighted Morrey space $L^{p,\varphi}(\omega)$ (see [21]).

(iii) If we take $\varphi(r) = r^\delta$ with $\delta \in (0,\infty)$, then the space $L^{p,\eta,\varphi}(\omega)$ is just weighted Morrey space $L^{p,\delta}(\omega)$ on $\mathbb{R}^n$, which is first introduced by Komori and Shirai in [16].

(iv) If we take $\omega \equiv 1$ and $\varphi(r) = r^\delta$ with $\delta \in (0,\infty)$, then the space $L^{p,\eta,\varphi}(\omega)$ is just classical Morrey space $L^{p,\delta}(\mathbb{R}^n)$ introduce by Morrey in [22].

The organization of this paper is as follows. In Section 2, we prove that bilinear pseudo-differential operator T_σ is bounded from the products of generalized fractional weighted Morrey space $L^{p_1,\eta_1,\varphi}(\omega_1) \times L^{p_2,\eta_2,\varphi}(\omega_2)$ into $L^{p,\eta,\varphi}(\vec{\omega})$, where $\vec{\omega} = (\omega_1,\omega_2) \in A_{\vec{P}}$, $\vec{P} = (p_1,p_2)$, $\eta = \eta_1+\eta_2$ and $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$, and bounded from the products of generalized weighted Morrey space $L^{p_1,\varphi}(\omega_1) \times L^{p_2,\varphi}(\omega_2)$ into space $L^{p,\varphi}(\nu_{\vec{\omega}})$ for all $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$. In Section 3, by establishing the sharp maximal estimate for the commutator $[b_1,b_2,T_\sigma]$ generated by T_σ and $b_1,b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, the authors prove that $[b_1,b_2,T_\sigma]$ is bounded on generalized fractional Morrey space $L^{p,\eta,\varphi}(\mathbb{R}^n)$ and on generalized Morrey space $L^{p,\varphi}(\mathbb{R}^n)$.

Finally, we make some conventions on notations. Throughout the whole paper, C represents a positive constant being independent of the main parameters, but it may vary from line to line. For any ball $B \subset \mathcal{X}$, we denote its center and radius, respectively, by c_B and r_B and, moreover, for any $\rho \in (0,\infty)$, we denote the ball $B(c_B,\rho r_B)$ by ρB. Given any $q \in (1,\infty)$, let $q' = q/(q-1)$ denote its conjugate index. For any set E, χ_E denotes its characteristic function, if E is also measurable and ω is a weight,

$$\omega(E) = \int_E \omega(x)dx.$$

2 Estimate for T_σ on $L^{p,\eta,\varphi}(\omega)$

In this section, we will mainly consider the boundedness of bilinear pseudo-differential operators T_σ on generalized fractional weighted Morrey spaces $L^{p,\eta,\varphi}(\omega)$ and on generalized weighted Morrey spaces $L^{p,\varphi}(\omega)$ is also obtained. First, we state the main theorems of this section as follows.

Theorem 2.1. *Let $0 \le \eta < n$, $\sigma \in BS^0_{1,0}$, $\vec{\omega} = (\omega_1,\omega_2) \in A_{\vec{P}}$ and $K(\cdot,\cdot,\cdot) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y,z) : x = y = z\})$ with satisfying*

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x,y,z)| \le \frac{C_{M,\alpha,\beta,\gamma}}{(|x-y|+|x-z|)^{2n+|\beta|+|\gamma|+M}} \tag{2.1}$$

for all $M > 0$ and multi-indices $\alpha,\beta,\gamma \in \mathbb{N}^n$. Then T_σ defined as in (1.5) is bounded from products of spaces $L^{p_1,\eta_1,\varphi}(\omega_1) \times L^{p_2,\eta_2,\varphi}(\omega_2)$ into spaces $L^{p,\eta,\varphi}(\vec{\omega})$, namely, there exists a constant $C > 0$ such that, for all $f_i \in L^{p_i,\eta_i,\varphi}(\omega_i)$, $1 < p_i < \frac{n}{\eta_i}$ with $i = 1,2$,

$$\|T_\sigma(f_1,f_2)\|_{L^{p,\eta,\varphi}(\vec{\omega})} \le C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_1,\eta_2,\varphi}(\omega_2)},$$

where $\frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}$ and $\eta = \eta_1+\eta_2$.

Theorem 2.2. *Let $\sigma \in BS^0_{1,0}$, $\vec{\omega} = (\omega_1, \omega_2) \in A_{\vec{P}}$ and $K(\cdot,\cdot,\cdot) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y,z) : x = y = z\})$ with satisfying* (2.1). *Then T_σ defined as in* (1.5) *is bounded from products of spaces $L^{p_1,\varphi}(\omega_1) \times L^{p_2,\varphi}(\omega_2)$ into spaces $L^{p,\varphi}(\nu_{\vec{\omega}})$, for all $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

Remark 2.1. By Remark 1.1, it is not difficult to see that Theorem 2.2 is a special case of Theorem 2.1, hence, in this section, we only state the proof of Theorem 2.1.

To prove Theorem 2.1, we need to recall the following lemmas (respectively, see [15, 20]).

Lemma 2.1. *Let $p \in [1,\infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Then there exist constant $C_1, C_2 \geq 1$ such that*

$$C_1^{-1}\left(\frac{|E|}{|B|}\right)^p \leq \frac{\omega(E)}{\omega(B)} \leq 1 - C_2^{-1}\left(1 - \frac{|E|}{|B|}\right)^p \tag{2.2}$$

for any ball B and measurable set $E \subset B$.

Lemma 2.2. *Let $1 < p_1,\ p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $\sigma \in BS^0_{1,0}$, $\vec{\omega} = (\omega_1, \omega_2) \in A_{\vec{P}}$ and $K(x,y,z) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y,z) : x = y = z\})$ with satisfying* (2.1). *Then T_σ defined as in* (1.5) *is bounded from products of weighted Lebesgue spaces $L^{p_1}_{\omega_1}(\mathbb{R}^n) \times L^{p_2}_{\omega_2}(\mathbb{R}^n)$ into space $L^p_{\nu_{\vec{\omega}}}(\mathbb{R}^n)$.*

Proof of Theorem 2.1. For the sake of convenience, decompose functions f_i as

$$f_i := f_i^1 + f_i^\infty := f_i\chi_{2B} + f_i\chi_{\mathbb{R}^n\setminus(2B)}, \quad i = 1,2.$$

Then, write

$$\begin{aligned}\|T_\sigma(f_1,f_2)\|_{L^{p,\eta,\varphi}(\nu_{\vec{\omega}})} \leq& \|T_\sigma(f_1^1,f_2^1)\|_{L^{p,\eta,\varphi}(\nu_{\vec{\omega}})} + \|T_\sigma(f_1^1,f_2^\infty)\|_{L^{p,\eta,\varphi}(\nu_{\vec{\omega}})} \\ &+ \|T_\sigma(f_1^\infty,f_2^1)\|_{L^{p,\eta,\varphi}(\nu_{\vec{\omega}})} + \|T_\sigma(f_1^\infty,f_2^\infty)\|_{L^{p,\eta,\varphi}(\nu_{\vec{\omega}})} \\ =:& d_1 + d_2 + d_3 + d_4.\end{aligned}$$

By applying (1.13) and Lemma 2.2, we obtain that

$$\begin{aligned} d_1 \leq& \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}} \left(\int_{\mathbb{R}^n} |T_\sigma(f_1^1,f_2^1)(x)|^p \nu_{\vec{\omega}}(x)dx\right)^{\frac{1}{p}} \\ \leq& C \sup_{x\in\mathbb{R}^n,\ r>0} \frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}} \|f_1^1\|_{L^{p_1}(\omega_1)} \|f_2^1\|_{L^{p_2}(\omega_2)} \\ \leq& C\|f_1\|_{L^{p_1,\eta_2,\varphi}(\omega_1)} \|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)} \sup_{x\in\mathbb{R}^n,\ r>0} \frac{[\varphi(2r)]^{\frac{1}{p}-\frac{\eta}{n}}}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}} \\ \leq& C\|f_1\|_{L^{p_1,\eta_2,\varphi}(\omega_1)} \|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}. \end{aligned}$$

For any $x \in B$, by applying (1.12), (1.13), (2.1) and Hölder inequality, we have

$$
\begin{aligned}
&|T_\sigma(f_1^1, f_2^\infty)(x)| \\
\leq & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y,z)||f_1^1(y)||f_2^\infty(z)|dydz \\
\leq & C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_1^1(y)||f_2^\infty(z)|}{(|x-y|+|x-z|)^{2n+M}}dydz \\
\leq & C\int_{\mathbb{R}^n\setminus(2B)} \frac{|f_2(z)|}{|c_B-z|^{2n+M}}dz\bigg(\int_{2B}|f_1(y)|dy\bigg) \\
\leq & C\sum_{k=1}^\infty \int_{2^{k+1}B\setminus(2^kB)} \frac{|f_2(z)|}{|c_B-z|^{2n+M}}dz\bigg\{\int_{2B}|f_1(y)|[\omega_1(y)]^{\frac{1}{p_1}}[\omega_1(y)]^{-\frac{1}{p_1}}dy\bigg\} \\
\leq & C\sum_{k=1}^\infty \frac{1}{(2^kr_B)^{2n+M}}\int_{2^{k+1}B}|f_2(z)|[\omega_2(z)]^{\frac{1}{p_2}}[\omega_2(z)]^{-\frac{1}{p_2}}dz \\
&\times\bigg\{\bigg(\int_{2B}|f_1(y)|^{p_1}\omega_1(y)dy\bigg)^{\frac{1}{p_1}}\bigg(\int_{2B}[\omega_1(y)]^{-\frac{p_1'}{p_1}}dy\bigg)^{\frac{1}{p_1'}}\bigg\} \\
\leq & C\sum_{k=1}^\infty \frac{1}{(2^kr_B)^{2n+M}}\bigg(\int_{2^{k+1}B}|f_2(z)|^{p_2}\omega_2(z)dz\bigg)^{\frac{1}{p_2}}\times\bigg(\int_{2^{k+1}B}[\omega_2(z)]^{-\frac{p_2'}{p_2}}dz\bigg)^{\frac{1}{p_2'}} \\
&\times\bigg\{\frac{1}{[\varphi(2r)]^{\frac{1}{p_1}-\frac{\eta_1}{n}}}\bigg(\int_{2B}|f_1(y)|^{p_1}\omega_1(y)dy\bigg)^{\frac{1}{p_1}}\bigg(\frac{1}{|2B|}\int_{2B}\omega_1(y)\mathrm{dy}\bigg)^{\frac{1}{p_1}} \\
&\times\bigg(\frac{1}{|2B|}\int_{2B}[\omega_1(y)]^{-\frac{p_1'}{p_1}}dy\bigg)^{\frac{1}{p_1'}}\frac{|2B|}{[\omega_1(2B)]^{\frac{1}{p_1}}}[\varphi(2r)]^{\frac{1}{p_1}-\frac{\eta_1}{n}}\bigg\} \\
\leq & C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\frac{|2B|}{[\omega_1(2B)]^{\frac{1}{p_1}}}[\varphi(2r)]^{\frac{1}{p_1}-\frac{\eta_1}{n}} \\
&\times\bigg\{\sum_{k=1}^\infty\frac{1}{(2^kr_B)^{2n+M}}\frac{1}{[\varphi(2^{k+1}r)]^{\frac{1}{p_2}-\frac{\eta_2}{n}}}\bigg(\int_{2^{k+1}B}|f_2(z)|^{p_2}\omega_2(z)dz\bigg)^{\frac{1}{p_2}} \\
&\times\bigg(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}[\omega_2(z)]^{-\frac{p_2'}{p_2}}dz\bigg)^{\frac{1}{p_2'}}\bigg(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\omega_2(z)dz\bigg)^{\frac{1}{p_2}} \\
&\times\frac{|2^{k+1}B|}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}[\varphi(2^{k+1}r)]^{\frac{1}{p_2}-\frac{\eta_2}{n}}\bigg\} \\
\leq & C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}\frac{|2B|}{[\omega_1(2B)]^{\frac{1}{p_1}}}[\varphi(2r)]^{\frac{1}{p_1}-\frac{\eta_1}{n}} \\
&\times\bigg\{\sum_{k=1}^\infty\frac{1}{(2^kr_B)^{2n+M}}\frac{|2^{k+1}B|}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}[\varphi(2^{k+1}r)]^{\frac{1}{p_2}-\frac{\eta_2}{n}}\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\leq & C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}\frac{|2B|^{1+\frac{M}{n}}}{[\omega_1(2B)]^{\frac{1}{p_1}}}[\varphi(2r)]^{\frac{1}{p_1}-\frac{\eta_1}{n}}\\
&\times\left\{\sum_{k=1}^{\infty}\frac{\widetilde{C}^{k(\frac{1}{p_2}-\frac{\eta_2}{n})}}{(2^k r_B)^{2n+M}}\frac{|2^{k+1}B|}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}[\varphi(2r)]^{\frac{1}{p_2}-\frac{\eta_2}{n}}\right\}\\
\leq & C[\varphi(2r)]^{\frac{1}{p}-\frac{\eta}{n}}\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}\frac{1}{[\omega_1(2B)]^{\frac{1}{p_1}}}\\
&\times\left\{\sum_{k=1}^{\infty}\frac{\widetilde{C}^{k(\frac{1}{p_2}-\frac{\eta_2}{n})}}{(2^k r_B)^{n+M}}\frac{|2B|^{1+\frac{M}{n}}}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}\right\}\\
\leq & C[\varphi(2r)]^{\frac{1}{p}-\frac{\eta}{n}}\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}\frac{1}{[\omega_1(2B)]^{\frac{1}{p_1}}}\\
&\times\left\{\sum_{k=1}^{\infty}\frac{\widetilde{C}^{k(\frac{1}{p_2}-\frac{\eta_2}{n})}}{2^{k(n+M)}}\frac{1}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}\right\}.
\end{aligned}
$$

Furthermore, by Definition 1.4 and $1 < p_2 < \frac{n}{\eta_2}$, we can deduce that

$$
\begin{aligned}
d_2 \leq & C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}\sup_{x\in\mathbb{R}^n,\ r>0}\left(\int_{B(x,r)}\nu_{\vec{\omega}}(x)dx\right)^{\frac{1}{p}}\\
&\times\frac{1}{[\omega_1(2B)]^{\frac{1}{p_1}}}\left\{\sum_{k=1}^{\infty}\frac{\widetilde{C}^{k(\frac{1}{p_2}-\frac{\eta_2}{n})}}{2^{k(n+M)}}\frac{1}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}\right\}\\
\leq & C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}\sup_{x\in\mathbb{R}^n,\ r>0}\left(\frac{1}{|B|}\int_{B(x,r)}\nu_{\vec{\omega}}(x)dx\right)^{\frac{1}{p}}|B|^{\frac{1}{p}}\\
&\times\prod_{i=1}^{2}\left\{\frac{1}{|B|}\int_B[\omega_i(x)]^{1-p_i'}dx\right\}^{\frac{1}{p_i'}}\left\{\frac{1}{|B|}\int_B[\omega_1(x)]^{1-p_1'}dx\right\}^{-\frac{1}{p_1'}}\frac{1}{[\omega_1(2B)]^{\frac{1}{p_1}}}\\
&\times\left\{\sum_{k=1}^{\infty}\frac{\widetilde{C}^{k(\frac{1}{p_2}-\frac{\eta_2}{n})}}{2^{k(n+M)}}\frac{1}{[\omega_2(2^{k+1}B)]^{\frac{1}{p_2}}}\left(\frac{1}{|B|}\int_B[\omega_2(x)]^{1-p_2'}dx\right)^{-\frac{1}{p_2'}}\right\}\\
\leq & C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)},
\end{aligned}
$$

where we have used the following fact that

$$\varphi(2t) \leq \widetilde{C}\varphi(t), \quad \widetilde{C} \text{ is a doubling constant.}$$

With an argument similar to that used in the above estimate of d_2, it is easy to obtain that

$$d_3 \leq C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}.$$

Now let us estimate d_4. For any $x\in B$, from (1.10), (1.12), (1.13), (2.1) and Hölder inequality, we obtain that

$$
\begin{aligned}
&|T_\sigma(f_1^\infty,f_2^\infty)(x)|\\
\le& C\int_{\mathbb{R}^n\setminus(2B)}\int_{\mathbb{R}^n\setminus(2B)}\frac{|f_1(y)||f_2(z)|}{(|x-y|+|x-z|)^{2n+M}}dydz\\
\le& C\prod_{i=1}^{2}\left[\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{|f_i(y_i)|}{|c_B-y_i|^{n+\frac{M}{2}}}dy_i\right]\\
\le& C\prod_{i=1}^{2}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{n+\frac{M}{2}}}\int_{2^{k+1}B}|f_i(y_i)|[\omega_i(y_i)]^{\frac{1}{p_i}}[\omega_i(y_i)]^{-\frac{1}{p_i}}dy_i\right\}\\
\le& C\prod_{i=1}^{2}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{n+\frac{M}{2}}}\left(\int_{2^{k+1}B}|f_i(y_i)|^{\frac{1}{p_i}}\omega_i(y_i)dy_i\right)^{\frac{1}{p_i}}\times\left(\int_{2^{k+1}B}[\omega_i(y_i)]^{-\frac{p_i'}{p_i}}dy_i\right)^{\frac{1}{p_i'}}\right\}\\
\le& C\prod_{i=1}^{2}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{n+\frac{M}{2}}}\left(\frac{1}{[\varphi(2^{k+1}r)]^{1-\frac{p_i\eta_i}{n}}}\int_{2^{k+1}B}|f_i(y_i)|^{\frac{1}{p_i}}\omega_i(y_i)dy_i\right)^{\frac{1}{p_i}}\right.\\
&\quad\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}\omega_i(y_i)dy_i\right)^{\frac{1}{p_i}}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}[\omega_i(y_i)]^{-\frac{p_i'}{p_i}}dy_i\right)^{\frac{1}{p_i'}}\\
&\quad\left.\times[\varphi(2^{k+1}r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}\frac{|2^{k+1}B|}{[\omega_i(2^{k+1}B)]^{\frac{1}{p_i}}}\right\}\\
\le& C\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\eta_i,\varphi}(\omega_i)}\left\{\sum_{k=1}^{\infty}\frac{|2^{k+1}B|}{(2^kr)^{n+\frac{M}{2}}}\frac{[\varphi(2^{k+1}r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}{[\omega_i(2^{k+1}B)]^{\frac{1}{p_i}}}\right\}\\
\le& C\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\eta_i,\varphi}(\omega_i)}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{\frac{M}{2}}}\frac{[\varphi(2^{k+1}r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}{[\omega_i(2^{k+1}B)]^{\frac{1}{p_i}}}\right\}.
\end{aligned}
$$

Further, by Definition 1.4, we can deduce that

$$
\begin{aligned}
d_4\le& C\sup_{x\in\mathbb{R}^n,\ r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\nu_{\vec{\omega}}(x)dx\right)^{\frac{1}{p}}|B(x,r)|^{\frac{1}{p}}\\
&\times\prod_{i=1}^{2}\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)}[\omega_i(x)]^{1-p_i'}dx\right\}^{\frac{1}{p_i'}}\\
&\times\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\eta_i,\varphi}(\omega_i)}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{\frac{M}{2}}}\frac{[\varphi(2^{k+1}r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}{[\omega_i(2^{k+1}B)]^{\frac{1}{p_i}}}\right\}\\
&\times\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)}[\omega_i(x)]^{1-p_i'}dx\right\}^{-\frac{1}{p_i'}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\eta_i,\varphi}(\omega_i)}\sup_{x\in\mathbb{R}^n,\, r>0}|B(x,r)|^{\frac{1}{p}}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^k r)^{\frac{M}{2}}}\frac{[\varphi(2^{k+1}r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}{[\varphi(r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}\right\}\frac{1}{[\omega_i(2^{k+1}B)]^{\frac{1}{p_i}}}\\
&\quad\times\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)}[\omega_i(x)]^{1-p_i'}dx\right\}^{-\frac{1}{p_i'}}\\
&\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\eta_i,\varphi}(\omega_i)}\sup_{x\in\mathbb{R}^n,\, r>0}\left\{\sum_{k=1}^{\infty}\frac{r^{\frac{2}{p_i}}}{(2^k r)^{\frac{M}{2}}}\frac{[\varphi(2^{k+1}r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}{[\varphi(r)]^{\frac{1}{p_i}-\frac{\eta_i}{n}}}\right\}\frac{[\omega_i(B)]^{\frac{1}{p_i}}}{[\omega_i(2^{k+1}B)]^{\frac{1}{p_i}}}\\
&\leq C\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\eta_i,\varphi}(\omega_i)}\left(\sum_{k=1}^{\infty}\frac{\widetilde{C}^{k(\frac{1}{p_i}-\frac{\eta_i}{n})}}{2^{\frac{kM}{2}}}\right)\\
&\leq C\|f_1\|_{L^{p_1,\eta_1,\varphi}(\omega_1)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\omega_2)}.
\end{aligned}
$$

Which, combing the estimates of d_1, d_2 and d_3, the proof of Theorem 2.1 is completed. □

3 Estimate for commutator $[b_1, b_2, T_\sigma]$ on $L^{p,\eta,\varphi}(\mathbb{R}^n)$

In this section, by establishing the sharp maximal estimate for commutator $[b_1, b_2, T_\sigma]$ which is generated by bilinear pseudo-differential operator T_σ and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, the authors prove that the commutator $[b_1, b_2, T_\sigma]$ is bounded from the products of generalized fractional Morrey space $L^{p_1,\eta_1,\varphi}(\mathbb{R}^n) \times L^{p_2,\eta_2,\varphi}(\mathbb{R}^n)$ into space $L^{p,\eta,\varphi}(\mathbb{R}^n)$, and bounded from the products of generalized Morrey space $L^{p_1,\varphi}(\mathbb{R}^n) \times L^{p_2,\varphi}(\mathbb{R}^n)$ into space $L^{p,\varphi}(\mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for $1 < p_1, p_2 < \infty$. First, we state the main theorems of this section as follows.

Theorem 3.1. *Let $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $K(\cdot,\cdot,\cdot)$ satisfy* (2.1) *and $1 < p_1, p_2 < \infty$ with satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Suppose that T_σ defined as in* (1.5) *is bounded from the products of spaces $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into space $L^{\frac{1}{2}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that, for all $f_i \in L^{p_i,\eta_i,\varphi}(\mathbb{R}^n)$, $i = 1, 2$,*

$$\|[b_1, b_2, T_\sigma](f_1, f_2)\|_{L^{p,\eta,\varphi}(\mathbb{R}^n)} \leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1,\eta_1,\varphi}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\eta_2,\varphi}(\mathbb{R}^n)}.$$

Theorem 3.2. *Let $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, $K(\cdot,\cdot,\cdot)$ satisfy* (2.1) *and $1 < p_1, p_2 < \infty$ with satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Suppose that T_σ defined as in* (1.5) *is bounded from the products of spaces $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into space $L^{\frac{1}{2}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that, for all $f_i \in L^{p_i,\varphi}(\mathbb{R}^n)$, $i = 1, 2$,*

$$\|[b_1, b_2, T_\sigma](f_1, f_2)\|_{L^{p,\varphi}(\mathbb{R}^n)} \leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1,\varphi}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\varphi}(\mathbb{R}^n)}.$$

Before stating the proof of main theorems, we should recall some necessary results given in [3, 15] as follows.

Lemma 3.1. *If σ is a symbol in $BS^0_{1,0}$, then T_σ defined as in* (1.5) *has a bounded extension from products of spaces $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, for all $1<p_1,p_2<\infty$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$.*

Lemma 3.2. *(1) Let $p\in(1,\infty)$ and $r\in(1,p)$. The non-centered maximal operators N and M_r are respectively defined by, for any $f\in L^1_{\rm loc}(\mathbb{R}^n)$,*

$$Nf(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|dy, \tag{3.1}$$

and

$$M_rf(x)=\sup_{B\ni x}\left(\frac{1}{|B|}\int_B|f(y)|^rdy\right)^{\frac{1}{r}}\quad for\quad 1<r<\infty, \tag{3.2}$$

are bounded on $L^p(\mathbb{R}^n)$ and also bounded from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

(2) For any $f\in L^1_{\rm loc}(\mathbb{R}^n)$, $\tau\in(0,1)$ and almost every $x\in\mathbb{R}^n$, the following inequality

$$|f(x)|\le N_\tau f(x) \tag{3.3}$$

holds true, where $N_\tau(f)(x)=[N(|f|^\tau)(x)]^{\frac{1}{\tau}}$.

For any $f\in L^1_{\rm loc}(\mathbb{R}^n)$, the sharp maximal function $M^\sharp(f)$ is defined by

$$M^\sharp f(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)-f_B|dy. \tag{3.4}$$

For any $0<\tau<1$, let

$$M^\sharp_\tau(f)(x)=[M^\sharp(|f|^\tau)(x)]^{\frac{1}{\tau}}.$$

Moreover, from [15,23], it is easy to see that sharp maximal function $M^\sharp f$ defined in (3.4) is equivalent to the following form

$$M^\sharp f(x)\approx\sup_{B\ni x}\inf_{c\in\mathbb{C}}\frac{1}{|B|}\int_B|f(y)-c|dy. \tag{3.5}$$

Lemma 3.3. *Let $0<p,\tau<\infty$ and $\omega\in\bigcup_{1\le r<\infty}A_r$. Then there exits a constant $C>0$ such that, for any smooth function f for which the left-hand side is finite,*

$$\int_{\mathbb{R}^n}[N_\tau(f)(x)]^p\omega(x)dx\le C\int_{\mathbb{R}^n}[M^\sharp_\tau(f)(x)]^p\omega(x)dx.$$

Corollary 3.1. *If $f\in{\rm BMO}(\mathbb{R}^n)$, then there exists a constant $C>0$ such that, for any balls B and $p\in[1,\infty)$,*

$$\left(\frac{1}{|B|}\int_B|f(x)-f_B|^pdx\right)^{\frac{1}{p}}\le C\|f\|_{{\rm BMO}(\mathbb{R}^n)}.$$

Lemma 3.4 (Kolmogorov's theorem)**.** *Let $0<p<q<\infty$ and for any measurable function f. Define that, for $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$,*

$$\|f\|_{L^{q,\infty}(\mathbb{R}^n)}=\sup_{\lambda>0}\lambda|\{x\in\mathbb{R}^n:|f(x)|>\lambda\}|^{\frac{1}{q}}$$

and

$$N_{p,q}(f)=\sup_B\frac{\|f\chi_B\|_{L^p(\mathbb{R}^n)}}{\|\chi_B\|_{L^r(\mathbb{R}^n)}},$$

where the sup is taken over all measurable sets B with $0<|B|<\infty$. Then there exists a positive constant C,

$$N_{p,q}(f)\le C\|f\|_{L^{q,\infty}(\mathbb{R}^n)}.$$

Also, we need to establish the following sharp maximal estimate for $[b_1,b_2,T_\sigma]$.

Lemma 3.5. *Let $b_1,b_2\in\mathrm{BMO}(\mathbb{R}^n)$, $1<p_1,p_2,p<\infty$, $1<s<p$ and $0<\eta<\frac{1}{2}$. Suppose that T_σ defined as in* (1.5) *is bounded from the products of spaces $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ into space $L^{\frac{1}{2}}(\mathbb{R}^n)$. Then there exists a constant $C>0$ such that, for any $x\in\mathbb{R}^n$, $f_1\in L^{p_1}(\mathbb{R}^n)$ and $f_2\in L^{p_2}(\mathbb{R}^n)$,*

$$\begin{aligned}
&M^\sharp_\eta[b_1,b_2,T_\sigma](f_1,f_2)(x)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r(T_\sigma(f_1,f_2))(x)\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r([b_2,T_\sigma](f_1,f_2))(x)+C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r([b_1,T_\sigma](f_1,f_2))(x)\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x), \qquad (3.6a)\\
&M^\sharp_\eta[b_1,T_\sigma](f_1,f_2)(x)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r(T_\sigma(f_1,f_2))(x)+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x), \qquad (3.6b)\\
&M^\sharp_\eta[b_2,T_\sigma](f_1,f_2)(x)\\
&\le C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r(T_\sigma(f_1,f_2))(x)+C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x). \qquad (3.6c)
\end{aligned}$$

Proof. Without loss of generality, we may assume that $f_1,f_2\in L^\infty_c(\mathbb{R}^n)$. And decompose function f_i as

$$f_i:=f_i^1+f_i^\infty:=f_i\chi_{2B}+f_i\chi_{\mathbb{R}^n\backslash 2B},\quad i=1,2. \qquad (3.7)$$

Since the methods for (3.6a), (3.6b) and (3.6c) are similar, so we only need to estimate (3.6a) in this paper. By the definition of $M^\sharp_\eta$, it only suffices to show that

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B\Big||[b_1,b_2,T_\sigma](f_1,f_2)(z)|^\eta-|h_B|^\eta\Big|dz\right)^{\frac{1}{\eta}}\\
\le&C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r(T_\sigma(f_1,f_2))(x)+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r([b_2,T_\sigma](f_1,f_2))(x)\\
&+C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r([b_1,T_\sigma](f_1,f_2))(x)+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x),
\end{aligned}$$

where

$$h_B := \Big(T_\sigma((b_1-(b_1)_B)f_1^\infty,(b_2-(b_2)_B)f_2^\infty) \Big)_B.$$

For any $z \in B$, since

$$\begin{aligned}
&[b_1,b_2,T_\sigma](f_1,f_2)(z)\\
=&(b_1(z)-(b_1)_B)(b_2(z)-(b_2)_B)T_\sigma(f_1,f_2)(z)\\
&-(b_1(z)-(b_1)_B)T_\sigma(f_1,(b_2(\cdot)-b_2(z)+b_2(z)-(b_2)_B)f_2)(z)\\
&-(b_1(z)-(b_1)_B)(b_2(z)-(b_2)_B)T_\sigma(f_1,f_2)(z)\\
&+(b_1(z)-(b_1)_B)[b_2,T_\sigma](f_1,f_2)(z)-(b_1(z)-(b_1)_B)(b_2(z)-(b_2)_B)T_\sigma(f_1,f_2)(z)\\
&+(b_2(z)-(b_2)_B)[b_1,T_\sigma](f_1,f_2)(z)+T_\sigma((b_1(\cdot)-(b_1)_B)f_1,(b_2(\cdot)-(b_2)_B)f_2)(z)\\
=&T_\sigma((b_1(\cdot)-(b_1)_B)f_1,(b_2(\cdot)-(b_2)_B)f_2)(z)+(b_1(z)-(b_1)_B)[b_2,T_\sigma](f_1,f_2)(z)\\
&+(b_2(z)-(b_2)_B)[b_1,T_\sigma](f_1,f_2)(z)-(b_1(z)-(b_1)_B)(b_2(z)-(b_2)_B)T_\sigma(f_1,f_2)(z),
\end{aligned}$$

then, we write

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B\Big||[b_1,b_2,T_\sigma](f_1,f_2)(z)|^\eta-|h_B|^\eta\Big|dz\right)^{\frac{1}{\eta}}\\
\le& C\left(\frac{1}{|B|}\int_B\Big|(b_1(z)-(b_1)_B)(b_2(z)-(b_2)_B)T_\sigma(f_1,f_2)(z)\Big|^\eta dz\right)^{\frac{1}{\eta}}\\
&+C\left(\frac{1}{|B|}\int_B\Big|(b_1(z)-(b_1)_B)T_\sigma(f_1,(b_2(\cdot)-(b_2)_B)f_2)(z)\Big|^\eta dz\right)^{\frac{1}{\eta}}\\
&+C\left(\frac{1}{|B|}\int_B\Big|(b_2(z)-(b_2)_B)T_\sigma((b_1(z)-b_1(\cdot))f_1,f_2)(z)\Big|^\eta dz\right)^{\frac{1}{\eta}}\\
&+C\left(\frac{1}{|B|}\int_B\Big|T_\sigma((b_1(\cdot)-(b_1)_B)f_1,(b_2(\cdot)-(b_2)_B)f_2)(z)-h_B\Big|^\eta dz\right)^{\frac{1}{\eta}}\\
=:&\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3+\mathrm{E}_4.
\end{aligned}$$

For any $1<r_1,r_2,r<\infty$ with satisfying $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r}=\frac{1}{\eta}$, by Hölder inequality and Corollary 3.1, we have

$$\begin{aligned}
\mathrm{E}_1\le& C\left(\frac{1}{|B|}\int_B|b_1(z)-(b_1)_B|^{r_1}|dz\right)^{\frac{1}{r_1}}\left(\frac{1}{|B|}\int_B|b_2(z)-(b_2)_B|^{r_2}dz\right)^{\frac{1}{r_2}}\\
&\times\left(\frac{1}{|B|}\int_B|T_\sigma(f_1,f_2)(z)|^r dz\right)^{\frac{1}{r}}
\end{aligned}$$

$$\leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r(T_\sigma(f_1,f_2))(x).$$

For any $\eta\in(0,\frac{1}{2})$, choosing a fit $s\in(1,\infty)$ with satisfying $\frac{1}{s}+\frac{1}{r}=\frac{1}{\eta}$, from Hölder inequality and Corollary 3.1, it follows that

$$\begin{aligned}\mathrm{E}_2\leq& C\left(\frac{1}{|B|}\int_B|b_1(z)-(b_1)_B|^s dz\right)^{\frac{1}{s}}\left(\frac{1}{|B|}\int_B|T_\sigma(f_1,(b_2(z)-b_2(\cdot))f_2)(z)|^r dz\right)^{\frac{1}{r}}\\ \leq& C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r([b_2,T_\sigma](f_1,f_2))(x).\end{aligned}$$

Similarly, it is not difficult to obtain that

$$\mathrm{E}_3\leq C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_r([b_1,T_\sigma](f_1,f_2))(x).$$

By (3.7), write

$$\begin{aligned}\mathrm{E}_4\leq& C\left(\frac{1}{|B|}\int_B\left|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^1)(z)\right|^\eta dz\right)^{\frac{1}{\eta}}\\ &+C\left(\frac{1}{|B|}\int_B\left|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^\infty)(z)\right|^\eta dz\right)^{\frac{1}{\eta}}\\ &+C\left(\frac{1}{|B|}\int_B\left|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^\infty,(b_2(\cdot)-(b_2)_B)f_2^1)(z)\right|^\eta dz\right)^{\frac{1}{\eta}}\\ &+C\left(\frac{1}{|B|}\int_B\left|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^\infty,(b_2(\cdot)-(b_2)_B)f_2^\infty)(z)-h_B\right|^\eta dz\right)^{\frac{1}{\eta}}\\ =&\mathrm{E}_{41}+\mathrm{E}_{42}+\mathrm{E}_{43}+\mathrm{E}_{44}.\end{aligned}$$

By the Kolmogorov's theorem, $(L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n),L^{\frac{1}{2},\infty}(\mathbb{R}^n))$-boundedness of T_σ, Hölder inequality and Corollary 3.1, we can deduce that

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B\left|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^1)(z)\right|^\eta dz\right)^{\frac{1}{\eta}}\\ =&\frac{\|\chi_B\|_{L^{\frac{\eta}{1-2\eta}}}}{|B|^{\frac{1}{\eta}}}\frac{\|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^1)\chi_B\|_{L^\eta}}{\|\chi_B\|_{L^{\frac{\eta}{1-2\eta}}}}\\ \leq&\frac{C}{|B|^2}\|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^1)\|_{L^{\frac{1}{2},\infty}(\mathbb{R}^n)}\\ \leq& C\frac{1}{|B|}\int_{2B}|b_1(y)-(b_1)_B||f_1(y_1)|dy_1\left(\frac{1}{|B|}\int_{2B}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\frac{1}{|B|}\left(\int_{2B}|b_1(y)-(b_1)_{2B}||f_1(y_1)|dy_1+|(b_1)_B-(b_1)_{2B}|\int_{2B}|f_1(y_1)|dy_1\right)\\
&\qquad\times\left(\frac{1}{|B|}\int_{2B}|b_2(z)-(b_2)_{2B}||f_2(y_2)|dy_2+|(b_2)_B-(b_2)_{2B}|\frac{1}{|B|}\int_{2B}|f_2(y_2)|dy_2\right)\\
&\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x),
\end{aligned}
$$

hence, we have

$$
\mathrm{E}_{41}\le C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x).
$$

To estimate E_{42}, we first consider

$$
|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^\infty)(z)| \quad \text{for} \quad z\in B.
$$

By (2.1), (3.2), Hölder inequality and Corollary 3.1, we obtain that

$$
\begin{aligned}
&|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^1,(b_2(\cdot)-(b_2)_B)f_2^\infty)(z)|\\
\le& C\int_{\mathbb{R}^n\setminus(2B)}\int_{2B}\frac{|b_1(y_1)-(b_1)_B||b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n+M}}dy_1dy_2\\
\le& C\int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|c_B-y_2|^{2n+M}}dy_2\\
\le& C|2B|\left(\frac{1}{|2B|}\int_{2B}|f_1(y_1)|^{p_1}dy_1\right)^{\frac{1}{p_1}}\left(\frac{1}{|2B|}\int_{2B}|b_1(y_1)-(b_1)_B|^{p_1'}dy_1\right)^{\frac{1}{p_1'}}\\
&\times\left(\sum_{k=1}^\infty\int_{2^{k+1}B\setminus(2^kB)}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|c_B-y_2|^{2n+M}}dy_2\right)\\
\le& C|2B|\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)\times\left[\sum_{k=1}^\infty\frac{1}{(2^kr)^{2n+M}}\left(\int_{2^{k+1}B}|b_2(y_2)-(b_2)_{2^{k+1}B}||f_2(y_2)|dy_2\right.\right.\\
&\left.\left.+\left|(b_2)_{2^{k+1}B}-(b_2)_B\right|\int_{2^{k+1}B}|f_2(y_2)|dy_2\right)\right]\\
\le& C|2B|\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)\times\left\{\sum_{k=1}^\infty\frac{1}{(2^kr)^{2n+M}}\left[\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_2(y_2)|^{p_2}dy_2\right)^{\frac{1}{p_2}}\right.\right.\\
&\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_2(y_2)-(b_2)_{2^{k+1}B}|^{p_2'}dy_2\right)^{\frac{1}{p_2'}}|2^{k+1}B|\\
&\left.\left.+|2^{k+1}B|\left|(b_2)_{2^{k+1}B}-(b_2)_B\right|\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_2(y_2)|^{p_2}dy_2\right)^{\frac{1}{p_2}}\right]\right\}\\
\le& C|2B|\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x)\left(\sum_{k=1}^\infty\frac{(k+1)|2^{k+1}B|}{(2^kr)^{2n+M}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\leq & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x)\left(\sum_{k=1}^{\infty}\frac{(k+1)|2^{k+1}B||2B|^{1+\frac{M}{n}}}{(2^kr)^{2n+M}}\right)\\
\leq & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x)\left(\sum_{k=1}^{\infty}\frac{(k+1)}{2^{k(n+M)}}\right)\\
\leq & C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x),
\end{aligned}
$$

where we have used the following fact

$$
|b_B - b_{2^kB}| \leq Ck\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{3.8}
$$

Further, we can deduce that

$$
\mathrm{E}_{42} \leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x).
$$

With a way similar to that used in the estimate of E_{42}, it is easy to obtain that

$$
\mathrm{E}_{43} \leq C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_1}f_1(x)M_{p_2}f_2(x).
$$

Finally, we estimate E_{44}. For $z \in B$, by applying (2.1), (3.2), Hölder inequality, Corollary 3.1 and (3.8), we have

$$
\begin{aligned}
&|T_\sigma((b_1(\cdot)-(b_1)_B)f_1^\infty,(b_2(\cdot)-(b_2)_B)f_2^\infty)(z)|\\
\leq & C\int_{\mathbb{R}^n\setminus(2B)}\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_1(y_1)-(b_1)_B||b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n+M}}dy_1dy_2\\
\leq & C\prod_{i=1}^{2}\int_{\mathbb{R}^n\setminus(2B)}\frac{|b_i(y_i)-(b_i)_B||f_i(y_i)|}{|c_B-y_i|^{n+\frac{M}{2}}}dy_i\\
\leq & C\prod_{i=1}^{2}\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus(2^kB)}\frac{|b_i(y_i)-(b_i)_B||f_i(y_i)|}{|c_B-y_i|^{n+\frac{M}{2}}}dy_i\right)\\
\leq & C\prod_{i=1}^{2}\left[\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{n+\frac{M}{2}}}\left(\int_{2^{k+1}B}|b_i(y_i)-(b_i)_{2^{k+1}B}||f_i(y_i)|dy_i\right.\right.\\
&\left.\left.+|(b_i)_{2^{k+1}B}-(b_i)_B|\int_{2^{k+1}B}|f_i(y_i)|dy_i\right)\right]\\
\leq & C\prod_{i=1}^{2}\left\{\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{n+\frac{M}{2}}}\left[\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_i(y_i)|^{p_i}dy_i\right)^{\frac{1}{p_i}}\right.\right.\\
&\times\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|b_i(y_i)-(b_i)_{2^{k+1}B}|^{p_i'}dy_i\right)^{\frac{1}{p_i}}|2^{k+1}B|\\
&\left.\left.+k\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|f_i(y_i)|^{p_i}dy_i\right)^{\frac{1}{p_i}}|2^{k+1}B|\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_i}(f_i)(x)\left(\sum_{k=1}^{\infty}\frac{(k+1)|2^{k+1}B|}{(2^k r)^{n+\frac{M}{2}}}\right)\\
&\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_i}(f_i)(x)\left(\sum_{k=1}^{\infty}\frac{(k+1)}{2^{\frac{kM}{2}}}\right)\\
&\leq C\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}(\mathbb{R}^n)}M_{p_i}(f_i)(x).
\end{aligned}
$$

Which, together with the estimates of E_{43}, E_{42}, E_{41}, E_3, E_2 and E_1, implies (3.6a). □

Proof of Theorem 3.1. From (1.13), Lemmas 3.3, 3.4 and 3.5, it then follows that

$$
\begin{aligned}
&\|[b_1,b_2,T_\sigma](f_1,f_2)\|^p_{L^{p,\eta,\varphi}(\mathbb{R}^n)}\\
\leq&\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|N_\tau([b_1,b_2,T_\sigma](f_1,f_2))(x)|^p dx\\
\leq&\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M^\sharp_\tau([b_1,b_2,T_\sigma](f_1,f_2))(x)|^p dx\\
\leq&C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_r(T_\sigma(f_1,f_2))(x)|^p dx\\
&+C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_r([b_2,T_\sigma](f_1,f_2))(x)|^p dx\\
&+C\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_r([b_1,T_\sigma](f_1,f_2))(x)|^p dx\\
&+C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_{p_1}f_1(x)M_{p_2}f_2(x)|^p dx\\
\leq&C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_r(T_\sigma(f_1,f_2))(x)|^p dx\\
&+C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_{p_1}f_1(x)M_{p_2}f_2(x)|^p dx\\
\leq&C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|T_\sigma(f_1,f_2)(x)|^p dx\\
&+C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{[\varphi(r)]^{\frac{1}{p}-\frac{\eta}{n}}}\int_{B(x,r)}|M_{p_1}f_1(x)M_{p_2}f_2(x)|^p dx\\
\leq&C\|b_1\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|^p_{L^{p_1,\eta_1,\varphi}(\mathbb{R}^n)}\|f_2\|^p_{L^{p_2,\eta_2,\varphi}(\mathbb{R}^n)},
\end{aligned}
$$

which is the desired result. □

Proof of Theorem 3.2. By applying (1.13), Lemmas 3.3, 3.4 and (3.6c), we have

$$\begin{aligned}
&\|[b_1,b_2,T_\sigma](f_1,f_2)\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
\leq&\|N_\tau([b_1,b_2,T_\sigma](f_1,f_2))\|_{L^{p,\varphi}(\mathbb{R}^n)}\leq C\|M^\sharp_\tau([b_1,b_2,T_\sigma](f_1,f_2))\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
\leq&C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_r(T_\sigma(f_1,f_2))\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_r([b_2,T_\sigma](f_1,f_2))\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
&\quad+C\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_r([b_1,T_\sigma](f_1,f_2))\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_{p_1}f_1M_{p_2}f_2\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
\leq&C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_r(T_\sigma(f_1,f_2))\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_{p_1}f_1M_{p_2}f_2\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
\leq&C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|T_\sigma(f_1,f_2)\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
&\quad+C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|M_{p_1}f_1M_{p_2}f_2\|_{L^{p,\varphi}(\mathbb{R}^n)}\\
\leq&C\|b_1\|_{\mathrm{BMO}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_1\|_{L^{p_1,\varphi}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\varphi}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, we complete the proof. □

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